Robust $\mathcal{D}_R$-admissibility of uncertain descriptor systems via LMI approach

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Abstract

This paper deals with the robust $\mathcal{D}_{\mathcal{R}}$-admissibility of descriptor systems. The research for robustness bounds for descriptor systems is addressed. The use of parameter-dependent Lyapunov functions allows ones to handle uncertain singular systems whose uncertainty is polytopic norm-bounded (*i.e.* the state matrix can be written $A + J\Delta L$ where A, J and L belong to a polytope of matrices and Δ is unknown). If the 2-norm of Δ is less than a robustness bound to be determined, the eigenvalues of any pencil $(E, A + J\Delta L)$ in the uncertainty domain are clustered in an Ellipsoidal Matrix Inequality (EMI)-region. The proposed bound is easy to compute by using LMI tools.

1 Introduction

Singular systems, also referred to as descriptor systems, both continuous and discrete, have been of great interest in the literature since they have many applications (see [4]), for instance in electrical circuits network, robotics and economics. It is fair to say that descriptor models give a more complete class of dynamical models than the conventional state-space systems.

Many classical concepts and results obtained for conventional systems have been extended to descriptor systems. Let us quote for instance controllability and observability, pole assignment, stability analysis [7, 5]) and stabilization techniques as well as results including robustness aspects [9, 6, 16].

The natural *Generalized Lyapunov Equation* (GLE) [7] was proven in [5] to fail unless the system is in its Weierstrass form and the author in [5] proposed a new GLE equivalent to that given in [13]. In [9], the authors modified the GLE from [13] and proposed an equivalent matrix inequality condition.

In a number of approaches, the system model is transformed into a special form and it is understandable that this way of doing is not very appropriate in the presence of uncertainty.

The admissibility property includes the stability as well as the regularity and the absence of impulses (or causality).

Concerning the stability analysis, a number of approaches assuming or not the regularity of the descriptor system have been proposed in the literature [2, 4, 14]. But stability and regularity are not always enough. Indeed, for conventional models, the location of the state matrix poles in the complex plane for a standard system is related to the performances of the system, let us quote for instance the rise or the settling time as well as the overshoot of a step response. However, strict location is not necessarily required and it can suffice that poles (the eigenvales of the state matrix) lie in some specified region $\mathcal{D}$ of the complex plane. Such a property is called matrix $\mathcal{D}$-stability or matrix root-clustering. For descriptor systems, the property has to be extended to the notion of $\mathcal{D}$-admissibility, which is defined as the satisfaction of the $\mathcal{D}$-stability of the state matrix pencil, the regularity and impulse freeness or causality. In this work, the clustering regions are the EMI (Ellipsoidal Matrix Inequality)-regions introduced in [11]. We denote any element of this set by a generic name $\mathcal{D}_R$.

Recently,the characterization of pole clustering via LMI has been extended to descriptor systems in [8]. In this paper, a Linear Matrix Inequality (LMI) formulation is adopted to express necessary and sufficient conditions for the $\mathcal{D}_R$-admissibility of continuous descriptor systems. The proposed approach can be understood as the LMI-correspondant formulation of the proposed GLE in [5]. It is known that strict inequality conditions are tractable and reliable especially with the available LMI software solvers. Moreover, the use of Linear Matrix Inequality formulation in expressing the poles clustering region of the complex plane has proved its efficiency [3]. These LMI enables us to describe EMI regions, for example, which encompass most practical performance specification regions in control theory.

Besides, it is even truer to claim that strict location is no longer what the designer is looking for when he has to take uncertainties into account. Since the used models are generally obtained from a non linear model, we consider simultaneously two kinds of uncertainties, that is, polytopic and unstructured uncertainties. Unfortunately, the uncertainties can generate unexpected pole migration in the complex plane. When stability is concerned, a solution to analyze robust stability is to find a bound on an additive uncertainty (on the 2-norm of a matrix for an unstructured uncertainty or on the modulus of the parameter variations for the structured uncertainty) such that stability is ensured. Such a bound is called a robust stability bound and is conservative most of the time [10]. In the present work, the uncertainty is both polytopic and norm-bounded (polytopic to encompass parameter deflection and norm-bounded to include neglected phenomena). The polytopic uncertainty will be accurately defined by the vertices of the polytope whereas the unstructured norm-bounded uncertainty is "less than" a bound which has to be maximized, in order to get large admissible uncertainty.

In the available literature we easily note that quadratic stability has taken a lion's share, especially in the LMI framework. The quadratic stability is characterized by a determination of a unique so-called Lyapunov matrix which gives the approach an inherent conservatism. Many results have been reported in quadratic stability analysis and/or stabilization see for instance [1, 18, 17] and the reference therein.

