Exponential stability for 2D systems: the linear case

Nima Yeganefar, Nader Yeganefar, Olivier Bachelier, and Emmanuel Moulay

***Abstract*—This short paper deals with a 2D discrete linear Roesser model. The results introduced here are a follow-up of a paper we proposed recently and where we explained and motivated the reasons we need to adopt a new definition of exponential stability for 2D systems. However this previous result left aside a crucial point that we would like to asses here: is our new definition of exponential stability coherent with the existing stability criterion in the linear case? We hereby show that, in the linear case, 1. our new definition of exponential stability is equivalent to asymptotic stability and 2. the characteristic polynomial-based stability criterion is a sufficient and necessary condition for the exponential stability we have introduced.**

keywords: 2D systems, linear discrete systems, exponential stability.

I. Introduction

A. Background

Multidimensional systems were first introduced in the mid 1970s. Working on digital filters, several authors proposed to extend the framework which was mainly based on transfer functions to the multidimensional case: instead of having one independent variable, they looked at polynomial functions of higher dimensions. The notion of stability was also discussed in the BIBO (bounded input bounded output) sense [1]. This allowed to extend a well-known stability criterion to the nD case and it was based on the characteristic polynomial equation of the studied system.

This famous criterion simply states that the transfer function is BIBO stable for systems devoid of poles outside the stability region, which seems natural as it is similar to the 1D case. However the criterion is not as simple as in the 1D case, the number of zeros are usually infinite and further techniques need to be used in order to have a tractable condition (see for instance [2]). It was quoted as a necessary and sufficient condition, led to several stability tests [3], [4], and broadened to include exponential stability in [5]. However Goodman later showed in [6] that the condition is necessary only if the transfer function does not present a particular type of point called nonessential singularity of the second kind.

State-space models for multidimensional systems were given by different authors around the same time (see [7] and [8], [9]). Fornasini-Marchesini also introduced a definition of asymptotic stability which is still in use today in the field. Introducing state space models allowed a large number of researchers to engage in the field of multidimensional systems as several techniques based on LMIs (linear matrix inequalities) and Lyapunov techniques were extended from the 1D case to the nD case ([10], [11]).

Exponential stability, on the other hand, has been seldom studied directly in the 2D systems literature. In the linear case, one of the only works dealing with this problem is given by Pandolfi in [5]. Pandolfi proved that asymptotic stability and exponential stability are equivalent, similarly to what happens in the 1D case. This equivalence could explain why exponential stability has not been very much investigated in the 2D field. Later in [12], the authors extended this definition to repetitive systems but we are not discussing this special type of multidimensional systems in this paper; this work could however impact the results given in [12]. In the continuous case, we also have proposed a definition in [13]. However, in a recently published paper [14] we have carefully explained the shortcomings of these definitions and strongly motivated the introduction of a new one (for the discrete case only).

B. Goal of the paper

As pointed out above, in [14], we have shown that the existing definition of exponential stability [5] used in the literature of 2D systems is not correct. We have therefore introduced and motivated the use of a new definition that allowed us to investigate the exponential stability of nonlinear 2D systems and to propose the first converse Lyapunov theorem for 2D systems under certain assumptions. By doing so, we have left unanswered several questions, especially in the linear case. This paper therefore tries to fill the gap introduced by our work.

The questions we would like to answer in this short paper are the following:

1) Do we still have equivalence between our newly defined exponential stability and asymptotic stability in the linear case?
2) There is a well-known characteristic polynomial-based stability criterion from which a lot stability results are derived. Is this stability criterion still valid (ie. necessary and sufficient) if one uses our newly proposed definition?

The answers to both of these questions are positive and the proofs of these answers are the main results of this article.

Let us now recall the results we are talking about.

C. Previous important results

In order to have a self-contained article, we think it is important to quickly recall a few of the important results we published in the previously mentioned work [14]. Moreover,

Nima Yeganefar and Olivier Bachelier are with the University of Poitiers, LIAS, Bâtiment B25, 2 rue Pierre Brousse, B.P. 633, 86022 Poitiers Cedex, France, surname.name@univ-poitiers.fr

Emmanuel Moulay is with the University of Poitiers, Xlim (UMR-CNRS 6172), 11 Bvd Marie et Pierre Curie - BP 30179, 86962 Futuroscope Chasseneuil Cedex, France, emmanuel.moulay@univ-poitiers.fr

