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Input-to-State Stability and Exponential Stability for Time-Delay Systems: Further Results

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***Abstract*— The main contribution of this paper is to establish a link between the exponential stability of an unforced system and the Input-to-State Stability (ISS) via the Liapunov-Krasovskii methodology. It is proved that a system which is (globally, locally) exponentially stable in the unforced case is (globally, locally) input-to-state stable when it is forced by a measurable and locally essentially bounded input, provided that the functional describing the dynamics in the unforced case is (globally, on bounded sets) Lipschitz and the functional describing the dynamics in the forced case satisfies a Lipschitz-like hypothesis with respect to the input. Moreover, a new feedback control law is provided for delay-free linearizable and stabilizable time-delay systems, whose dynamics is described by locally Lipschitz functionals, by which the closed loop system is ISS with respect to disturbances adding to the control law, a typical problem due to actuator errors.**

Keywords: **Input-to-State Stability, Exponential Stability, Nonlinear Time-Delay Systems, Liapunov-Krasovskii Theorem.**

I. Introduction

For non-delayed systems, the Input-to-State Stability (ISS) property has been widely studied and its efficiency has been proved in practical applications such as networked control and robot manipulators (see for instance [1], [2], [3], [4], [5], [6], [7], [8] and for a survey [9]). The main point here is to focus on the robustness problem of nonlinear perturbed systems with possible large perturbations. ISS implies not only that the unperturbed system is asymptotically stable in the Liapunov sense but also that its behavior remains bounded when its inputs (eg. exogenous perturbations) are bounded. This is due to the contribution of Sontag in [10], who was the first to harmonize the Liapunov state and the input-output approaches (see [11], [12], [13], [14]).

Recently, some authors have attempted to address the lack of results regarding time-delay systems. Until 2003, only the work [15] by Teel had been devoted to the ISS property. In Teel's paper, a definition of the input-to-state stability for time-delay systems was given and sufficient conditions were stated using a Razumikhin-type theorem. In [16], Pepe and Jiang extended the definition of the ISS-Liapunov function to Liapunov-Krasovskii functional and presented a sufficient condition to guarantee the ISS property. Also, a recent paper by Liberzon [17] is devoted to the quantized approach and ISS using Teel's propositions.

The scientific community's interest in the ISS property for time-delay systems is now rapidly increasing. In this context, we hope that this work will open even more perspectives with regard to this topic. Specifically, in this paper, we will exhibit a link between exponential stability and the ISS property. Exponential stability has proved its efficiency in networked control (see eg. [18]). However, the influence of disturbances on the solutions behavior have to be more deeply analyzed from both a qualitative and a quantitative point of view. For networked control systems, a first work in this direction is the paper [19] by Polushin and Marquez relying on Teel's results, which may be somewhat conservative due to the use of Razumikhin-Liapunov functions. Characterization of ISS for nonlinear time-delay systems is still a hard task despite recent results ([16], [20], [21].

We show in this paper a link between ISS and exponential stability for a large class of systems. It is proved that a system which is (globally, locally) exponentially stable in the unforced case is (globally, locally) input-to-state stable when it is forced by a measurable and locally essentially bounded input, provided that the functional describing the dynamics in the unforced case is (globally, on bounded sets) Lipschitz and the functional describing the dynamics in the forced case satisfies a Lipschitz-like hypothesis with respect to the input. Moreover, a new feedback control law is provided for delay-free linearizable and stabilizable time-delay systems, whose dynamics is described by locally Lipschitz functionals, by which the closed loop system is ISS with respect to disturbances adding to the control law, a typical problem due to actuator errors.