In the present work, the robustness bound not only ensures stability but also ensures the $\mathcal{D}_R$-admissibility of the system, for all matrices Δ such that $||\Delta||_2 \leq \rho$. It is computed through the use of LMI conditions for robust $\mathcal{D}_R$-admissibility, which, unlike the nominal case, are conservative.
This paper is organized as follows. Section 2 gives the problem formulation. Section 3 gives the result on $\mathcal{D}_R$-admissibility for nominal singular system whereas Section 4 presents the result for uncertain singular systems. Section 5 presents an illustrative example. Section 6 concludes the paper.

Notation

We denote by $X^\top$ the conjugate transpose of matrix X, and by $\mathsf{Sym}\{X\}$ the Hermitian expression $= X + X^\top$. The Kronecker product is denoted by $\otimes$. $||X||_2$ is the 2-norm of matrix X, I is the identity matrix of suitable order, 0 is a null matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Löewner *i.e.* "< 0" ("≤ 0") means negative (semi-)definite and "> 0" ("≥ 0") positive (semi-)definite.

2 Preliminaries

In this section, firstly we give some basic definitions concerning descriptor systems. Secondly, the definition of an EMI-region is recalled.

2.1 Descriptor system

Consider the following continuous-time descriptor system,

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \\ y(t) & = & Cx(t), \end{array} \tag{1}$$

where $x(t)$ is a $\mathbb{R}^n$ state vector, $u(t)$ is a $\mathbb{R}^m$ the control input. The matrix E may be singular, we shall assume that $rank(E) = r \leq n$. A and B are known real constant matrices with appropriate dimensions.

Definition 2.1 *The pair (E, A) is said to be regular when $det(pE - A)$ is not identically zero and impulse free when $deg(det(pE - A)) = rank(E)$.*

Note that the regularity guarantees the existence and uniqueness of solution $x(.)$. Also, since the impulsive modes tend to generate undesired impulsive behaviors, they should be eliminated.

Provided (E, A) is regular there exist two non singular matrices U and V such that, [4]

$$\bar{E} = UEV = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{A} = UAV = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \tag{2}$$

2.2 Characterization of an EMI-region $\mathcal{D}_R$

The general definition of EMI-region is given:

Definition 2.2 *: Let $R \in \mathbb{C}^{2d \times 2d}$ be the Hermitian matrix defined by*

$$\begin{cases} R \;=\; R^\top \;=\; \begin{bmatrix} R_1 & R_3 \\ R_3^\top & R_2 \end{bmatrix}, \\ \\ R_2 \geq 0 \in \mathbb{C}^{d \times d}. \end{cases}$$

The set of points $\mathcal{D}_R$ defined by

$$\mathcal{D}_R \;=\; \{z \in \mathbb{C} \,|\, f_{\mathcal{D}_R}(z) = \; R_1 + \mathsf{Sym}\,\{R_3 z\} + R_2 z z' \;<\; 0\} \tag{3}$$

is called an EMI-region (EMI for Ellipsoidal Matrix Inequality) of degree d.

Such a region is convex. For a real R, this formulation is close to the one of LMI regions (see [11] for more details). The set of EMI-regions includes, for instance, shifted and half planes, classical and hyperbolic sectors, vertical and horizontal strips, discs or ellipses.

Remark 2.1 *The intersection of two EMI regions $\mathcal{D}_{R1}$ and $\mathcal{D}_{R2}$ is an EMI region which characteristic function is given by*

$$f_{\mathcal{D}_{R1}\cap\mathcal{D}_{R2}} = diag(f_{\mathcal{D}_{R1}}, f_{\mathcal{D}_{R2}})$$

Definition 2.3 *Matrix A is said to be $\mathcal{D}_R$-stable if all its eigenvalues are in the region $\mathcal{D}_{\mathcal{R}}$.*

The $\mathcal{D}_R$-stability of a matrix A is characterized by an LMI condition as follows:
If there exists a symmetric positive definite matrix $X = X^\top > 0$ such that the LMI

$$R_1 \otimes (X) + \mathsf{Sym}\left\{R_3 \otimes (AX)\right\} + R_2 \otimes (AXA^\top) < 0$$

is satisfied then matrix A is $\mathcal{D}_R$ stable.

Definition 2.4 *Let $\mathcal{D}$ be any subset of $\mathbb{C}$. System (1) is said to be $\mathcal{D}$-admissible if it is regular, impulse free and if A is $\mathcal{D}$-stable.*

Fact 2.1 *The singular system is said to be $\mathcal{D}_R$-admissible if and only if there exits a symmetric and positive matrix $\bar{X}_{11}$ and a non singular matrix $\bar{Y}_{22}$ such that the following conditions*

$$\begin{aligned} R_1 \otimes (\bar{X}_{11}) + \mathsf{Sym}\left\{R_3 \otimes (\hat{A}\bar{X}_{11})\right\} + R_2 \otimes (\hat{A}\bar{X}_{11}\hat{A}^\top) &< 0 \quad (4)\\ \mathsf{Sym}\left\{\bar{A}_{22}\bar{Y}_{22}\right\} &< 0 \quad (5) \end{aligned}$$

hold with $\hat{A} = (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21})$.

