On the Kalman-Yakubovich-Popov lemma and the multidimensional models*

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Abstract

This paper focuses on Kalman-Yakubovich-Popov lemma for multidimensional systems described by Roesser model that possibly includes both continuous and discrete dynamics. It is shown that, similarly to the standard 1-D case, this lemma can be studied through the lens of S-procedure. Furthermore, by virtue of this lemma, we will examine robust stability, bounded and positive realness of multidimensional systems.

Keywords : KYP lemma, Hybrid n-D Roesser model, S-procedure, Polynomial matrix $\partial\mathcal{D}$-regularity, LMI.

1 Introduction

In the past three decades, a large attention has been paid to the study of multidimensional (n-D) systems [7, 8, 23, 28]. These systems are characterized by rational functions, or matrices of several independent variables which can represent different space coordinates or mixed time and space variables. This is a result of information propagation in more than one independent direction which is the essential difference from the classical, or one-dimensional (1-D) case, where information propagates only in one direction.

The interest in n-D systems has predominantly been motivated by a wide variety of applications, arising in both theory and practical applications. Particular applications include n-D filtering [5, 31], n-D coding and decoding [37], image processing [10], and multidimensional signal processing [14]. n-D systems theory also successfully applies for analysis and synthesis of processes with repetitive dynamics, see for example [35] and the references therein.

Two basic state-space models for n-D systems have been developed. The first is credited to Roesser [34] and clearly has a first order structure. In what follows, the state vector is partitioned into sub-vectors - one for each of n directions of information propagation. Another

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commonly used state-space model for n-D systems has been proposed by Fornasini and Marchesini [16]. Note, however, that these models are not fully independent and it is possible to transform one into the other.

The unquestioned popularity of the state-space methods in n-D system theory stems from the fact that they are well understood and efficient numerical linear algebra routines exist (required when manipulating state-space models). In what follows, the stability problem, which is a main requirement for n-D systems, can be solved within Lyapunov's framework [21, 25, 28, 30], which is naturally performed in the state space. The most important fact associated with such an approach is that an n-D system stability condition can be recast into a linear matrix inequality (LMI) feasibility problem [9], i.e. finite dimensional convex optimization problem involving LMI constraints. It has to be mentioned that most of the proposed LMI stability conditions, when tractable, are only sufficient. Recently, some preliminary results on formulating necessary and sufficient stability condition have been proposed in [15, 17]. However, these conditions are only formulated for 2-D discrete system case and cannot be easily extended to controller design case.

However, the most known conditions relevant to the stability of n-D systems are expressed in terms of characteristic polynomial root-clustering [27]. In this framework, it becomes of interest to derive theoretical results on polynomial matrices [36]. Once again, convex optimization techniques over LMI constraints have turned to be a good tool to tackle some problems induced by polynomial matrices, especially $\mathcal{D}$-stability analysis [24]. This paper lies in this framework.

The connection between the polynomial, or, in a more restrictive way, frequency approach, and the matrix inequalities is quite well established in the context of 1-D systems. The strongest result is the celebrated Kalman-Yakubovich-Popov (KYP) lemma [33, 26] which gives equivalences between crucial frequency domain inequalities and LMIs.

To date, no work has been reported on a solution to this problem in terms of n-D systems and therefore it is a natural question to ask if it is possible to provide a version of the KYP lemma which can be exploited in the realm of n-D systems. Recently, much effort has been dedicated to establishing KYP lemma for n-D systems. Although few special instances of the KYP lemma have been implicitly addressed for 2-D models [38, 39], no real general result has been provided yet. This paper aims at filling this gap.

The paper is organized as follows. After this introduction, Section 2 provides the preliminary background. It recalls the Roesser model and shortly highlights its various extensions. These models are very popular to describe the behaviour of multidimensional systems. Besides, it exploits some descriptions (encountered in the literature) of regions of the complex plane as well as the notion of $\partial\mathcal{D}$-regularity for multivariate matrix functions. Section 3 states the main result which is some extension of the generalized KYP lemma. In Section 4, this lemma is applied to analyze the behaviour of n-D systems described with Roesser model introduced in Section 2. Some numerical illustration is provided in Section 5 to highlight the relevance of the approach. Finally, results are summarized and conclusions are stated in Section 6.

The following notation will be used throughout the paper. M' denotes the transpose conjugate of a matrix M. Hence, λ' is the conjugate of complex number λ. $||M||_2$ is the matrix 2-norm (the maximum singular value) induced by the Euclidean vector norm. I_n is the identity matrix of dimension n and I (resp. 0) is the identity (resp. a null) matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Löwner, i.e. > 0 (resp. < 0) means positive (resp. negative) definite and ≥ 0 (resp. ≤ 0) means positive (resp. negative) semi-definite. The notation $\mathcal{H}_n$ stands for the set of Hermitian matrices of dimension n. $\mathcal{H}_n^+ \subset \mathcal{H}_n$ is the subset of positive definite matrices and $\mathcal{H}_n^- \subset \mathcal{H}_n$ that of negative definite matrices. Also let the following notations be defined:

$$\bigoplus_{i=1}^{k} M_i = \underset{i=1,\dots k}{\text{diag}} \{M_i\}$$

For a given subset $\mathcal{S}$ of the set $\mathcal{X}$, the set $\mathcal{S}^C$ is the complementary set of $\mathcal{S}$ such that $\mathcal{S} \cup \mathcal{S}^C = \mathcal{X} \,\&\, \mathcal{S} \cap \mathcal{S}^C = \emptyset$. At last, the sets of indices are denoted as follows:

$$\mathbb{I}(q) := \{1, \dots, q\}, \;\; q \in \mathbb{N}.$$

2 Preliminaries

This section introduces some preliminaries, some of them being borrowed from the literature. The first part is dedicated to the presentation of the Roesser models and to some associated stability conditions. The second provides the description of a class of regions of the "multidimensional" complex plane which will be used in the next section. The third part briefly introduces the notion of $\partial\mathcal{D}$-regularity of a multivariate matrix function, inspired from the concept of matrix $\partial\mathcal{D}$-regularity.

2.1 Hybrid Roesser model

One of the most commonly used model for n-D discrete systems is the Roesser model (RM) that has been originally introduced in [34]. One of the key features of this model is that the state vector is partitioned into horizontal and vertical components.

Furthermore, it is also possible to define the continuous or hybrid versions of Roesser model [20]. The general version of the hybrid Roesser model for n-D systems takes the following form

$$E \left[\begin{array}{c} \dfrac{\partial}{\partial t_1} x^1(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \vdots \\ \dfrac{\partial}{\partial t_r} x^r(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \hline x^{r+1}(t_1, \dots, t_r, j_{r+1}+1, \dots, j_k) \\ \vdots \\ x^k(t_1, \dots, t_r, j_{r+1}, \dots, j_k+1) \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} x^1(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \vdots \\ x^r(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \hline x^{r+1}(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \vdots \\ x^k(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] u(t_1, \dots, t_r, j_{r+1}, \dots, j_k)$$

$$y(t_1, \dots, t_r, j_{r+1}, \dots, j_k) = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] \left[\begin{array}{c} x^1(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \vdots \\ x^r(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \hline x^{r+1}(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \\ \vdots \\ x^k(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \end{array}\right] + Du(t_1, \dots, t_r, j_{r+1}, \dots, j_k) \tag{1}$$

where $\sum_{i=1}^{k} n_i = n$.