Notations:

For $y \in \mathrm{R}^n$, $|y|$ denotes the Euclidean norm of the vector y. The essential supremum norm of an essentially bounded function is indicated with the symbol $\|\cdot\|_\infty$. A function u is said to be *essentially bounded* if $ess\sup_{t\geq 0}|u(t)| < \infty$; for given times $0 \leq T_1 < T_2$, we indicate with $u_{[T_1,T_2)} : [0,+\infty) \to R^m$ the function given by $u_{[T_1,T_2)}(t) = u(t)$ for all $t \in [T_1, T_2)$ and $= 0$ elsewhere. An input u is said to be *locally essentially bounded* if, for any $T > 0$, $u_{[0,T)}$ is essentially bounded. A function $w : [0, b) \to R$, $0 < b \leq +\infty$, is said to be locally absolutely continuous if it is absolutely continuous in any interval $[0, c]$, $0 < c < b$; a continuous function $\omega : [0,\infty) \to [0,\infty)$ is of class $\mathcal{K}$ if it is strictly increasing and $\omega(0) = 0$, is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A function $\beta : [0,\infty)^2 \to [0,\infty)$ is of class $\mathcal{KL}$ if for each fixed t the function $s \to \beta(s,t)$ is of class $\mathcal{K}$ and for each fixed s the function $t \to \beta(s,t)$ is non-increasing and goes to zero as $t \to \infty$.

For a given $\tau > 0$, $\mathcal{C}$ denotes the space of continuous functions mapping the interval $[-\tau, 0]$ into R^n and for $\varphi \in \mathcal{C}$, $\|\varphi\|_c = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$. For a given positive real H, $\mathcal{C}_H$ denotes the space of continuous functions φ mapping the interval $[-\tau, 0]$ into R^n such that $\|\varphi\|_c < H$.

With the symbol $\|\cdot\|_a$ (see [16]) we indicate any semi-norm in $\mathcal{C}$, such that, for some positive reals $\gamma_a, \bar{\gamma}_a$, the following inequalities hold

$$\gamma_a|\phi(0)| \leq \|\phi\|_a \leq \bar{\gamma}_a \|\phi\|_c, \ \forall \phi \in \mathcal{C} \tag{1}$$

For any continuous function $x(s)$ defined on $-\tau \leq s < A$, $A > 0$, and any fixed t, $0 \leq t < A$, the standard symbol x_t will denote the element of $\mathcal{C}$ defined by $x_t(\theta) = x(t+\theta)$, $-\tau \leq \theta \leq 0$.

II. Preliminaries

A. Exponential stability of unforced systems

We will consider the exponential stability problem for the following equation in R^n with bounded delay $\tau > 0$:

$$\begin{cases} \dot{x}(t) = f(x_t), \quad t \geq 0, \quad f(0) = 0; \\ \qquad x_0 = \psi, \end{cases} \tag{2}$$

where $\psi \in \mathcal{C}$, and $f : \mathcal{C} \to \mathrm{R}^n$ is continuous and Lipschitz on bounded sets.

In this paper, we shall denote by $x(t,\psi)$ (a vector of R^n) the solution at time t of system (2) with the initial condition ψ at 0. We shall, by convenient abuse of notation, consider $x_t(\psi)$ (a function of $\mathcal{C}$) also a solution of (2). Observe that $x_0(\psi) = \psi$. We recall here the definition of exponential stability in the case of time-delay systems.

Definition 2.1: The solution $x(t) = 0$ of (2) is *exponentially stable* if there exists positive reals H, A, B such that for every $\psi \in \mathcal{C}_H$ the solution $x_t(\psi)$ of (2) exists $\forall\ t \geq 0$ and furthermore satisfies

$$\|x_t(\psi)\|_c \leq Ae^{-Bt}\|\psi\|_c \tag{3}$$

Definition 2.2: The solution $x(t) = 0$ of (2) is *globally exponentially stable* if there exists positive reals A, B such that for every $\psi \in \mathcal{C}$ the solution $x_t(\psi)$ of (2) exists $\forall\ t \geq 0$ and furthermore satisfies

$$\|x_t(\psi)\|_c \leq Ae^{-Bt}\|\psi\|_c \tag{4}$$

The next two theorems (here reported for the time invariant case), proved in [22, Lemma 33.1], will play an important role in the proof of our main theorems in the next section.