Indeed, condition (5) means that matrix $\bar{A}_{22}$ is non singular which implies impulse freeness and regularity, whereas condition (4) states that matrix $(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21})$ is $\mathcal{D}_R$-stable. Positive definiteness is not required for a global Lyapunov matrix $\bar{X}$ but only for one smaller block $\bar{X}_{11}$ that has to satisfy (4). Definition 2.4 is an adaptation of that given in [4, 6].

3 $\mathcal{D}_R$-admissibility analysis

First, two lemmas are recalled that will be useful for the various reasonings and proofs in the sequel. Then a necessary and sufficient condition for the $\mathcal{D}_R$-admissibility of a descriptor system is presented.

Lemma 3.1 *[15] Let Z, E, Δ and F be complex matrices with appropriate dimensions. Assume that Z is Hermitian then*

$$Z + \mathsf{Sym}\{E\Delta F\} \prec 0 \quad \forall \Delta \,|\, \Delta'\Delta \preceq \mathbf{I}, \tag{6}$$

if and only if there exists a scalar number $\epsilon > 0$ satisfying

$$Z + \epsilon EE' + \frac{1}{\epsilon}F'F \prec 0. \tag{7}$$

Lemma 3.2 *[12] Let $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{k\times n}$ and $Q = Q' \in \mathbb{C}^{n\times n}$ be given. Then the following statements are equivalent:*

(i) There exists a matrix Y satisfying

$$Q + \mathsf{Sym}\{BYC\} \prec 0. \tag{8}$$

(ii) The following two conditions hold

$$\left\{ \begin{array}{l} B^{\perp}QB^{\perp'} \prec 0 \quad \text{or} \quad BB' \succ 0 \\ C'^{\perp}QC'^{\perp'} \prec 0 \quad \text{or} \quad C'C \succ 0. \end{array} \right. \tag{9}$$

Consider the singular system described by the pair (E, A). Let $E^{\perp}$, $E^{\ddagger}$ and $E^{\dagger}$ be defined as

$$\begin{array}{rcl} E^{\perp} & = & V\,(I - UEV)\,U, \quad E^{\ddagger} = U^{\top}\,(I - UEV)\,U^{-\top}, \\ E^{\dagger} & = & U^{-1}\,(I - UEV)\,U. \end{array} \tag{10}$$

with U and V some non singular matrices satisfying the first equality in (2) Note that we have $EE^{\perp} = 0$ and $E^{\top}E^{\ddagger} = 0$ and $E^{\dagger}E = 0$.

Theorem 3.1 *The singular system* (E, A) *is* $\mathcal{D}_R$*-admissible if and only if there exist three matrices* X*,* Y *and* Z *such that*

$$EXE^{\top} + \mathsf{Sym}\left\{E^{\dagger}Z\right\} > 0 \tag{11}$$

$$\begin{array}{l} M = R_1 \otimes (EXE^{\top}) + \mathsf{Sym}\left\{R_3 \otimes (AXE^{\top}) + I \otimes (AE^{\perp}YE^{\ddagger})\right\} \\ +R_2 \otimes (AXA^{\top}) < 0 \end{array} \tag{12}$$

Proof of Theorem 3.1

Sufficiency:

For the sufficiency part we assume that conditions (11) and (12) hold. Notice that (11), if satisfied, implies that the 11-block of $\bar{X} = V^{-1}XV^{-\top}$ is strictly positive definite. Indeed, we have

$$U\left(EXE^{\top} + \mathsf{Sym}\left\{E^{\dagger}Z\right\}\right)U^{\top} \;=\; \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{22} \end{bmatrix} > 0$$

and matrix Z is introduced precisely to write a strict LMI condition.