The vectors $x'^i(t_1,\ldots,t_r,j_{r+1},\ldots,j_k) \in \mathbb{R}^{n_i}$, $i = 1,..,k$, $u(t_1,\ldots,t_r,j_{r+1},\ldots,j_k) \in \mathbb{R}^m$ and $y(t_1,\ldots,t_r,j_{r+1},\ldots,j_k) \in \mathbb{R}^p$ are the local state subvectors, the input vector and the output vector respectively. The matrices

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \in \mathbb{R}^{n\times n},\ B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \in \mathbb{R}^{n\times m},$$
$$C = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}$$

are the state, control, observation and direct transfer matrices respectively. The matrix $E \in \mathbb{R}^{n\times n}$ is introduced to enable the description of some singularities in order to extend the notion of 1-D linear descriptor systems. Obviously, the systems represented by the above model have continuous dynamics along r dimensions and discrete dynamics along $(k-r)$ dimensions. If the matrix E is equal to I (no singularity in the dynamics), then the model of [6] is recovered. If, furthermore, $r = 0$, then it reduces to the classical Roesser model [34] whereas if $r = k$ then one gets its continuous counterpart. At last, if $r = 1$ and $k = 2$ (still with $E = I$), then the obtained 2-D model is particularly suitable to describe the differential repetitive processes [22].

Note that relationships between polynomial matrix theory and state-space description are very strong in the n-D linear case. As a result of application the Laplace transform and the $\mathcal{Z}$-transform the following frequency characterization of the system (1) is obtained

$$Y(\lambda) = G(\lambda)U(\lambda),$$

where

$$G(\lambda) = C(EH(\lambda) - A)^{-1}B + D, \quad H(\lambda) := \bigoplus_{i=1}^{k} \lambda_i I_{n_i}. \tag{2}$$

Now let the function $c(\lambda, E, A)$ be defined by

$$c(\lambda, E, A) := \det(EH(\lambda) - A) \tag{3}$$

We begin with presenting the stability condition for n-D systems, which is just a reformulation of a proposition in [6].

Lemma 1. *Consider a multidimensional system represented by (1) and assume that $E = I_n$. Then, such a system is asymptotically stable if and only if*

$$c(\lambda, I, A) \neq 0 \quad \forall \lambda \in \mathcal{S}^C, \tag{4}$$

where $c(.,.,.)$ is defined by (3) and where $\mathcal{S}^C$, the complementary set of the "asymptotic stability region", is defined by

$$\mathcal{S}^C = \left\{ \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} \in \mathbb{C}^k : \mathrm{Re}(\lambda_i) \geq 0, i = 1,...,r \ \&\ |\lambda_i| \geq 1, i = r+1,...,k \right\} \tag{5}$$

It can be seen that the above Lemma covers the existing results for both discrete and continuous n-D systems. Obviously, if $r = 0$, then the condition for the stability of a discrete Roesser model proposed in [1, 27] is recovered. Furthermore, by taking $r = k$ the result of [32] is obtained.

Remark 1. *The reader may have noticed that the matrix E has not been taken into account in Lemma 1. The reason is that the stability of singular n-D systems has not yet been clearly connected to the notion of root-clustering although some extended conditions could seem natural. Besides, the asymptotic stability of descriptor systems is not always sufficient to ensure some satisfactory behavior.* Regularity *and* impulse-free responses *are also required [11]. Few papers address both multidimensional and singular systems. Let however reference [19] be quoted.*

In Lemma 1, the stability condition is expressed in terms of polynomial root-clustering which generally does not lead to computationally feasible conditions. One of the ways to obtain stability conditions that result in a significant reduction in computational complexity is to formulate them as convex optimization problems involving LMIs. Therefore, the next part provides some preliminaries on point clustering in order to prepare the derivation of some LMI-based conditions in the next section.

2.2 Point-clustering

Unlike the 1-D system case, zeros of n-D system characteristic polynomial (i.e. system poles) are not isolated and generally they cannot be a finite set. Furthermore, it is accompanied by difficulties in applying the pole placement technique for n-D systems, since there is no link between pole location and the dynamic response of n-D system. Therefore, it becomes clear that only the imaginary axis and the unit circle, which are the boundaries of $\mathcal{S}^C$ defined in (5), can be considered.

To proceed, consider the following matrices

$$\begin{bmatrix} R_{i_{11}} & R'_{i_{10}} \\ R_{i_{10}} & R_{i_{00}} \end{bmatrix} \in \mathbb{C}^{2\times 2}. \tag{6}$$

Let the sets $\partial\mathcal{D}_i$ be described by

$$\partial\mathcal{D}_i := \{s \in \mathbb{C} \ : \mathcal{F}_{R_i}(s) = 0, \forall i \in \mathbb{I}(k)\} \tag{7}$$

where the functions $\mathcal{F}_{R_i}(s)$ are defined by

$$\mathcal{F}_{R_i}(s) := \begin{bmatrix} sI \\ I \end{bmatrix}' R_i \begin{bmatrix} sI \\ I \end{bmatrix} \quad \forall i \in \mathbb{I}(k). \tag{8}$$

Due to the fact that the only meaningful sets are lines or circles we limit our consideration to sets described by the equalities $\mathcal{F}_{R_i}(s) = 0$.

Now define the "k-region" $\partial\mathcal{D}$ as

$$\partial\mathcal{D} := \partial\mathcal{D}_1 \times \partial\mathcal{D}_2 \times ... \times \partial\mathcal{D}_k. \tag{9}$$

In the remaining part of the paper, we shall restrict the study to matrices that comply with

$$R_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \forall i \in \mathbb{I}(r) \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \forall i \in \{r+1, \dots, k\}, \tag{10}$$

meaning that, respectively, only the imaginary axis and the unit circle are considered as instances of $\partial\mathcal{D}_i$. Indeed, as it has been mentioned, they are the ones of practical interest for the study of nD models. More precisely, with such a choice, $\partial\mathcal{D}$ is the boundary of $\mathcal{S}^C$ defined in (5). But note that another work considering a much larger class of regions has been completed in a technical report which is available upon request to the authors [4].

Let λ be a complex vector that can be written

$$\lambda := \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} \in \mathbb{C}^k. \tag{11}$$

In the sequel, define the following matrices

$$\Lambda := \bigoplus_{i=1}^{k} \lambda_i;$$

$$\mathbf{R} := \begin{bmatrix} \bigoplus_{i=1}^{k} R_{i_{11}} & \bigoplus_{i=1}^{k} R'_{i_{10}} \\ \bigoplus_{i=1}^{k} R_{i_{10}} & \bigoplus_{i=1}^{k} R_{i_{00}} \end{bmatrix};$$

to obtain

$$\lambda \in \partial\mathcal{D} \Leftrightarrow \begin{bmatrix} \Lambda \\ I_k \end{bmatrix}' \mathbf{R} \begin{bmatrix} \Lambda \\ I_k \end{bmatrix} = 0. \tag{12}$$

Before proceeding further, we give the following lemma which will be useful in the sequel.