Nader Yeganefar is with University of Provence, CMI (UMR 6632), Technopole Chateau-Gombert, 39, rue F. Joliot Curie, 13453 Marseille Cedex 13, France nader.yeganefar@cmi.univ-mrs.fr

to make it easier to follow for the reader, we are going to use a Fornasini-Marchesini type model [8], [9] contrary to [14] where the model is a nonlinear version of the well-known Roesser model [7]. Both models are equivalent and therefore working with one or the other is the same. That being said, our system is described as follows:

$$x(i+1,j+1) = Ax(i,j+1) + Bx(i+1,j) \qquad (1)$$

where x is a real vector of dimension n and matrices A, B are of appropriate dimensions. i and j are positive integers so that the initial conditions are taken in the first quadrant ($\varphi(i) = x(i,0)$, and $\psi(j) = x(0,j)$ for $i,j \in \mathbb{N}$), and we study the behavior of trajectories in $\mathbb{N}^2$ as $i+j \to +\infty$. Note that the approach adopted by Fornasini-Marchesini ([8], [9]) for 2D systems is different: they consider initial conditions given on the subset $\{(i,j) \in \mathbb{Z}^2,\, i+j=0\}$, and they study the behavior of trajectories as $i+j \to +\infty$. This choice of initial conditions is not without consequence on the stability definitions as we have pointed out in [14], for further remarks on the importance of the initial conditions for 2D systems one can refer to eg. [15], [16].

For such a system, observe that the definition of asymptotic stability we adopted in [14] is more restrictive than the one usually found in the literature, as we define the concept of stability and attractivity (as in the 1D case).

Definition 1.1 ([17], [14]): The point $x_e = 0$ is said to be asymptotically stable (in the sense of Lyapunov) if:

1) $x_e = 0$ is stable (for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $\|\varphi(i)\| < \delta$, $\|\psi(j)\| < \delta$ then $\|x(i,j)\| < \epsilon$ for all $i,j > 0$),
2) $\lim_{i+j\to\infty} x(i,j) = 0$ when $\lim_{i\to\infty} \varphi(i) = 0$ and $\lim_{j\to\infty} \psi(j) = 0$. This property will be called *attractivity* as in the 1D case.

Exponential stability is defined as follows:

Definition 1.2 ([14]): The equilibrium point $x_e = 0$ of system (1) is said to be exponentially stable if there are constants $q \in (0,1)$ and $M > 0$ such that for any initial conditions φ and ψ, and for all $(i,j) \in \mathbb{N}^2$, we have:

$$\|x(i,j)\| \leq M \left(\sum_{r=0}^{i} \frac{\|\varphi(r)\|}{q^{r+1}} + \sum_{s=0}^{j} \frac{\|\psi(s)\|}{q^{s+1}} \right) q^{i+j} \qquad (2)$$

Remark 1: This definition was mainly supported by two observations in [14]. First, notice that each term $x(i,j)$ is determined by the knowledge of $\varphi(r)$ and $\psi(s)$, for $r = 0,\ldots i$, and $s = 0,\ldots,j$ respectively, so that it is natural to express the bound (2) in terms of these quantities (see for instance (5) which gives the full expression of the solutions). Second, analyze the following special example of an upper triangular matrix with a single eigenvalue $0 < q < 1$:

$$x(i+1,j+1) = \begin{bmatrix} q & 1 \\ 0 & 0 \end{bmatrix} x(i,j+1) + \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} x(i+1,j) \quad (3)$$

As expected, because $q \in (0,1)$, the trajectories are exponentially decreasing. However, the trivial solution of system (3) is not exponentially stable using the definition introduced in [5] but it is using Definition 1.2 (for further details, please see [14]).

II. Main results

Proposition 2.1: Exponential stability implies asymptotic stability (Definition 1.1).

Proof: Let us first show that exponential stability implies stability. This comes from the fact that for a given $\epsilon > 0$, there exists a $\delta = \frac{\epsilon(1-q)}{4M} > 0$ such that if for all $i,j \geq 0$, $\|\varphi(i)\| \leq \delta$ and $\|\psi(j)\| \leq \delta$, then

$$\begin{aligned} \|x(i,j)\| &\leq M \left(\sum_{k=0}^{j} \frac{\|\psi(k)\|}{q^{k+1}} + \sum_{k=0}^{i} \frac{\|\varphi(k)\|}{q^{k+1}} \right) q^{i+j} \\ &\leq M\delta \left(\sum_{k=0}^{j} \frac{1}{q^{k+1}} + \sum_{k=0}^{i} \frac{1}{q^{k+1}} \right) q^{i+j} \\ &\leq M\delta \left(\frac{q^{j-1} + q^{i-1} - 2q^{i+j}}{1-q} \right) \leq \epsilon \end{aligned}$$

Attractivity follows by similar arguments. Note that this proof is independent of the linearity of the system. ■