To prove the admissibility of the considered system we begin by proving that if (12) is satisfied then the 22-block of UAV is invertible. For this purpose we transform M according to

$$\begin{array}{l} = R_1 \otimes (\bar{E}\bar{X}\bar{E}^{\top}) + \mathsf{Sym}\left\{R_3 \otimes (\bar{A}\bar{X}\bar{E}^{\top}) + I \otimes (\bar{A}\bar{E}^{\perp}\bar{Y}\bar{E}^{\ddagger})\right\} \\ +R_2 \otimes (\bar{A}\bar{X}\bar{A}^{\top}) \end{array}$$

with

$$\bar{X} = V^{-1}XV^{-\top} = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} \\ \bar{X}_{21} & \bar{X}_{22} \end{bmatrix}, \quad \bar{Y} = UYU^\top = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} \\ \bar{Y}_{21} & \bar{Y}_{22} \end{bmatrix}$$

and

$$\bar{E}^\perp = V^{-1}E^\perp U^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} = \bar{E}^\ddagger = U^{-\top}E^\ddagger U^\top.$$

Since $\bar{X}_{11}$ is invertible then $\bar{X}$ can be written as

$$\bar{X} = \begin{bmatrix} I & 0 \\ \bar{X}_{21}\bar{X}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12} \end{bmatrix} \begin{bmatrix} I & \bar{X}_{11}^{-1}\bar{X}_{12} \\ 0 & I \end{bmatrix},$$

the square matrix $\bar{A}\bar{X}\bar{A}^\top$ is written as

$$\begin{aligned} \bar{A}\bar{X}\bar{A}^\top &= \begin{bmatrix} * & \bar{A}_{12} \\ A_x & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12} \end{bmatrix} \begin{bmatrix} * & A_x^\top \\ \bar{A}_{12}^\top & \bar{A}_{22}^\top \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ * & A_x\bar{X}_{11}A_x^\top + \bar{A}_{22}\left(\bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12}\right)\bar{A}_{22}^\top \end{bmatrix} \end{aligned}$$

where the * corresponds to entries with no much relevance at this step and $A_x = \bar{A}_{21} + \bar{A}_{22}\bar{X}_{k21}\bar{X}_{k11}^{-1}$. It comes, from $\bar{M}$, by permutations on the rows and on the columns,

$$\begin{aligned} \bar{M}_0 = & \begin{bmatrix} R_1 \otimes (\bar{X}_{11}) & 0 \\ 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} * & * \\ * & R_2 \otimes \left(A_x\bar{X}_{11}A_x^\top + \bar{A}_{22}\left(\bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12}\right)\bar{A}_{22}^\top\right) \end{bmatrix} \\ & + \mathsf{Sym}\left\{\begin{bmatrix} R_3 \otimes (\bar{A}_{11}\bar{X}_{11} + \bar{A}_{12}\bar{X}_{21}) & 0 \\ R_3 \otimes (\bar{A}_{21}\bar{X}_{11} + \bar{A}_{22}\bar{X}_{21}) & 0 \end{bmatrix}\right\} \\ & + \mathsf{Sym}\left\{\begin{bmatrix} 0 & I \otimes (\bar{A}_{12}\bar{Y}_{22}) \\ 0 & I \otimes (\bar{A}_{22}\bar{Y}_{22}) \end{bmatrix}\right\} < 0. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} S &= R_2 \otimes \left(A_x\bar{X}_{11}A_x^\top + \bar{A}_{22}\left(\bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12}\right)\bar{A}_{22}^\top\right) \\ &+ \mathsf{Sym}\left\{I \otimes (\bar{A}_{22}\bar{Y}_{22})\right\} < 0. \end{aligned}$$

If $\bar{A}_{22}$ was singular then there would exist a non zero vector ξ such that $\xi \in \mathrm{Ker}\bar{A}_{22}^T$. Therefore, one would get

$$(I \otimes \xi^\top) S (I \otimes \xi) = \xi^\top \left(R_2 \otimes (A_x \bar{X}_{11} A_x^\top)\right) \xi < 0$$

which is impossible since $R_2 \geq 0$ and $\bar{X}_{11} > 0$. Hence, $\bar{A}_{22}$ cannot be singular.

Now we have to show that the system is $\mathcal{D}_R$-stable or precisely that matrix $\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$ is $\mathcal{D}_R$-stable. For this purpose, let us consider the two matrices

$$\Sigma = \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & \bar{A}_{22}^{-1} \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} I & -\bar{A}_{12}\bar{A}_{22}^{-1} \\ 0 & I \end{bmatrix},$$

that transform matrix $\bar{A}$ in a block diagonal form as

$$\bar{\bar{A}} = \Gamma \bar{A} \Sigma = \begin{bmatrix} \bar{\bar{A}}_{11} & 0 \\ 0 & I \end{bmatrix}$$

with $\bar{\bar{A}}_{11} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$ and transform $\bar{M}$ into $\bar{\bar{M}}$ as

$$\begin{aligned}
&\bar{\bar{M}} = (I \otimes \Gamma)\, \bar{M}\, \left(I \otimes \Gamma^\top\right) \\
&= R_1 \otimes (\bar{\bar{E}}\bar{\bar{X}}\bar{\bar{E}}^\top) + \mathsf{Sym}\left\{R_3 \otimes (\bar{\bar{A}}\bar{\bar{X}}\bar{\bar{E}}^\top)\right\} + \mathsf{Sym}\left\{I \otimes (\bar{\bar{A}}\bar{\bar{E}}^\perp\bar{\bar{Y}}\bar{\bar{E}}^\ddagger)\right\} \\
&+ R_2 \otimes (\bar{\bar{A}}\bar{\bar{X}}\bar{\bar{A}}^\top) < 0
\end{aligned} \tag{13}$$