Lemma 2. *Let a k-region $\partial\mathcal{D}$ be defined as in (9) with (6)–(8), (10) and $\lambda \in \mathbb{C}^k$ comply with (11). The two following statements are equivalent.*

i)

$$\lambda \in \partial\mathcal{D} \tag{13}$$

ii)

$$\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} = 0 \ \forall \mathcal{P} \in \mathbb{H} \tag{14}$$

where $\mathbb{H} = \{\mathcal{H}_{n_1} \times \cdots \times \mathcal{H}_{n_k}\}$, $H(\lambda)$ is defined in (2) and $\overline{\mathbf{R}}(\mathcal{P})$ stands for

$$\overline{\mathbf{R}}(\mathcal{P}) := \left[\begin{array}{c|c} \bigoplus_{i=1}^{k} P_i R_{i_{11}} & \bigoplus_{i=1}^{k} P_i R'_{i_{10}} \\ \hline \bigoplus_{i=1}^{k} P_i R_{i_{10}} & \bigoplus_{i=1}^{k} P_i R_{i_{00}} \end{array}\right], \forall l \in \mathbb{I}(k), \tag{15}$$

and with $P_{n_i} \in \mathcal{H}_{n_i}$, $\forall i \in \mathbb{I}(k)$.

Proof. **i)** $\Rightarrow$ **ii)** : if $\lambda \in \partial\mathcal{D}$ then the equality in (12) holds or equivalently

$$\bigoplus_{i=1}^{k} \mathcal{F}_{R_i}(\lambda_i) = 0,$$

leading to

$$\bigoplus_{i=1}^{k} \mathcal{F}_{R_i}(\lambda_i) P_i = 0, \tag{16}$$

with considering any set $\mathcal{P}$ of k matrices $P_i \in \mathcal{H}_{n_i}$, $i \in \mathbb{I}(k)$, one gets

$$\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} = 0. \tag{17}$$

ii) $\Rightarrow$ **i**): Assume that (14) holds. Let a vector z comply with

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} \text{ with } z_i \in \mathbb{C}^{n_i}, \forall i \in \mathbb{I}(k). \tag{18}$$

Left and right multiplying (14) by z' and z respectively yields

$$\sum_{i=1}^{k} ((z_i' P_i z_i) \mathcal{F}_{R_i}(\lambda_i)) = 0. \tag{19}$$

The above inequality holds for *any* set $\mathcal{P}$ meaning that $z_i' P_i z_i$ can take any value. Thus, it is clear that $\mathcal{F}_{R_i}(\lambda_i) = 0 \, \forall i \in \mathbb{I}(k)$. □

2.3 Multivariate matrix $\partial\mathcal{D}$-regularity

To conclude this section, the definition of the $\partial\mathcal{D}$-regularity of a multivariate matrix function is proposed, accompanied by a short discussion.

Definition 1. *Let $\partial\mathcal{D}$ be a subset of $\mathbb{C}^k$ and $\lambda \in \mathbb{C}^k$ comply with (11). Also consider a k-variate matrix function $\mathbf{A}(\lambda)$. Then, $\mathbf{A}(\lambda)$ is said to be*

- *$\partial\mathcal{D}$-regular if $\{\lambda \in \mathbb{C}^k : \det(\mathbf{A}(\lambda)) = 0\} \cap \partial\mathcal{D} = \emptyset$;*
- *$\partial\mathcal{D}$-singular if $\{\lambda \in \mathbb{C}^k : \det(\mathbf{A}(\lambda)) = 0\} \cap \partial\mathcal{D} \neq \emptyset$.*

In Section 4, a particular attention will be paid to the k-D polynomial matrix of the form

$$\mathbf{A}(\lambda) := EH(\lambda) - A, \tag{20}$$

since, in this case, $\det(\mathbf{A}(\lambda)) = c(\lambda, E, A)$ corresponds to the characteristic polynomial of Roesser model (1). If $E = I$, from Lemma 1, it can be seen that the asymptotic stability of (1) is a special instance of the $\partial\mathcal{D}$-regularity of $\mathbf{A}(\lambda)$ as defined in (20). More precisely, for (1) with $E = I$ to be asymptotically stable, it is *necessary* that $\mathbf{A}(\lambda)$ be $\partial\mathcal{D}$-regular with (6)–(10).

In such a case, the difference between $\partial\mathcal{D}$-regularity and asymptotic stability lies in the distribution of the roots of $c(\lambda, I, A)$ with respect to $\partial\mathcal{D}$. More generally, $\mathcal{D}$-stability requires $\partial\mathcal{D}$-regularity when $\partial\mathcal{D}$ is the boundary of the region $\mathcal{D}$.

For the classical 1-D case ($r = k = 1$ or $k = k - r = 1$), then $\mathbf{A}(\lambda)$ can be written $\mathbf{A}(\lambda) = (\lambda E - A)$ and, if $E = I$, Definition 1 becomes equivalent to the definition of the $\partial\mathcal{D}$-regularity of matrix A proposed in [3]. If $E \neq I$, then the definition in [3] can be directly extended to the notion of $\partial\mathcal{D}$-regularity of the pencil (E, A).

3 A version of the KYP lemma

This section is devoted to the derivation of the main result which is some sort of k-D version of the celebrated KYP-lemma. This version will only provide a sufficient condition but the necessity will be briefly discussed with the help of the so-called generalized S-procedure. A large reference is made to the seminal paper [26] where generalized KYP lemma for 1-D linear models and the generalized S-procedure are presented in a very elegant way.

Before stating the result, let us introduce the following notation useful for a further analysis (note that various notations are borrowed from [26]).

Consider some matrix $F \in \mathbb{C}^{(n+m)\times(n+m)}$ and let another matrix Θ belong to $\mathcal{H}_{n+m}$. Next, let the set $\mathcal{P}$ be made up by k matrices $P_i \in \mathcal{H}_{n_i}$, $\forall i \in \mathbb{I}(k)$. With each set $\mathcal{P}$ and with a set $\partial\mathcal{D}$ given by (9) with (6)–(8) and (10), one can associate a matrix $\overline{\mathbf{R}}(\mathcal{P})$, as in Lemma 2, defined by (15). The set $\mathbf{M}$ is the set of all complex matrices M associated with F and $\overline{\mathbf{R}}(\mathcal{P})$ in the following way:

$$\mathbf{M} := \{M \in \mathcal{H}_{n+m} : M = F'\overline{\mathbf{R}}(\mathcal{P})F\}. \tag{21}$$

¿From the above formulation, the subset $\tilde{\mathbf{M}}$ can be described by

$$\tilde{\mathbf{M}} := \{M \in \mathbf{M} : (M + \Theta) \in \mathcal{H}^-_{n+m}\}. \tag{22}$$

At last, for a complex vector λ defined as in (11), the matrices $\Gamma(\lambda)$ and $N(\lambda)$ are defined by

$$\Gamma(\lambda) := \begin{bmatrix} I & -H(\lambda) \end{bmatrix} \text{ and } \mathrm{Span}(N(\lambda)) := \mathrm{Ker}(\Gamma(\lambda)F)$$

respectively.