Theorem 2.3: If the system (2) is exponentially stable (with initial conditions in $\mathcal{C}_H$, $0 < H < +\infty$) then there exists a continuous functional $V(\varphi)$ defined on $\mathcal{C}_{\frac{H}{A}}$ and positive constants C_i, $i = 1,2,3,4$, such that the following conditions hold $\forall \varphi, \xi \in \mathcal{C}_{\frac{H}{A}}$:

$$C_1\|\varphi\|_c \leq V(\varphi) \leq C_2\|\varphi\|_c, \tag{5}$$

$$\limsup_{h\to 0^+} \frac{V(x_h(\varphi)) - V(\varphi)}{h} \leq -C_3\|\varphi\|_c, \tag{6}$$

$$|V(\varphi) - V(\xi)| \leq C_4\|\varphi - \xi\|_c \tag{7}$$

Theorem 2.4: If the system (2) is globally exponentially stable then there exists a continuous functional $V(\varphi)$ defined on $\mathcal{C}$ which satisfies the following conditions in $\mathcal{C}$:

$$C_1\|\varphi\|_c \leq V(\varphi) \leq C_2\|\varphi\|_c, \tag{8}$$

$$\limsup_{h\to 0^+} \frac{V(x_h(\varphi)) - V(\varphi)}{h} \leq -C_3\|\varphi\|_c, \tag{9}$$

where C_i, $i = 1,2,3$ are some positive constants. Moreover, if the functional f is globally Lipschitz, then there exists a positive real C_4, such that the following inequality holds $\forall \varphi, \xi \in \mathcal{C}$:

$$|V(\varphi) - V(\xi)| \leq C_4\|\varphi - \xi\|_c \tag{10}$$

Remark 1: In Theorems 2.3, 2.4, the coefficients C_i, $i = 1,2,3,4$ depend on the positive reals A, B and on the Lipschitz coefficient of the functional f in $\mathcal{C}_H$ (or in $\mathcal{C}$) and can be easily computed (see [22]).

B. Input-to-State Stability

As previously stated, a definition of input-to-state stability for time-delay systems has been given in [15] and a useful characterization has been presented in [16]. For the reader's convenience, and to make our work self-contained, we report here the definition of ISS for time delay systems and its characterization with an ISS-Liapunov-Krasovskii functional (see [10],[16]).

Consider the system

$$\begin{cases} \dot{x}(t) = f(x_t, u(t)), \quad t \geq 0, \\ \qquad x_0 = \psi, \end{cases} \tag{11}$$

where f is a continuous functional defined on $\mathcal{C} \times \mathrm{R}^m$, Lipschitz on bounded sets. The input u is a measurable and locally essentially bounded function of t for all $t \geq 0$. Consider also the unforced system

$$\begin{cases} \dot{x}(t) = f(x_t, 0), \quad t \geq 0, \\ \qquad x_0 = \psi, \end{cases} \tag{12}$$

For the reader's convenience Sontag's definition of ISS is reported below (see also [16]).

Definition 2.5: The system (11) is locally input-to-state stable if there exists two positive reals r and r_u, a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that, $\forall \psi \in C_r$ and $\forall u$ such that $ess\sup_{t\geq 0}|u(t)| < r_u$, the solution exists for all $t \geq 0$ and furthermore satisfies

$$|x(t,\psi)| \leq \beta(\|\psi\|_c, t) + \gamma(\|u_{[0,t]}\|_\infty). \tag{13}$$

Definition 2.6: The system (11) is input-to-state stable if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that, for any initial state ψ and any locally essentially bounded input u, the solution $x_t(\psi)$ exists for all $t \geq 0$ and furthermore satisfies

$$|x(t,\psi)| \leq \beta(\|\psi\|_c, t) + \gamma(\|u_{[0,t]}\|_\infty). \tag{14}$$

In the following, the continuity of a functional $V : \mathcal{C} \to R^+$ is intended with respect to the supremum norm.