with

$$\bar{\bar{E}} = \Gamma \bar{E} \Sigma = \bar{E}, \bar{\bar{X}} = \Sigma^{-1}\bar{X}\Sigma^{\top-1}, \bar{\bar{E}}^\perp = \Sigma^{-1}\bar{E}^\perp\Gamma^{-1} = \bar{E}^\perp, \bar{\bar{Y}} = \Gamma\bar{Y}\Gamma^\top.$$

From (13), by permutation on the rows and on the columns, one gets

$$\begin{aligned}
\bar{\bar{M}}_0 \;\; = \;\; & \begin{bmatrix} R_1 \otimes \bar{\bar{X}}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \mathsf{Sym}\left\{\begin{bmatrix} R_3 \otimes (\bar{\bar{A}}_{11}\bar{\bar{X}}_{11}) & 0 \\ * & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}\right\} \\
& + \begin{bmatrix} R_2 \otimes \left(\bar{\bar{A}}_{11}\bar{\bar{X}}_{11}\bar{\bar{A}}_{11}^\top\right) & * \\ * & * \end{bmatrix} < 0,
\end{aligned}$$

which implies that if $\bar{\bar{M}}_0$ or equivalently $\bar{\bar{M}}$ is negative definite and then we have necessarily

$$R_1 \otimes \bar{\bar{X}}_{11} + \mathsf{Sym}\left\{R_3 \otimes (\bar{\bar{A}}_{11}\bar{\bar{X}}_{11})\right\} + R_2 \otimes \left(\bar{\bar{A}}_{11}\bar{\bar{X}}_{11}\bar{\bar{A}}_{11}^\top\right) < 0. \tag{14}$$

Inequality (14) means that the singular system is $\mathcal{D}_{\mathcal{R}}$-stable bearing in mind that $\bar{X}_{11}$ is positive definite thanks to condition (11) and this ends the proof of the sufficiency part.

Necessity

Assume that the system is $\mathcal{D}_R$-admissible or in other words that matrix $\bar{A}_{22}$ is invertible and there exists a positive definite matrix $\bar{X}_{11}$ such that

$$R_1 \otimes \bar{X}_{11} + \mathsf{Sym}\left\{R_3 \otimes (\bar{\bar{A}}_{11}\bar{X}_{11})\right\} + R_2 \otimes \left(\bar{\bar{A}}_{11}\bar{X}_{11}\bar{\bar{A}}_{11}^{\top}\right) < 0$$

we have

$$\bar{E} = \bar{U}E\bar{V} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \bar{A} = \bar{U}A\bar{V} = \begin{bmatrix} \bar{\bar{A}}_{11} & 0 \\ 0 & I \end{bmatrix}.$$

Let $E^{\perp}$, $E^{\ddagger}$ and $E^{\dagger}$ be defined as

$$E^{\perp} = \bar{V}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\bar{U} \qquad E^{\ddagger} = \bar{U}^{\top}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\bar{U}^{-\top} \qquad E^{\dagger} = \bar{U}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\bar{U}.$$

Note that in this part of the proof the matrices below do not comply with the same expression as in (2) and take

$$X = \bar{V}\begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} \end{bmatrix}\bar{V}^{\top}, \qquad Y = \bar{U}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \bar{Y}_{22} \end{bmatrix}\bar{U}^{-\top}.$$

It is clear that there exist $\bar{X}_{11}$ and $\bar{Y}_{22}$ such that (4) can be written as

$$\begin{aligned} &\begin{bmatrix} R_1 \otimes \bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \mathsf{Sym}\left\{\begin{bmatrix} R_3 \otimes (\bar{\bar{A}}_{11}\bar{X}_{11}) & 0 \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \begin{bmatrix} R_2 \otimes (\bar{\bar{A}}_{11}\bar{X}_{11}\bar{\bar{A}}_{11}^{\top}) & 0 \\ * & * \end{bmatrix} + \mathsf{Sym}\left\{\begin{bmatrix} 0 & * \\ 0 & I \otimes (\bar{Y}_{22}) \end{bmatrix}\right\} < 0, \end{aligned}$$

and, with permutations on rows and columns, we rewrite the above condition as follows (noting that $\bar{A}\bar{E}^{\perp}\bar{Y}\bar{E}^{\ddagger} = \bar{E}^{\perp}\bar{Y}\bar{E}^{\ddagger}$)

$$\begin{aligned} &\bar{M} = R_1 \otimes (\bar{E}\bar{X}\bar{E}^{\top}) + \mathsf{Sym}\left\{R_3 \otimes (\bar{A}\bar{X}E^{\top})\right\} + R_2 \otimes (\bar{A}\bar{X}\bar{A}^{\top}) \\ &+\mathsf{Sym}\left\{I \otimes \left(\bar{A}\bar{E}^{\perp}\bar{Y}\bar{E}^{\ddagger}\right)\right\} < 0 \end{aligned} \tag{15}$$