Now, the main result, which can be considered as some extension of the KYP lemma in its "strict inequality" version [33] is stated.

Theorem 1. *With the notations and assumptions detailed above, consider the two following statements:*

i)
$$N'(\lambda)\Theta N(\lambda) \in \mathcal{H}^-_m \ \forall\lambda \in \partial\mathcal{D}; \tag{23}$$

ii)
$$\tilde{\mathbf{M}} \neq \emptyset. \tag{24}$$

Then **ii)** *is sufficient for* **i)**.

Before giving the proof to the above theorem, the comments to the both conditions are provided. The first statement corresponds to some property to be checked such as robust stability, $\mathcal{H}_\infty$ performance level, and others (see paragraph 4.1.1 to understand how a suitable choice of Θ can link $N(\lambda)$ and Θ to a property to be satisfied by the system transfer matrix). The second statement corresponds to some numerically tractable condition which simply means that there exists some set $\mathcal{P}$ (which is usually seen as a set of Lyapunov matrices), such that

$$M + \Theta < 0. \tag{25}$$

The idea is then to handle inequality (25) rather than directly tackle the original property **i**) which cannot be easily exploited from a numerical point of view.

Proof. Firstly, observe that (24), as mentioned above, means that there exists a set $\mathcal{P}$ such that (25) holds. Further, based on the definition of $N(\lambda)$, it can be seen that

$$\Gamma(\lambda)FN(\lambda) = 0, \ \ \forall\lambda \in \partial\mathcal{D}. \tag{26}$$

Next, inequality (25) implies that

$$N'(\lambda)(M + \Theta)N(\lambda) < 0, \ \ \forall\lambda \in \partial\mathcal{D} \tag{27}$$

and hence

$$N'(\lambda)F'\overline{\mathbf{R}}(\mathcal{P})FN(\lambda) + N'(\lambda)\Theta N(\lambda) < 0, \ \ \forall\lambda \in \partial\mathcal{D}. \tag{28}$$

Moreover, we can see from (26) that Span$\,(FN(\lambda))$ is completely characterized by the relation

$$\mathrm{Span}(FN(\lambda)) = \mathrm{Span}\left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix} X\right), \tag{29}$$

where X is any full rank matrix. Thus, inequality (28) can be equivalently written as

$$X'\begin{bmatrix} H(\Lambda) \\ I \end{bmatrix}'\overline{\mathbf{R}}(\mathcal{P})\begin{bmatrix} H(\Lambda) \\ I \end{bmatrix}X + X'N'(\lambda)\Theta N(\lambda)X < 0, \ \ \forall\lambda \in \partial\mathcal{D}. \tag{30}$$

Finally, taking the result of Lemma 2 into account, it is clear that the first term of the left handside member of inequality (30) is zero when $\lambda \in \partial\mathcal{D}$. Hence, the second term is negative definite which is equivalent to **i)**. This completes the proof. □

The important point to note is that the 1-D version of KYP lemma is known to provide a necessary and sufficient condition whereas Theorem 1 only proposes a sufficient condition **ii)** for property **i)** to hold. More than proving Theorem 1, it is important to emphasize why condition **ii)** might not be necessary. The 1-D version of KYP lemma [33] can be proven through the so-called S-procedure [40] in its generalized form (see [26] and the references therein). In the remaining part of the section, Theorem 1 is studied through the lens of S-procedure, which is now recalled in its generalized strict inequality version.

Lemma 3. (S-procedure, from [26]) *Let $\mathbf{\Theta}$ be an Hermitian matrix and $\mathbf{M}$ be an arbitrary subset of $\mathcal{H}_q$. Moreover let $\tilde{\mathbf{M}} \subset \mathbf{M}$ be defined by*

$$\tilde{\mathbf{M}} := \{M \in \mathbf{M} : (M + \mathbf{\Theta}) \in \mathcal{H}_q^-\} \tag{31}$$

and the set $\mathbf{S}$ be defined by

$$\mathbf{S} := \{S \in \mathcal{H}_q : S \neq 0, \, \mathrm{rank}(S) = 1, \, S \geq 0, \mathrm{tr}(MS) \geq 0 \, \forall M \in \mathbf{M}\}. \tag{32}$$

Consider the two following statements:

a)

$$\mathrm{tr}(\mathbf{\Theta}S) < 0, \quad \forall S \in \mathbf{S}; \tag{33}$$

b)

$$\tilde{\mathbf{M}} \neq \emptyset. \tag{34}$$

Then **b)** *is sufficient for* **a)** *and if the set* $\mathbf{M}$ *is* rank-one separable *(see the definition in [26]) then* **b)** *is also necessary and the S-procedure is said* lossless.

A possible approach to understand why Theorem 1 only provides a sufficient condition unlike the classical KYP lemma is to try to see why S-procedure is either useless or is simply not lossless when applied to the assumptions of Theorem 1. According to the used notations, it is clear that the idea is to compare the sets $\mathbf{M}$ and $\tilde{\mathbf{M}}$ defined, on one hand, in Theorem 1 and, on the other hand, in Lemma 3 with $q = n + m$, and then to address two issues.

1. Is it possible to exhibit a set $\mathbf{S} = \mathbf{S}_a$ from the assumptions of Theorem 1 that could match the set $\mathbf{S} = \mathbf{S}_b$ in Lemma 3 (this would enable ones to apply the S-procedure, at least in the sense **b)** $\Rightarrow$ **a)**) ?
2. *If so*, is the S-procedure lossless (this would enable ones to apply the S-procedure in the sense **a)** $\Rightarrow$ **b)**)?