Theorem 2.2: The linear system (1) is exponentially stable if and only if

$$\det(I - z_1 A - z_2 B) \neq 0 \qquad (4)$$

in the closed set $\left\{(z_1,z_2) \in C^2, |z_1| \leq 1, |z_2| \leq 1\right\}$

Proof: The proof is very similar to the one given in [5] (and inspired by [9]) only the final steps are different. The idea of the proof can be easily explained following this path:

$$(ES) \Rightarrow (AS) \Leftrightarrow (4) \Rightarrow (ES)$$

Indeed from Proposition 2.1, exponential stability (ES) implies asymptotic stability (AS). From the work of Fornasini-Marchesini [9], we know that asymptotic stability is equivalent to the characteristic polynomial condition (4). We still need to prove that if (4) is verified then the system is exponentially stable. Note that this will also prove that exponential and asymptotic stability are equivalent in the linear case.

To do so, we need to explicitly find the solutions of the linear system (1). Following Pandolfi's work in [5], we know that the solutions of system (1) can be explicitly computed using a (z,ζ)-transform of $x(i,j)$:

$$x(i,j) = \sum_{r=0}^{i} E(i-r,j)\Phi(r,0) + \sum_{s=0}^{j} E(i,j-s)\Psi(0,s) \qquad (5)$$

where $E(i,j)$, $\Phi(i,j)$ and $\Psi(i,j)$ are defined by their (z,ζ)-transform.

$$\begin{aligned} E(z,\zeta) &= \left(I - \frac{A}{z} - \frac{B}{\zeta} \right)^{-1} \\ \Phi(z,\zeta) &= \left(I - \frac{A}{z} \right) \varphi(z) \\ \Psi(z,\zeta) &= \left(I - \frac{B}{\zeta} \right) \psi(\zeta). \end{aligned}$$

Let us also give the expression of Φ (Ψ is analogous, see [5] for details). $\Phi(i,j) = \Phi_1(i,j) - A\Phi_2(i,j)$ with

$$\Phi_1(i,j) = \begin{cases} \varphi(i), & i \geq 0, j = 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$\Phi_2(i,j) = \begin{cases} \varphi(i-1), & i \geq 1, j = 0 \\ 0, & \text{otherwise.} \end{cases}$$

From inequality (5) and the definitions of Φ and Ψ, we can derive the following bound:

$$\begin{aligned} \|x(i,j)\| \leq & \sum_{r=0}^{i} \|E(i-r,j)\| \, \|\varphi(r)\| \\ & + \|A\| \sum_{r=0}^{i} \|E(i-r,j)\| \, \|\varphi(r-1)\| \\ & + \sum_{s=0}^{j} \|E(i,j-s)\| \, \|\psi(s)\| \\ & + \|B\| \sum_{s=0}^{j} \|E(i,j-s)\| \, \|\psi(s-1)\| \, . \end{aligned} \tag{6}$$

As shown in [9], [5], if condition (4) is verified then there exists a number $\alpha \in (0,1)$ such that the series

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} z^{-r} \zeta^{-s} E(r,s)$$

is absolutely convergent for $|z| > \alpha$, $|\zeta| > \alpha$.

Choosing $\alpha < q < 1$, then

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} q^{-r} q^{-s} \|E(r,s)\| < M.$$

As we are dealing only with positive terms, the following inequality holds:

$$q^{-(i+j)} \|E(i,j)\| \leq \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} q^{-r} q^{-s} \|E(r,s)\| \, ,$$

so that

$$\|E(i,j)\| \leq M q^{i+j}.$$

Therefore we can bound the different terms in (6) as follows:

$$\sum_{r=0}^{i} \|E(i-r,j)\| \, \|\varphi(r)\| \leq M \sum_{r=0}^{i} q^{i+j-r} \|\varphi(r)\|$$

$$\begin{aligned} \sum_{r=0}^{i} \|E(i-r,j)\| \, \|\varphi(r-1)\| &= \sum_{r=0}^{i-1} \|E(i-r-1,j)\| \, \|\varphi(r)\| \\ \leq M \sum_{r=0}^{i-1} q^{i+j-r-1} \|\varphi(r)\| &= M \frac{1}{q} \sum_{r=0}^{i-1} q^{i+j-r} \|\varphi(r)\| \end{aligned}$$

and similarly for the other two terms in (6). Playing with the indices and the constants, we can easily show that there exists a constant M' such that

$$\|x(i,j)\| \leq M' \left(\sum_{r=0}^{i} \frac{\|\varphi(r)\|}{q^{r+1}} + \sum_{s=0}^{j} \frac{\|\psi(s)\|}{q^{s+1}} \right) q^{i+j}$$

Thus, (4) implies ES which concludes the proof. ■

Remark 2: The attentive reader could notice that Pandolfi in [5] showed that his definition of ES is equivalent to the criterion (4). We have just proved that using our definition of ES, we also have equivalence with (4). However we pointed out earlier that both definitions are not equivalent (see Remark 1). The problem lies in the fact that Pandolfi wrongly assumed that his definition of ES implies AS: this is not true; this means that his proof of equivalence is not complete. But as indirectly pointed out several times in this paper, he did all the hard work and his contribution to our paper is invaluable.