Pre and postmultiplying (15) respectively by $(I \otimes U^{-1})$ and $(I \otimes U^{-\top})$ yields

$$R_1 \otimes (EXE^\top) + \mathsf{Sym}\left\{R_3 \otimes (AXE^\top)\right\} + \mathsf{Sym}\left\{I \otimes (AE^\perp Y E^\ddagger)\right\}$$
$$+R_2 \otimes (AXA^\top) < 0.$$

This allows us to recover condition(12).
Condition (11) is easily recovered since for

$$Z = \bar{U}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \bar{Z}_{22}\end{bmatrix}\bar{U}^{-\top}$$

with $\bar{Z}_{22} > 0$, it comes

$$\bar{E}\bar{X}\bar{E}^\top + \mathsf{Sym}\left\{\begin{bmatrix} 0 & 0 \\ 0 & \bar{Z}_{22}\end{bmatrix}\right\} = \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{Z}_{22}\end{bmatrix} > 0,$$

which implies that $\bar{X}_{11} > 0$. This closes the proof of the theorem. $\nabla\nabla\nabla$

4 Robust $\mathcal{D}_R$-admissibility analysis

In this section the singular system is characterized by the pair

$$(E,\ A(\Delta, \bar{\alpha}))$$

where matrix $A(\Delta, \bar{\alpha})$ reads

$$A(\bar{\alpha}) + J(\bar{\alpha})\Delta L(\bar{\alpha}).$$

Matrix Δ is a real or a complex unstructured norm bounded uncertainty satisfying

$$\Delta^\top \Delta \le \rho^2 I, \tag{16}$$

and matrices $A(\bar{\alpha})$, $J(\bar{\alpha})$ and $L(\bar{\alpha})$ belong to a polytope in the following way:

$$\begin{bmatrix} A(\bar{\alpha}) & J(\bar{\alpha}) \\ L(\bar{\alpha}) & 0 \end{bmatrix} = \sum_{i=1}^{p} \alpha_i \begin{bmatrix} A_i & J_i \\ L_i & 0 \end{bmatrix}$$

with

$$\alpha_i \geq 0,\ i = 1,\ \dots,\ p,$$
$$\sum_{i=1}^{p} \alpha_i = 1 \qquad \text{and} \qquad \bar{\alpha} = [\,\alpha_1 \quad \dots \quad \alpha_p\,]\,.$$

Matrices $A_i,\ J_i,\ L_i$ are known matrices that make the vertices of the polytope up.

The problem for a system corrupted by uncertainty is to preserve its performances for all admissible uncertainties or in other terms for every instance of matrix $A(\Delta, \bar{\alpha})$.

Definition 4.1 *The uncertain system is robustly $\mathcal{D}_R$-admissible if it is $\mathcal{D}_R$-admissible for all admissible uncertainties Δ and $\bar{\alpha}$.*

Explicitly, the uncertain singular system will be $\mathcal{D}_R$-admissible if for every instance Δ and α, matrix $\bar{A}_{22}(\Delta, \bar{\alpha})$ is invertible and there exists a positive definite matrix $\bar{X}_{11}(\Delta, \bar{\alpha})$ such that

$$\begin{aligned} &R_1 \otimes (\bar{X}_{11}(\Delta, \bar{\alpha})) + \mathsf{Sym}\left\{R_3 \otimes (\hat{A}(\Delta, \bar{\alpha})\bar{X}_{11}(\Delta, \bar{\alpha}))\right\} \\ &+R_2 \otimes (\hat{A}(\Delta, \bar{\alpha})\bar{X}_{11}(\Delta, \bar{\alpha})\hat{A}^\top(\Delta, \bar{\alpha})) < 0 \end{aligned} \tag{17}$$

with $\hat{A}(\Delta, \bar{\alpha}) = (\bar{A}_{11}(\Delta, \bar{\alpha}) - \bar{A}_{12}(\Delta, \bar{\alpha})\bar{A}_{22}^{-1}(\Delta, \bar{\alpha})\bar{A}_{21}(\Delta, \bar{\alpha}))$.