Firstly, it can be seen that the statement **i)** in Theorem 1 is equivalent to

$$y'N'(\lambda)\Theta N(\lambda)y < 0,\ \forall y \neq 0, \forall \lambda \in \partial\mathcal{D}. \tag{35}$$

Next, substitute $\xi = N(\lambda)y$ into (35) to obtain

$$\mathrm{tr}(\Theta S) < 0,\ \forall S \in \mathbf{S}_a := \{S = \xi\xi' : \xi \neq 0,\ \Gamma(\lambda)F\xi = 0,\ \lambda \in \partial\mathcal{D}\}. \tag{36}$$

With the change $\eta = F\xi$, it is easy to see that η complies with

$$\eta = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z,\ z \neq 0, \tag{37}$$

and hence the set $\mathbf{S}_a$ can be written

$$\mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0, \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z,\ \lambda \in \partial\mathcal{D}\right\}. \tag{38}$$

In the following, based on Lemma 2, it can be deduced that

$$\begin{aligned}\mathbf{S}_a = \Bigg\{&\xi\xi' : \xi \neq 0, \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z, \\ &\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} = 0\ \forall \mathcal{P} \in \mathbb{H}\Bigg\}.\end{aligned} \tag{39}$$

$$\begin{aligned}\Leftrightarrow \mathbf{S}_a = \Bigg\{&\xi\xi' : \xi \neq 0, \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z, \\ &q'\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} = 0\ \forall\{\mathcal{P}; q\} \in \mathbb{H} \times \mathbb{C}^n\Bigg\}.\end{aligned} \tag{40}$$

so, it is possible to express **i)** as **a)** with $\boldsymbol{\Theta} = \Theta$ and with $\mathbf{S}$ matching $\mathbf{S}_a$ as described above. Secondly, the condition **ii)** in Theorem 1 can be expressed as **b)** in Lemma 3 with $\boldsymbol{\Theta} = \Theta$ and with the set $\mathbf{M}$ as defined in (21). Next, application of Lemma 3 yields

$$\mathrm{tr}(\Theta S) < 0,\ \forall S \in \mathbf{S}_b := \{S = \xi\xi' : \xi \neq 0,\ \mathrm{tr}(M\xi\xi') \geq 0\,\forall M \in \mathbf{M}\},$$

where $\mathbf{S}_b$ can also be written as

$$\mathbf{S}_b = \{S = \xi\xi' : \xi \neq 0,\, \xi' F' \overline{\mathbf{R}}(\mathcal{P}) F \xi \geq 0 \;\forall \mathcal{P} \in \mathbb{H}\}. \tag{41}$$

Now, since $\mathcal{P}$ is any set in $\mathbb{H}$ then the inequality involved in the above description of $\mathbf{S}_b$ holds for any choice $\mathcal{P}_+ = \{P_i\} \in \mathbb{H}$ as well as for any choice $\mathcal{P}_- = \{-P_i\} \in \mathbb{H}$. In the following, observe that $\overline{\mathbf{R}}(\mathcal{P})$ is linear with respect to various matrices P_i and therefore the description of $\mathbf{S}_b$ can be modified as follows

$$\mathbf{S}_b = \{S = \xi\xi' : \xi \neq 0,\, \xi' F' \overline{\mathbf{R}}(\mathcal{P}) F \xi = 0 \;\forall \mathcal{P} \in \mathbb{H}\}. \tag{42}$$

Now, the issue 1 mentioned above is to compare $\mathbf{S}_a$ in (40) and $\mathbf{S}_b$ in (42). $\mathbf{S}_a$ can be written

$$\Leftrightarrow \mathbf{S}_a = \left\{ \xi\xi' : \xi \neq 0 : \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z, \right.$$
$$\left. q' \begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} q = 0 \;\forall \{\mathcal{P}; q\} \in \mathbb{H} \times \mathbb{C}^n \right\}.$$

which, with $q = Xz$, also writes (X being any full rank matrix)

$$\Leftrightarrow \mathbf{S}_a = \left\{ \xi\xi' : \xi \neq 0 : \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z, \right.$$
$$\left. z'X' \begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} Xz = 0 \,\forall \{\mathcal{P}; X : \det(X) \neq 0\} \in \mathbb{H} \times \mathcal{H}_n \right\}.$$

Now, assume that $k = 1$ which corresponds to the 1-D system case, then, with no loss of generality, the matrix P_1 can be substituted with $X^{-1} P_1 X^{-1}$ and $\mathbf{S}_a$ can then be described by

$$\Leftrightarrow \mathbf{S}_a = \left\{ \xi\xi' : \xi \neq 0 : \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z, \right.$$
$$\left. z' \begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z = 0 \;\forall \mathcal{P} \in \mathbb{H} \right\},$$

which is simplified in

$$\Leftrightarrow \mathbf{S}_a = \left\{ \xi\xi' : \xi \neq 0 : \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z,\, \eta' \overline{\mathbf{R}}(\mathcal{P}) \eta = 0 \;\forall \mathcal{P} \in \mathbb{H} \right\}.$$

or equivalently in $\mathbf{S}_b$. But this equality $\mathbf{S}_a = \mathbf{S}_b$ holds when $k = 1$. It means that the S-procedure can be directly applied from **ii)** to **i)** to prove Theorem 1 only when one comes back to the 1-D case. In this case only, shall one ask the question of necessity *i.e.* of the losslessness of the S-procedure. So it is lossless if the set $\mathbf{M}$ defined in (21) is rank-one separable. Actually this set is the one studied in [26] so the answer is yes, it is rank-one separable.

As a summary of the above discussion, here are four points:

- Theorem 1 provides a sufficient condition that is generally not necessary.
- $\mathbf{S}_a \subset \mathbf{S}_b$, so **b**) is clearly more constraining than **a**); Thus, **b**) $\Rightarrow$ **a**). But it is not a direct application of the S-procedure. However **ii**) $\Leftrightarrow$ **b**) with $\mathbf{S} = \mathbf{S}_b \Rightarrow$ **b**) with $\mathbf{S} = \mathbf{S}_a \overset{\text{S-procedure}}{\Rightarrow}$ **a**) with $\mathbf{S} = \mathbf{S}_a \Leftrightarrow$ **i**). Hence, **ii**) $\Rightarrow$ **i**).
- At last, when $k = 1$ (1-D case), then $\mathbf{S} = \mathbf{S}_a = \mathbf{S}_b$ which enables ones to directly apply the S-procedure. Moreover, it has been proven that it is lossless in this case so **ii**) $\Leftrightarrow$ **i**). One recovers the classical KYP lemma [33].

4 Application to the analysis of Roesser models

In this section, the relevance of Theorem 1 for the study of multidimensional hybrid state-space Roesser model (1) is emphasized. The idea is simply to make a suitable choice of matrices F and θ. Matrix F is chosen as follows:

$$F := \begin{bmatrix} A & B \\ E & 0 \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{43}$$

where the matrices A, B and E are of course those involved in (1). It can be noticed that matrix F is here real which corresponds to a special case of Theorem 1. Condition **i)** in Theorem 1 must be considered as a performance level or a property of (1) to be tested. In the sequel, three properties are considered, depending on the choice of Θ.

4.1 Robust $\partial\mathcal{D}$-regularity against an LFT-based uncertainty

This part is devoted to the analysis of the $\partial\mathcal{D}$-regularity of the uncertain k-variate polynomial matrix described by

$$\mathbb{A}(\lambda) = \mathbf{A}(\lambda) - \bar{A} = EH(\lambda) - A_c, \tag{44}$$

where $\mathbf{A}(\lambda)$ is given by (20), where A, B, C, and E have the same dimensions as those involved in (1), where $A_c = A + \bar{A}$ and where $\bar{A}$, given by

$$\bar{A} = B\bar{\Delta}C, \tag{45}$$

is an uncertain matrix complying with the so-called LFT (Linear Fractional Transform)-based structure:

$$\bar{\Delta} = \Delta(I_p - D\Delta)^{-1}, \quad \det(I_p - D\Delta) \neq 0. \tag{46}$$

The above full rank assumption is referred to as the well posedness of the uncertainty. Matrix Δ is the actual uncertainty matrix which belongs to $\mathcal{B}(\rho)$, the ball of complex matrices Δ verifying $||\Delta||_2 \leq \rho$.