Given a continuous functional $V : \mathcal{C} \to R^+$, the upper-right hand Dini derivative (as proposed by [23] and used in [24], [16] and, in a generalized version, in [25]) is given by

$$D^+V(\varphi, v) = \limsup_{h\to 0^+} \frac{1}{h}(V(\varphi_{h,v}) - V(\varphi)), \tag{15}$$

where $\varphi_{h,v} \in \mathcal{C}$ is given by

$$\varphi_{h,v}(s) = \begin{cases} \varphi(s+h), \quad s \in [-\tau, -h), \\ \varphi(0) + (s+h)f(\varphi, v), \quad s \in [-h, 0]. \end{cases} \tag{16}$$

It is proved in [26] that, under Carathéodory conditions, if the functional V is locally Lipschitz, then, for any $\varphi \in \mathcal{C}$, almost everywhere in t,

$$D^+V(x_t(\varphi), u(t)) = \limsup_{h\to 0^+} \frac{V(x_{t+h}(\varphi)) - V(x_t(\varphi))}{h} \tag{17}$$

Moreover, it is proved in [27] that the problem of the absolute continuity of the function $t \to V(x_t(\varphi))$ (see the hypothesis Hp_1 in [16]) is overcome if V is locally Lipschitz.

Taking into account the above two facts, a main contribution in [16] is here reported by the following definitions and theorems.

Definition 2.7: A locally Lipschitz continuous functional $V : \mathcal{C} \to \mathrm{R}^+$ is a local ISS Liapunov-Krasovskii functional for system (11) if there exist two positive reals k_1, k_2, $\mathcal{K}_\infty$-functions a, b, and $\mathcal{K}$-functions χ and α such that, $\forall \varphi \in \mathcal{C}_{k_1}$, $\forall u$ with $|u| < k_2$,

1) $a(|\varphi(0)|) \le V(\varphi) \le b(\|\varphi\|_a)$
2) $D^+V(\varphi, u) \le -\alpha(\|\varphi\|_a), \quad \forall \|\varphi\|_a \ge \chi(|u|)$

Theorem 2.8: If system (11) admits a local ISS Liapunov-Krasovskii functional, then it is locally ISS with $\gamma = a^{-1} \circ b \circ \chi$.

Definition 2.9: A locally Lipschitz continuous functional $V : \mathcal{C} \to \mathrm{R}^+$ is an ISS Liapunov-Krasovskii functional for system (11) if there exist $\mathcal{K}_\infty$-functions a, b, and $\mathcal{K}$-functions χ and α such that

1) $a(|\varphi(0)|) \le V(\varphi) \le b(\|\varphi\|_a)$
2) $D^+V(\varphi, u) \le -\alpha(\|\varphi\|_a), \quad \forall \|\varphi\|_a \ge \chi(|u|)$

Theorem 2.10: If system (11) admits an ISS Liapunov-Krasovskii functional, then it is ISS with $\gamma = a^{-1} \circ b \circ \chi$.

III. Main Results

A. Links between ISS and Exponential Stability

Our main results can now be stated as follows.

Theorem 3.1: Let there exist positive reals H, D and ℓ and a non negative real $p < 1$, such that:

1) the unforced system (12) is exponentially stable (initial conditions in $\mathcal{C}_H$);
2) for all $\varphi \in \mathcal{C}_H$, for all $u \in \mathrm{R}^m$ with $|u| < D$, the following inequality holds

$$|f(\varphi, u) - f(\varphi, 0)| \le \ell \max\{\|\varphi\|_c^p, 1\} |u| \tag{18}$$

Then, the perturbed system (11) is locally input-to-state stable, with r and r_u in the definition 2.6 being any positive reals satisfying the inequalities

$$r < \frac{H}{A}, \qquad r_u < D, \tag{19}$$

$$max \left\{ \frac{2C_4 l r_u}{C_3}, \left(\frac{2C_4 l r_u}{C_3}\right)^{\frac{1}{1-p}} \right\} \frac{C_2}{C_1} + \frac{C_2}{C_1} r < \frac{H}{A}, \tag{20}$$

C_i, $i = 1, 2, 3, 4$, being the positive constants given in Theorem 2.3 and A being the constant in (3) for the unforced system (12).

Remark 2: A large class of systems verifies the condition (18). For instance, systems $\dot{x}(t) = f(x_t) + g(x_t)u(t)$, with any f and $|g(\varphi)| \le \ell \max\{\|\varphi\|_c^p, 1\}$, verify the condition (18). The term $\max\{\|\varphi\|_c^p, 1\}$ may be substituted by a positive constant (included in l), but this would reduce, in general, the region where the input-to-state stability holds (see second inequality in 19).