It is understandable that computing matrices $\bar{X}_{11}(\Delta, \bar{\alpha})$ for every instance $(\Delta, \bar{\alpha})$ is inconceivable. Thus one can find other alternatives to check the robust $\mathcal{D}_R$-admissibility.

If we make the assumption that matrix $\bar{A}_{22}(\Delta, \bar{\alpha}))$ is invertible and there exists a unique matrix X_{11} over the uncertainty set, or in other words matrix $\bar{X}_{11}(\Delta, \bar{\alpha}) = \bar{X}_{11}$ for every instance $(\Delta, \bar{\alpha})$ such that we have

$$R_1 \otimes (\bar{X}_{11}) + \mathsf{Sym}\left\{R_3 \otimes (\hat{A}(\Delta, \bar{\alpha})\bar{X}_{11})\right\} + R_2 \otimes (\hat{A}(\Delta, \bar{\alpha})\bar{X}_{11}\hat{A}(\Delta, \bar{\alpha})^\top) < 0$$

In this case the system will be termed as quadratically $\mathcal{D}_R$-admissible.

From above, we easily understand that quadratic $\mathcal{D}_R$-admissibility implies robust $\mathcal{D}_R$-admissibility but the $\mathcal{D}_R$-admissibility converse is, in general, false.
Nevertheless, to reduce the conservatism inherent to quadratic approach, we now propose a theorem in which the conditions implicitely involve matrices X, Y and Z that match the polytopic structure of A, J and L.

Theorem 4.1 *The uncertain singular system is robustly $\mathcal{D}_R$-admissible if there exist matrices X_i, Y_i and Z_i, $i = 1, \ldots, p$, with*

$$EX_iE^\top + \mathsf{Sym}\left\{E^\dagger Z_i\right\} > 0 \quad \forall i \in \{1, \ldots, p\} \tag{18}$$

and matrices $\mathbb{G}_1$ and $\mathbb{G}_2$ such that the condition

$$\begin{gathered}\begin{bmatrix} R_1 \otimes (EX_iE^\top) & W^\top & \rho(I \otimes J_i) & ((I \otimes L_i)\mathbb{G}_1)^\top \\ W & R_2 \otimes X_i & 0 & ((I \otimes L_i)\mathbb{G}_2)^\top \\ \rho(I \otimes J_i)^\top & 0 & -I & 0 \\ ((I \otimes L_i)\mathbb{G}_1) & ((I \otimes L_i)\mathbb{G}_2) & 0 & -I \end{bmatrix} \\ +\mathsf{Sym}\left\{\begin{bmatrix} I \otimes A_i \\ -I \otimes I \\ 0 \\ 0 \end{bmatrix} [\,\mathbb{G}_1 \quad \mathbb{G}_2 \quad 0 \quad 0\,]\right\} < 0 \quad \forall i \in \{1, \ldots, p\}\end{gathered} \tag{19}$$

with $W = \left(R_3 \otimes (X_iE^\top) + I \otimes (E^\perp Y_i E^\ddagger)\right)^\top$ is satisfied.

Proof:

Assume that condition (19) is satisfied and let

$$[\,X(\bar\alpha) \quad Y(\bar\alpha) \quad Z(\bar\alpha)\,] = \sum_{i=1}^{p} \alpha_i\,[\,X_i \quad Y_i \quad Z_i\,],$$

then one can deduce that

$$\begin{bmatrix} R_1 \otimes (EX(\bar{\alpha})E^\top) & * & * & * \\ R_3 \otimes (X(\bar{\alpha})E^\top) + I \otimes (E^\perp Y(\bar{\alpha})) & R_2 \otimes X(\bar{\alpha}) & * & * \\ \rho(I \otimes J(\bar{\alpha}))^\top & 0 & -I & * \\ ((I \otimes L(\bar{\alpha}))\mathbb{G}_1) & ((I \otimes L(\bar{\alpha}))\mathbb{G}_2) & 0 & -I \end{bmatrix}$$
$$+\mathsf{Sym}\left\{ \begin{bmatrix} I \otimes A(\bar{\alpha}) \\ -I \otimes I \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbb{G}_1 & \mathbb{G}_2 & 0 & 0 \end{bmatrix} \right\} < 0$$

Now let

$$\mathbb{G} = \begin{bmatrix} \mathbb{G}_1 & \mathbb{G}_2 \end{bmatrix}$$

then after a Schur complement we get

$$\begin{gathered} \begin{bmatrix} R_1 \otimes (EX(\bar{\alpha})E^\top) & * \\ R_3 \otimes (X(\bar{\alpha})E^\top) + I \otimes (E^\perp Y(\bar{\alpha})) & R_2 \otimes X(\bar{\alpha}) \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} I \otimes A(\bar{\alpha}) \\ -I \otimes I \end{bmatrix} \mathbb{G} \right\} \\ +\rho^2 \begin{bmatrix} I \otimes J(\bar{\alpha}) \\ 0 \end{bmatrix} \begin{bmatrix} I \otimes J^\top(\bar{\alpha}) & 0 \end{bmatrix} + (I \otimes (L(\bar{\alpha}))\mathbb{G})^\top (I \otimes (L(\bar{\alpha}))\mathbb{G}) < 0 \end{gathered} \tag{20}$$