4.1.1 Complex $\partial\mathcal{D}$-regularity radius

Nominal matrix $\mathbf{A}(\lambda)$ is assumed to be $\partial\mathcal{D}$-regular with $\partial\mathcal{D}$ as defined by (9) with (6)–(8) and (10). The purpose is here to derive what can be called the complex $\partial\mathcal{D}$-regularity radius $\varrho_{\partial\mathcal{D}}$, which is the largest value of the radius ρ such that the uncertain polynomial matrix $\mathbb{A}(\lambda)$ defined by (44) remains $\partial\mathcal{D}$-regular over $\mathcal{B}(\rho)$. To reach such a goal, Θ is chosen as follows:

$$\Theta := \begin{bmatrix} C'C & C'D \\ D'C & D'D - \gamma I \end{bmatrix}, \ \gamma = \rho^{-1/2}. \tag{47}$$

Corollary 1. *Let an uncertain k-variate matrix $\mathbb{A}(\lambda)$ be defined as in (44) and a set $\partial\mathcal{D}$ be defined by (9) with (6)–(8) and (10). $\mathbb{A}(\lambda)$ is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exists a set $\mathcal{P} \in \mathbb{H}$ such that (25) holds with M defined as in (21), $\overline{\mathbf{R}}(\mathcal{P})$ defined by (15), F given by (43) and Θ given by (47).*

Proof. It consists in applying Theorem 1. With the choice (47), the property **i)** in Theorem 1 can be expressed

$$||G(\lambda)||_2 < \sqrt{\gamma} \ \forall \lambda \in \partial\mathcal{D}, \tag{48}$$

(with $G(\lambda)$ given by (2)), by noting that the choice of F proposed in (43) leads to

$$\mathrm{Span}(N(\lambda)) = \mathrm{Span}\left(\begin{bmatrix} (EH(\lambda)-A)^{-1}B \\ I \end{bmatrix}\right). \tag{49}$$

Then, inequality (48) can be written as

$$\{\sup_{\lambda\in\partial\mathcal{D}} \bar{\sigma}(G(\lambda))\}^{-1} > \rho. \tag{50}$$

Simple arguments on singular values show that

$$\bar{\sigma}(G(\lambda)) = \mu_{\mathbb{C}}(G(\lambda)) \tag{51}$$

where $\mu_{\mathbb{C}}(.)$ denotes the celebrated complex structured singular value introduced in [12]. Then it comes

$$\{\sup_{\lambda\in\partial\mathcal{D}} \mu_{\mathbb{C}}(G(\lambda))\}^{-1} > \rho. \tag{52}$$

Taking the fact that

$$\mu_{\mathbb{C}}(G(\lambda)) = \left[\inf_{\Delta}\{\bar{\sigma}(\Delta) : \det(I - G(\lambda)\Delta) = 0\}\right]^{-1} \tag{53}$$

into account, one gets

$$\inf_{\lambda\in\partial\mathcal{D}}\left\{\inf_{\Delta}\{\bar{\sigma}(\Delta) : \det(I - G(\lambda)\Delta) = 0\}\right\} > \rho. \tag{54}$$

Besides,

$$\begin{aligned} &\det(I - G(\lambda)\Delta) = 0 \\ &\Leftrightarrow \det(I - C(EH(\lambda)-A)^{-1}B\Delta - D\Delta) = 0 \\ &\Leftrightarrow \det(I - C(EH(\lambda)-A)^{-1}B\Delta(I_p - D\Delta)^{-1})\det(I_p - D\Delta) = 0. \end{aligned}$$

From the well posedness assumption, $\det(I_p - D\Delta) \neq 0$, the above equality is equivalent to

$$\begin{aligned} &\det(I - C(EH(\lambda)-A)^{-1}B\bar{\Delta}) = 0 \\ &\Leftrightarrow \det(I - (EH(\lambda)-A)^{-1}B\bar{\Delta}C) = 0 \\ &\Leftrightarrow \det((EH(\lambda)-A)^{-1})\det(EH(\lambda) - A - B\bar{\Delta}C) = 0. \end{aligned}$$

Since, for $\Delta = 0$, $\mathbb{A}(\lambda) = \mathbf{A}(\lambda)$ is implicitly assumed to be $\partial\mathcal{D}$-regular (otherwise why testing robust $\partial\mathcal{D}$-regularity?), the left factor of the left handside member is non zero so it follows that

$$\det(EH(\lambda) - A - \bar{A}) = 0. \tag{55}$$

From (54) and (55), it can be deduced that

$$\inf_{\lambda\in\partial\mathcal{D}}\left\{\inf_{\Delta}\{\bar{\sigma}(\Delta) : \det(\mathbb{A}(\lambda)) = 0\}\right\} > \rho, \tag{56}$$

which proves that $\mathbb{A}(\lambda)$ remains $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$. □

It is clear that $\rho^\star$, the maximum value of $\rho = \gamma^{-1/2}$ obtained by computing $\mathcal{P}$ proving (24), is a lower bound of $\varrho_{\partial\mathcal{D}}$. If $k = 1$, then $\rho^\star = \varrho_{\partial\mathcal{D}}$. Note, however, that matrix Δ is here assumed to be complex. In practice, it would be interesting to take its possible realness into account.

4.1.2 Application to asymptotic stability analysis of hybrid Roesser model

In this part, the model (1) is considered in association with the uncertainty structure defined by the feedback equation

$$u(t_1,\ldots,t_r,j_{r+1},\ldots,j_k)=\Delta y(t_1,\ldots,t_r,j_{r+1},\ldots,j_k),\ \Delta\in\mathcal{B}(\rho)\cap\mathbb{R}^{m\times p}, \tag{57}$$

The above equation corresponds to the LFT formalism mentioned in the previous subsection.

Application of (57) to (1) yields the closed-loop state matrix

$$A_c=A+\bar{A}, \tag{58}$$

where $\bar{A}$ is given by (45)–(46).