Theorem 3.2: Let the system (12) be globally exponentially stable. Let there exist positive reals L, l such that

1) the following inequality holds $\forall \varphi_1, \varphi_2 \in \mathcal{C}$

$$|f(\varphi_1, 0) - f(\varphi_2, 0)| \le L \|\varphi_1 - \varphi_2\|_c ; \tag{21}$$

2) for all $\varphi \in \mathcal{C}$, for all $u \in \mathrm{R}^m$, the following inequality holds

$$|f(\varphi, u) - f(\varphi, 0)| \le \ell \max\{\|\varphi\|_c^p, 1\} |u| \tag{22}$$

Then, the perturbed system (11) is input-to-state stable.

B. An input-to-state stabilizing feedback

In this section we consider nonlinear systems

$$\begin{cases} \dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), & t \ge 0, \\ x_0 = \psi, \end{cases} \tag{23}$$

where f, g are locally Lipschitz continuous functionals defined on $\mathcal{C}$, $u(t) \in R^m$ is the control input, $d(t) \in R^m$ is a measurable and locally essentially bounded disturbance. In recent literature (see [28] and [29]), many results concerning the elementary theory of nonlinear feedback for time-delay systems have been achieved. On the basis of these results, we suppose here that there exists a feedback control law $u(t) = k(x_t)$, where $k : \mathcal{C} \to \mathcal{C}$ is a continuous mapping, such that, by this feedback control law, the system (23) is transformed into the system

$$\begin{cases} \dot{x}(t) = Fx(t) + g(x_t)d(t), & t \ge 0, \\ x_0 = \psi, \end{cases} \tag{24}$$

with F a Hurwitz matrix.

The feedback control k is proved to exist for instance for the class of nonlinear time-delay systems (with discrete delays, functions f, g smooth) which admit full uniform (type-III) vector relative degree with respect to some smooth output function h (a suitable change of variables may be necessary), see [28] [29]. Though the feedback control law may involve longer delays then the ones involved in the system dynamics, there is no loss of generality to consider the same maximum delay in the system and in the control law (zero terms with higher delays may be added in the system equations as well as initial conditions defined on longer delay intervals may be considered, see the example in [28]).

Theorem 3.3: Consider the feedback control law

$$u(t) = k(x_t) - g^T(x_t)Qx(t)x^T(t)Px(t), \tag{25}$$

where $P \in R^{n\times n}$ is any symmetric positive definite matrix and $Q \in R^{n\times n}$ is the symmetric positive definite matrix solution of $F^TQ+QF=-P$. Then, the closed loop system (23), (25) is input-to-state stable with respect to the measurable, locally essentially bounded disturbance $d(t)$.

Remark 3: Note that in this case the functional g does not have to satisfy the condition (18) as reported in Remark 2. As well known, given the symmetric positive definite matrix P, the matrix Q is equal to $\int_0^{+\infty} e^{F^T t}Pe^{Ft}dt$.

Remark 4: Note that, for the class of systems studied in this section, the new feedback control law is the same proposed by Sontag in the celebrated paper [10], when g is a function of $x(t)$ and not of x_t.

C. Proof of Theorem 3.1

We view system (11) as a perturbation of the unforced system (12). The main idea of the proof will be to show that there exists an ISS Liapunov-Krasovskii functional for system (11); theorem 2.10 will then insure that our system is ISS. Moreover, the functional we are looking for is the same as the one for the unforced system (using Theorem 2.3). The details of the proof are reported below.

Proof: The converse Liapunov theorem 2.3 shows that the unforced system (12) has a Liapunov-Krasovskii functional $V(\varphi)$ that satisfies, in $\mathcal{C}_{\frac{H}{A}}$, the inequalities (5-7). Note that V is Lipschitz in $\mathcal{C}_{\frac{H}{A}}$. Let $\|\varphi\|_c < \frac{H}{A}$ and $|u| < D$. Computing the upper right-hand Dini derivative of the functional V as in (15), we get:

$$\begin{aligned}D^+V(\varphi,u) &= \limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi))\\ &= \limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi_{h,0})\\ &\qquad -V(\varphi)+V(\varphi_{h,0}))\\ &\le D^+V(\varphi,0)+\limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi_{h,0}))\\ &\le -C_3\|\varphi\|_c+\limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi_{h,0})).\end{aligned}$$