III. Conclusion

In this short paper, we wanted to complete the results introduced in a recently published work on exponential stability [14]. Whereas [14] was focused on nonlinear systems, this paper deals with linear systems and deals with several questions left unanswered. Indeed, by introducing a new definition of exponential stability, it was important to show that this definition is in coherence with the huge literature in the linear case. This is done by showing, first, that exponential stability and asymptotic stability are equivalent in the linear case, second, that a necessary and sufficient condition for exponential stability is given by the well-known characteristic polynomial condition. Whether this work impacts the definition of exponential stability given for repetitive systems remains to be investigated.

$$\|x(i,j)\| > M \left(q^i \max_{0<j'<j} \|\psi(j)\| + q^j \max_{0<i'<i} \|\varphi(i)\| \right).$$

References

[1] J. Shanks, S. Treitel, and J. Justice, "Stability and synthesis of two-dimensional recursive filters," *IEEE Trans. on Audio and Electroacoustics*, vol. AU-20, no. 2, pp. 115–128, 1972.

[2] M. Ghamgui, N. Yeganefar, O. Bachelier, and D. Mehdi, "Robust stability of hybrid Roesser models against parametric uncertainty : A general approach," *Multidimensional Systems and Signal Processing*, vol. Accepted for publication, pp. doi:10.1007/s11045--012--0213--4, 2013.

[3] T. Huang, "Stability of two-dimensional recursive filters," *IEEE Trans. on Audio and Electroacoustics*, vol. AU-20, no. 2, pp. 158–163, 1972.

[4] B. Anderson and E. Jury, "Stability test for two-dimensional recursive filters," *IEEE Trans. on Audio and Electroacoustics*, vol. AU-21, no. 4, pp. 366–372, 1973.

[5] L. Pandolfi, "Exponential stability of 2-D systems," *Systems and Control Letters*, vol. 4, pp. 381–385, 1984.

[6] D. Goodman, "Some stability properties of two dimensional linear shift-invariant digital filters," *IEEE Trans. on Circuits and Systems*, vol. CAS-24, no. 4, pp. 201–208, 1977.

[7] R. Roesser, "Discrete state-space model for linear image processing." *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 1–10, 1975.

[8] E. Fornasini and G. Marchesini, "State-space realization theory of two-dimensional filters," *IEEE Transactions on Automatic Control*, vol. 21, no. 4, pp. 484–492, 1976.

[9] ——, "Doubly-indexed dynamical systems: State-space models and structural properties," *Mathematical Systems Theory*, vol. 12, pp. 59–72, 1978.

[10] K. Galkowski, J. Lam, S. Xu, and Z. Lin, "LMI approach to state-feedback stabilization of multidimensional systems," *International Journal of Control*, vol. 76, no. 14, pp. 1428–1436, 2003.

[11] T. Kaczorek, "LMI approach to stability of 2D positive systems," *Multidimensional Systems and Signal Processing*, vol. 20, no. 1, pp. 39–54, 2009.

[12] M. Dymkov, I. Gaishun, K. Galkowski, E. Rogers, and D. Owens, "Exponential stability of discrete linear repetitive processes," *International Journal of Control*, vol. 75, no. 12, pp. 861–869, 2002.

[13] M. Ghamgui, N. Yeganefar, , O. Bachelier, and D. Mehdi, "Exponential stability conditions for 2D continuous state-delayed systems," in *nDS'11, 7th International Workshop on Multidimensional Systems*, 2011.

[14] N. Yeganefar, N. Yeganefar, M. Ghamgui, and E. Moulay, "Lyapunov theory for 2D nonlinear roesser models: application to asymptotic and exponential stability," *IEEE Trans. Aut. Control*, vol. 58, no. 5, pp. 1299–1304, 2013.

[15] M. Valcher, "Characteristic cones and stability properties of two-dimensional autonomous behaviors," *IEEE Trans. on Circuits and Systems I: Fundamental Theory and Applications*, vol. 47, no. 3, pp. 290–302, 2000.
[16] D. Napp, P. Rapisarda, and P. Rocha, "Time-relevant stability of 2D systems," *Automatica*, vol. 47, no. 11, pp. 2373–2382, 2011.
[17] D. Liu and A. N. Michel, "Stability analysis of state-space realizations for two-dimensional filters with overflow nonlinearities," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 41, pp. 127–137, 1994.