Before to study various performance levels in the next subsection, the main property is considered, namely asymptotic stability. As mentioned in Remark 1, the stability of descriptor nD-models is rather still an open problem so this discussion on stability will be led with assuming $E=I$. In this case, Corollary 1 enables ones to test, with conservatism, if matrix $\mathbb{A}(\lambda)=H(\lambda)-A_c$ is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$. As mentioned at the end of subsection 2.3, asymptotic stability of Roesser model (1) with $E=I$ depends on the root-clustering of polynomial $c(\lambda,I,A)$. Thus, robust stability of the uncertain hybrid Roesser model described by (1) with $E=I$ and (57) against $\mathcal{B}(\rho)$ depends on the root-clustering of $c(\lambda,I,A_c)=\det(\mathbb{A}(\lambda))$. More precisely, for $c(\lambda,I,A_c)$ to match the robust stability conditions induced by Lemma 1, it is necessary that $\mathbb{A}(\lambda)$ be at least $\partial\mathcal{D}$-regular with the choice (10). Following this idea, the next theorem is stated

Theorem 2. *Suppose that a multidimensional hybrid system of the form described by (1) with $E=I$ is subject to uncertainty (57). Then the resulting uncertain system is robustly asymptotically stable against $\mathcal{B}(\rho)$ if there exists a set $\mathcal{P}$ of k matrices $P_i\in\mathcal{H}^+_{n_i}$, $i=1,\ldots,k$, such that the following LMI holds*

$$F'\overline{\mathbf{R}}(\mathcal{P})F+\Theta<0 \tag{59}$$

where F is given by (43), $\overline{\mathbf{R}}(\mathcal{P})$ is given by (15) with (10), and Θ is given by (47).

Proof. Based on Corollary 1, we conclude that the LMI (59) implies that $\mathbb{A}(\lambda)$ is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ with $\partial\mathcal{D}$ characterized by (10). But it is not just another corollary. Since the matrices P_i are assumed to be positive definite, then the first block in (59), which is

$$\begin{bmatrix}A\\I\end{bmatrix}'\overline{\mathbf{R}}(\mathcal{P})\begin{bmatrix}A\\I\end{bmatrix}+C'C<0\Rightarrow\begin{bmatrix}A\\I\end{bmatrix}'\overline{\mathbf{R}}(\mathcal{P})\begin{bmatrix}A\\I\end{bmatrix}<0,$$

proves asymptotic stability of the nominal system (1) with $E=I$. To see this, note that the columns of

$$\begin{bmatrix}A\\I_n\end{bmatrix},$$

span the kernel of

$$\begin{bmatrix}I_n & -A\end{bmatrix}, \tag{60}$$

and, in the sequel, invoke the matrix elimination procedure [9] to observe that there exists a matrix $G=G'$ such that

$$\mathbf{N}=\overline{\mathbf{R}}(\mathcal{P})+\begin{bmatrix}I_n\\-A'\end{bmatrix}G\begin{bmatrix}I_n & -A\end{bmatrix}<0. \tag{61}$$

Further, let a vector λ (as introduced in (11)) comply with $c(\lambda, I, A) = 0$. It will be proven that λ cannot lie in $\mathcal{S}^C$ defined as in (5). Indeed, if $c(\lambda, I, A) = 0$, there exists a vector $v \in \mathbb{C}^n$ such that

$$\mathbf{A}(\lambda)v = 0. \tag{62}$$

Let the vector q be defined by

$$q = \begin{bmatrix} H(\lambda) \\ I_n \end{bmatrix} v.$$

Note that, if v is written

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}, \quad \text{with} \quad v_i \in \mathbb{C}^{n_i} \ \forall i \in \mathbb{I}(k),$$

then q becomes

$$q = \begin{bmatrix} \lambda_1 v_1 \\ \vdots \\ \lambda_k v_k \\ v_1 \\ \vdots \\ v_k \end{bmatrix}.$$

Now, based on (61), it can be deduced that

$$q'\mathbf{N}q < 0$$

which can be rewritten as

$$\sum_{i=1}^{k} \left(v_i' P_i v_i \mathcal{F}_{R_i}(\lambda_i) \right) + v'\mathbf{A}'(\lambda) G \mathbf{A}(\lambda) v < 0. \tag{63}$$

From (62), it can be seen that the second term in the above inequality is zero. Moreover, the set $\mathcal{S}^C$ can be defined (with the choice (10)), by $\mathcal{F}_{R_i} \geq 0$. Thus, if λ belongs to $\mathcal{S}^C$, then the first term in (63) is positive or zero, which contradicts (63). Hence, λ cannot belong to $\mathcal{S}^C$. Then $c(\lambda, I, A) = \det(\mathbf{A}(\lambda))$ satisfies the condition of Lemma 1, implying nominal satbility. It is clear that $c(\lambda, I, A_c) = \det(\mathbb{A}(\lambda))$ also satisfies the condition of Lemma 1 for any Δ otherwise, by continuity of λ with respect to Δ, $\mathbb{A}(\lambda)$ would becomes $\partial\mathcal{D}$-singular. Hence, (1) with $E = I$ and (57) is robustly asymptotically stable against $\mathcal{B}(\rho)$. □

It can be noted again that the maximum value of ρ while satisfying LMI (59), denoted by $\rho^\star$, is a lower bound of the complex stability radius.

The strong difference between $\partial\mathcal{D}$-regularity and asymptotic stability lies in the distribution of the roots of $c(\lambda, I, A_c)$ with respect to $\partial\mathcal{D}$. Basically, Theorem 2 ensures robust $\partial\mathcal{D}$-regularity. However, if nominal k-variate polynomial matrix is "stable", meaning that $c(\lambda, I, A)$ fills the conditions given by Lemma 1 or equivalently that (1) with $E = I$ is asymptotically stable, then this property is obviously preserved when Δ describes $\mathcal{B}(\rho)$ if $\mathbb{A}(\lambda)$ remains $\partial\mathcal{D}$-regular. Nevertheless, if, for some Δ, matrix $\mathbb{A}(\lambda)$ becomes $\partial\mathcal{D}$-singular, it does not necessarily imply

instability. Thus, the conservatism of Theorem 2 is due both to this reason and to the fact that Corollary 1 also proposes a sufficient condition. Another reason is that the realness of Δ is not taken into account.
In the 1-D case, only this last reason induces conservatism. Indeed, in this case, not only Theorem 1 and thus Corollary 1 provide a necessary and sufficient condition but moreover, $\partial\mathcal{D}$-singularity implies instability.

4.2 Positive realness of hybrid Roesser model

Since the work of Popov on hyperstability, positive realness [2] of transfer matrices have been widely investigated, especially for classical 1-D systems, noting that, for such models, positive realness means passivity and enables the designer to include sector-bounded nonlinearities. For multidimensional linear models, few contributions exist. However, the 2-D discrete Roesser models are considered from the viewpoint of positive realness in [38]. Hence, we make an attempt to extend those results to investigate positive realness of hybrid Roesser models.

We will first state the following definition.

Definition 2. *Let the set $\partial\mathcal{D}$ be defined by (9) with (6)–(8) and (10). The model (1) is said to be strictly positive real over $\partial\mathcal{D}$ if*

$$G(\lambda) + G'(\lambda) > 0 \quad \forall \lambda \in \partial\mathcal{D} \tag{64}$$

Following this idea, the next corollary is proposed.

Corollary 2. *Consider a hybrid Roesser model and the set $\partial\mathcal{D}$ be described by (1) and (9) with (6)–(8) and (10) respectively. Suppose also that M is defined in (21), $\overline{\mathbf{R}}(\mathcal{P})$ is defined by (15), F is given by (43) and*

$$\Theta := \begin{bmatrix} 0 & -C' \\ -C & -D - D' \end{bmatrix}. \tag{65}$$

Then, the model (1) is strictly positive real over $\partial\mathcal{D}$ if there exists a set $\mathcal{P} \in \mathbb{H}$ such that (25) holds.