Taking into account the condition (18) on f, the following inequalities hold for sufficiently small h:

$$\begin{aligned}|V(\varphi_{h,u})-V(\varphi_{h,0})| &\le C_4\|\varphi_{h,u}-\varphi_{h,0}\|_c\\ &= C_4\sup_{s\in[-\tau,0]}|\varphi_{h,u}(s)-\varphi_{h,0}(s)|\\ &\le C_4\sup_{s\in[-h,0]}|s+h||f(\varphi,u)-f(\varphi,0)|\\ &\le C_4|h|\ell\max\{\|\varphi\|_c^p,1\}|u|\end{aligned}$$

Let $\omega : R^+ \to R^+$ be the class $\mathcal{K}_\infty$ function defined as: $\omega(s)=\theta\, min\{s,s^{1-p}\}$, where $0<\theta<\frac{C_3}{C_4\ell}$. Then, if $\|\varphi\|_c \ge \omega^{-1}(|u|)$, the following inequalities hold:

$$D^+V(\varphi,u)\le -C_3\|\varphi\|_c+C_4\ell\theta\|\varphi\|_c\le-\delta\|\varphi\|_c,$$

where $\delta=C_3-C_4\ell\theta>0$. Let us choose $\theta=\frac{C_3}{2C_4 l}$, so that $\delta=\frac{C_3}{2}$.

Hence, the conditions of theorem 2.8 are satisfied using the norm $\|\cdot\|_c$ as a $\|\cdot\|_a$ norm, $a(s)=C_1s$, $b(s)=C_2s$, $\alpha(s)=\delta s$ and $\chi(s)=\omega^{-1}(s)$. We can conclude that the system (11) is locally input-to-state stable. The positive reals $r<\frac{H}{A}$ and $r_u<D$ can be computed by requiring that the solution $x_t(\varphi)$ corresponding to initial conditions φ with $\|\varphi\|_c<r$ and to input $u(t)$ with $ess\sup_{t\ge 0}|u(t)|<r_u$ satisfies the following inequality

$$\|x_t(\varphi)\|_c\le\frac{H}{A},\qquad t\ge 0 \tag{26}$$

To prove the inequality (26), we have just to consider that, in the ISS inequality (13) the functions β and γ are here given, for $s,t\ge 0$, by

$$\beta(s,t)=s\frac{C_2}{C_1}exp\left(-\frac{C_3}{2C_2}t\right), \tag{27}$$

$$\gamma(s)=\frac{C_2}{C_1}max\left\{\frac{2C_4ls}{C_3},\left(\frac{2C_4ls}{C_3}\right)^{\frac{1}{1-p}}\right\} \tag{28}$$

Therefore, the inequality (26) follows from the inequality (19) and the theorem is proved.

D. Proof of Theorem 3.2

Proof: The converse Liapunov theorem 2.4 shows that the unforced system (12) has a Liapunov-Krasovskii functional $V(\varphi)$ that satisfies, in $\mathcal{C}$, the inequalities (8-10). From the hypothesis 1) on the functional f it follows that V is globally Lipschitz in $\mathcal{C}$. Computing the upper right-hand Dini derivative of the functional V as in (15), we get:

$$\begin{aligned}D^+V(\varphi,u) &= \limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi))\\ &= \limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi_{h,0})\\ &\qquad -V(\varphi)+V(\varphi_{h,0}))\\ &\le D^+V(\varphi,0)+\limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi_{h,0}))\\ &\le -C_3\|\varphi\|_c+\limsup_{h\to 0}\frac{1}{h}(V(\varphi_{h,u})-V(\varphi_{h,0})).\end{aligned}$$

Taking into account the hypothesis 2) on the functional f, the following inequalities hold:

$$\begin{aligned}|V(\varphi_{h,u})-V(\varphi_{h,0})| &\le C_4\|\varphi_{h,u}-\varphi_{h,0}\|_c\\ &= C_4\sup_{s\in[-\tau,0]}|\varphi_{h,u}(s)-\varphi_{h,0}(s)|\\ &\le C_4\sup_{s\in[-h,0]}|s+h||f(\varphi,u)-f(\varphi,0)|\\ &\le C_4|h|\ell\max\{\|\varphi\|_c^p,1\}|u|\end{aligned}$$

Let $\omega : R^+ \to R^+$ be the class $\mathcal{K}_\infty$ function defined as: $\omega(s)=\theta\, min\{s,s^{1-p}\}$, where $0<\theta<\frac{C_3}{C_4\ell}$. Then, if $\|\varphi\|_c \ge \omega^{-1}(|u|)$, the following inequalities hold:

$$D^+V(\varphi,u)\le -C_3\|\varphi\|_c+C_4\ell\theta\|\varphi\|_c\le-\delta\|\varphi\|_c,$$

where $\delta=C_3-C_4\ell\theta>0$.

Hence, the conditions of theorem 2.10 are satisfied using the norm $\|\cdot\|_c$ as a $\|\cdot\|_a$ norm, $a(s) = C_1 s$, $b(s) = C_2 s$, $\alpha(s) = \delta s$ and $\chi(s) = \omega^{-1}(s)$. We can conclude that the system (11) is input-to-state stable.

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E. Proof of Theorem 3.3

Let us apply the theorem 2.10 to the closed loop system (23), (25). Let us consider the Liapunov-Krasovskii functional $V(\varphi) = \varphi^T(0)Q\varphi(0)$. The following inequalities hold for $|d| \le \varphi(0)^T P\varphi(0)$:

$$\begin{aligned}
D^+V(\varphi,d) &= \varphi^T(0)QH\varphi(0) + \varphi^T(0)H^TQ\varphi(0)\\
&- \varphi^T(0)Qg(\phi)g^T(\varphi)Q\varphi(0)\varphi^T(0)P\varphi(0)\\
&- \varphi^T(0)P\varphi(0)\varphi^T(0)Qg(\phi)g^T(\varphi)Q\varphi(0)\\
&+ \varphi^T(0)Qg(\varphi)d + d^Tg^T(\varphi)Q\varphi(0) \le\\
&- \varphi^T(0)P\varphi(0)\\
&- 2\varphi^T(0)Qg(\phi)g^T(\varphi)Q\varphi(0)\varphi^T(0)P\varphi(0)+\\
&2\varphi^T(0)Qg(\phi)d \le\\
&- \varphi^T(0)P\varphi(0) - 2|\varphi^T(0)Qg(\phi)|^2\varphi^T(0)P\varphi(0)+\\
&2\varphi^T(0)Qg(\phi)d \le\\
&- \varphi^T(0)P\varphi(0) - 2|\varphi^T(0)Qg(\phi)|^2\varphi^T(0)P\varphi(0)+\\
&2|\varphi^T(0)Qg(\phi)|^2\varphi^T(0)P\varphi(0) + \frac{1}{2}\varphi^T(0)P\varphi(0) \le\\
&- \frac{1}{2}\varphi^T(0)P\varphi(0)
\end{aligned}$$

Therefore, by theorem 2.10, it follows that the closed loop system (23),(25) is input-to-state stable with respect to measurable and locally essentially bounded disturbances $d(t)$. Note that in this case the Euclidean norm $|\varphi(0)|$ is used as a $\|\varphi\|_a$ semi-norm.

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IV. Examples

Theorems 3.1, 3.2 and 3.3 have nice applications in the nonlinear feedback control of time delay systems, when a disturbance adds to the control law, which usually happens because of actuator errors.