Proof. It follows immediately form Theorem 1 that with (65), the expression 23 corresponds to (64). □

It is clear that the notion of strict positive realness over $\partial\mathcal{D}$ might not be sufficient to appreciate the performances of a nD-model and one can come back to the more classical notion of strict positive realness.

Definition 3. *The model (1) is strictly positive real if $G(\lambda)$ is analytic on $\mathcal{S}^C$ defined by (5) and if inequality*

$$G(\lambda) + G'(\lambda) > 0 \tag{66}$$

hold on $\mathcal{S}^C$.

Theorem 3. *Consider a multidimensional hybrid system described by Roesser model (1) with $E = I$. Suppose also that F is given by (43), $\overline{\mathbf{R}}(\mathcal{P})$ is given by (15) with (10) and Θ is given by (65). Then, the hybrid n-D system is strictly positive real if there exists a set $\mathcal{P}$ of k matrices $P_i \in \mathcal{H}_{n_i}^+$, $i = 1, \dots, k$, such that LMI (59) holds.*

Proof. This follows immediately from the result of Corollary 2 except that the matrices P_i are assumed to be positive definite in order to guarantee asymptotic stability as in the proof of Theorem 2. □

It is significant to note that Theorem 3 can be seen as an extension of the LMI version of the classical positive real lemma (see [9] for example).

4.3 Bounded realness of hybrid Roesser model

The idea is about the same as in the above subsection but the performance to be considered is the so-called $\mathcal{H}_\infty$-level. As some extension of the $\mathcal{H}_\infty$-norm proposed in [13] for the 2-D discrete Roesser models, the following definition introduces the $\mathcal{L}_\infty$-norm of model (1) with respect to $\partial\mathcal{D}$.

Definition 4. *Let a hybrid Roesser model and the set $\partial\mathcal{D}$ be described by (1) and (9) with (6-8) and (10) respectively. Then, the $\mathcal{L}_\infty$-norm of (1) with respect to $\partial\mathcal{D}$ is defined by*

$$||G||_\infty = \sup_{\lambda \in \partial\mathcal{D}} ||G(\lambda)||_2. \tag{67}$$

From this definition and from Theorem 1, the next corollary is stated.

Corollary 3. *Let the hybrid Roesser model and the set $\partial\mathcal{D}$ be described by (1) and (9) with (6-8)and (10) respectively. The $\mathcal{L}_\infty$-norm of (1) with respect to $\partial\mathcal{D}$ is strictly lower than $\sqrt{\gamma}$ if there exists a set $\mathcal{P} \in \mathbb{H}$ such that (25) holds with M defined as in (21), $\overline{\mathbf{R}}(\mathcal{P})$ defined by (15), F given by (43) and Θ given by (47).*

Proof. Direct from the discussion in subsection 4.1.1 and especially inequality (48). □

A particular attention can be paid to the case where $E = I$. Assuming that (1) is asymptotically stable (*i.e.* $G(\lambda)$ is analytic on $\mathcal{S}^C$), then the $\mathcal{L}_\infty$-norm becomes the so-called $\mathcal{H}_\infty$-norm.

Definition 5. *Let the hybrid Roesser model (1) with $E = I$ comply with Lemma 1. Its $\mathcal{H}_\infty$-norm is defined by (67) with (9) and (6-8) together with (10).*

This $\mathcal{H}_\infty$-norm actually equals the so-called $\mathcal{L}_2$-gain but this connection is not detailed here for the sake of conciseness.

From the above definition, the next corollary is formulated.

Theorem 4. *Let the hybrid Roesser model described by (1) with $E = I$. It is asymptotically stable and its $\mathcal{H}_\infty$-norm is strictly lower than $\sqrt{\gamma}$ if there exists a set $\mathcal{P}$ of k matrices $P_i \in \mathcal{H}^+_{n_i}$, $i = 1, \ldots, k$, such that LMI (59) holds, where F is given by (43), $\overline{\mathbf{R}}(\mathcal{P})$ is given by (15) with (10) and Θ is given by (47).*

Proof. The proof follows immediately from the result of Corollary 3 combined with (10) and under assumption that the matrices P_i are positive definite to ensure asymptotic stability. □

Theorem 4 can be seen as an extension of the classical LMI version of the Bounded real lemma [18] to hybrid Roesser models. It has to be mentioned that the $\mathcal{H}_\infty$-control design is considered in [39] through an equivalent approach.

Remark 2. *Corollaries 1 and 3 actually involve the same conditions. The same for Theorems 2 and 4. In Corollary 1 (resp. Theorem 2), robust $\partial\mathcal{D}$-regularity (resp. robust stability) is concerned whereas in Corollary 3 (resp. Theorem 4), $\mathcal{L}_\infty$-norm on $\partial\mathcal{D}$ (resp. $\mathcal{H}_\infty$ performance level) is addressed. This connection between the $\mathcal{L}_\infty$ or $\mathcal{H}_\infty$-norm of a system and the quadratic stability against an LFT-based uncertainty is already well known in the* 1*-D system case [29].*

5 Numerical Example

In this section, the results developed above are illustrated by one numerical example.

Example 1. *Consider the hybrid 2-D system (1) with $r = 1$ (it means that the system has continuous dynamics along one of two dimensions) and the state space model matrices given by*

$$A = \begin{bmatrix} -0.5 & 0.2 \\ 0.1 & -0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 & 1.1 \\ 0.6 & 0.1 \end{bmatrix}$$
$$C = \begin{bmatrix} -0.1 & -0.1 \\ -0.2 & 0.6 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & -0.2 \\ -0.1 & -0.3 \end{bmatrix}.$$

The purpose of this example is to use the result of Theorem 4 to compute the minimum upper bound on $\mathcal{H}_\infty$ norm of the system described by the above matrices.

With computations performed with LMI CONTROL TOOLBOX*, it is verified that this system is asymptotically stable and the minimum $\mathcal{H}_\infty$ performance level $\rho = \sqrt{\gamma}$ is* 1.3224*. Furthermore, the matrices that solves the LMI (59) are*

$$P_1 = 0.4233, \; P_2 = 0.9447.$$

Following Remark 2, it also means that ρ is a lower bound of the complex stability radius.

6 Conclusion

In this paper, a KYP-like theorem has been proposed with a purpose of application to the very general hybrid state-space Roesser models. As a special instance of the result, the KYP lemma for classical 1-D linear models is recovered. Unlike for this 1-D case, for multidimensional models, some conservatism appears. This is due to the nature of Roesser models themselves. The origin of this conservatism has been highlighted.

Some possibilities of exploiting this new theorem in terms of robust stability, positive realness and $\mathcal{H}_\infty$-analysis of hybrid Roesser models have been emphasized. The authors hope that this point of view will be seen as an attempt to unify numerous contributions dealing with LMI approach applied to n-D models.

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