As an application of theorem 3.2, consider the following time-delay system:

$$\left\{\begin{array}{l}
\dot{x}_1(t) = x_2(t) + \int_{-1}^{0} 0.1\theta x_1(t+\theta)d\theta\\
\dot{x}_2(t) = x_1(t-1)x_2(t-1)+\\
\qquad \left(1 + |x_1(t-1)|^{\frac{1}{2}}\right)(u(t) + d(t))
\end{array}\right. \tag{29}$$

where u is the control input and d is an unknown measurable, locally essentially bounded disturbance. The following control law

$$u(t) = \frac{-x_1(t-1)x_2(t-1) + \left[\begin{array}{cc} -2 & -3 \end{array}\right]\left[\begin{array}{c} x_1(t)\\ x_2(t)\end{array}\right]}{1 + |x_1(t-1)|^{\frac{1}{2}}},$$

is such that the closed loop system becomes

$$\left\{\begin{array}{l}
\dot{x}_1(t) = x_2(t) + \int_{-1}^{0} 0.1\theta x_1(t+\theta)d\theta\\
\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + \left(1 + |x_1(t-1)|^{\frac{1}{2}}\right)d(t)
\end{array}\right. \tag{30}$$

The closed loop system (30) with zero disturbance ($d(t) \equiv 0$) is a globally exponentially stable linear time-delay system (it can be checked by Proposition 5.15, pp. 171 in [30]). If the disturbance is present, then the theorem 3.2 allows to say that the closed loop system (30) has the important ISS property with respect to the disturbance $d(t)$.

As an application of Theorem 3.3, consider the following time-delay system:

$$\left\{\begin{array}{l}
\dot{x}_1(t) = x_2(t)\\
\dot{x}_2(t) = x_1(t-\tau)x_2(t-\tau)+\\
\qquad (1 + x_1^2(t)x_2^2(t-\tau))(u(t) + d(t))
\end{array}\right. \tag{31}$$

where u is the control input and d is an unknown measurable, locally essentially bounded disturbance. The following control law ([28], [29])

$$u_1(t) = \frac{-x_1(t-\tau)x_2(t-\tau) + k^T\left[\begin{array}{c} x_1(t)\\ x_2(t)\end{array}\right]}{1 + x_1^2(t)x_2^2(t-\tau)},$$

with $k \in R^2$, is such that the closed loop system becomes

$$\left\{\begin{array}{l}
\dot{x}_1(t) = x_2(t)\\
\dot{x}_2(t) = k^T\left[\begin{array}{c} x_1(t)\\ x_2(t)\end{array}\right] + \left(1 + x_1^2(t)x_2^2(t-\tau)\right)d(t)
\end{array}\right. \tag{32}$$

By an easy choice of the vector k, the system (32) with zero disturbance ($d(t) \equiv 0$) is an exponentially stable linear delay-free system . But, if the disturbance is present, the time-delay closed loop system (32) is not ISS. Actually, any constant disturbance may cause the state variables to go to ∞. By theorem 3.3, the following control law

$$u(t) = u_1(t) - \left[\begin{array}{cc} 0 & 1 + x_1^2(t)x_2^2(t-\tau) \end{array}\right]Qx(t)x^T(t)x(t), \tag{33}$$

where Q is the symmetric positive matrix satisfying

$$\left[\begin{array}{cc} 0 & k_1\\ 1 & k_2\end{array}\right]Q + Q\left[\begin{array}{cc} 0 & 1\\ k_1 & k_2\end{array}\right] = -I,$$

I is the identity matrix in $R^{2\times 2}$, k_1, k_2 are the components of the vector k, is such that the closed loop system (31, 33) is ISS with respect to the disturbance $d(t)$. Note that the functional g is (only) locally Lipschitz, and that the delay τ can be arbitrarily large.

V. Conclusions

The ISS theory, recently being adapted to time-delay systems, is one of the best tools for analysis and control of nonlinear systems. In this paper, we establish a connection between ISS and exponential stability of time-delay systems. It is proved that a system which is (globally, locally) exponentially stable in the unforced case is (globally, locally) input-to-state stable when it is forced by a measurable and locally essentially bounded input, provided that the functional describing the dynamics in the unforced case is (globally, on bounded sets) Lipschitz and the functional describing the dynamics in the forced case satisfies a Lipschitz-like hypothesis with respect to the input. Moreover, a new feedback control law is provided for delay-free linearizable and stabilizable time-delay systems, whose dynamics is described by locally Lipschitz functionals, by which the closed loop system is ISS with respect to disturbances adding to the control law, a typical problem due to actuator errors.

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