At this step, the use of lemma 3.1 in the sense (7) to (6) allows us to state that

$$\begin{gathered} \begin{bmatrix} R_1 \otimes (EX(\bar{\alpha})E^\top) & * \\ R_3 \otimes (X(\bar{\alpha})E^\top) + I \otimes (E^\perp Y(\bar{\alpha})) & R_2 \otimes X(\bar{\alpha}) \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} I \otimes A(\bar{\alpha}) \\ -I \otimes I \end{bmatrix} \mathbb{G} \right\} \\ +\mathsf{Sym}\left\{ \begin{bmatrix} I \otimes J(\bar{\alpha}) \\ 0 \end{bmatrix} (I \otimes (\Delta)) (I \otimes (L(\bar{\alpha}))) \mathbb{G} \right\} < 0 \end{gathered} \tag{21}$$

holds for every Δ satisfying (16). Note that the implication (21) $\Rightarrow$ (20) might not hold because $(I \otimes \Delta)$ is not full block. Notice that the condition above can be written as follows

$$\begin{bmatrix} R_1 \otimes (EX(\bar{\alpha})E^\top) & * \\ R_3 \otimes (X(\bar{\alpha})E^\top) + I \otimes (E^\perp Y(\bar{\alpha})) & R_2 \otimes X \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} I \otimes A(\Delta, \bar{\alpha}) \\ -I \otimes I \end{bmatrix} \mathbb{G} \right\} < 0 \tag{22}$$

The use of matrix elimiation procedure [12] enables ones to state that the previous inequality implies

$$\begin{gathered} \tilde{M} = \begin{bmatrix} I \otimes I \\ I \otimes A^\top(\Delta, \bar{\alpha}) \end{bmatrix}^\top \begin{bmatrix} R_1 \otimes (EX(\bar{\alpha})E^\top) & * \\ R_3 \otimes (X(\bar{\alpha})E^\top) + I \otimes (E^\perp Y(\bar{\alpha})) & R_2 \otimes X(\bar{\alpha}) \end{bmatrix} \\ \begin{bmatrix} I \otimes I \\ I \otimes A^\top(\Delta, \bar{\alpha}) \end{bmatrix} < 0 \end{gathered}$$

or explicitly

$$\begin{array}{r}\tilde{M} = R_1 \otimes (EX(\bar{\alpha})E^\top) + R_2 \otimes (A(\Delta,\bar{\alpha})X(\bar{\alpha})A^\top(\Delta,\bar{\alpha})) \\ +\mathsf{Sym}\left\{R_3 \otimes (A(\Delta,\bar{\alpha})X(\bar{\alpha})E^\top) + I \otimes (A(\Delta,\bar{\alpha})E^\perp Y(\bar{\alpha}))\right\} < 0\end{array}$$

which is in fact condition (12) expressed for the uncertain system.

Note also that according to the definition of $X(\bar{\alpha})$ and $Z(\bar{\alpha})$ we easily deduce that

$$EX(\bar{\alpha})E^\top + \mathsf{Sym}\left\{E^\dagger Z(\bar{\alpha})\right\} > 0$$

which is in fact condition (11) expressed for the uncertain system. This ends the proof. $\nabla\nabla\nabla$

It must be noticed that (18)-(19) makes an LMI system with respect to X_i, Y_i, Z_i and ρ that can be solved while maximizing ρ.

In [8] a sufficent condition to check the robustness of the pole clustering in the presence of a Linear Fractional Transform (LFT)-based uncertainty is presented. However, in [8] no parametric uncertainty is considered. The main interest of the present approach, compared with the reference mentioned herebefore, is that, not only a polytopic uncertainty is considered but the derived conditions implicity involve parameter-dependent Lyapunov matrices which is less conservative than the quadratic approach used in [8]. It is enabled by the use of EMI formulation of regions rather than LMI formulation. But it should be noted that, although that it does not tackle parametric uncertainties, the result of [8] is for an LFT-based uncertainty, which is a bit more general than a norm-bounded one.

5 Illustrative example

Consider the descriptor system defined by

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

and the state matrix belongs to the polytope whose vertices are given by

$$A_1 = \begin{bmatrix} 0.1376 & -0.0264 & -0.0021 \\ 0.0000 & 0.0000 & -1.0000 \\ -0.8624 & -0.0264 & -0.0021 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0000 & -1.0000 & 0.0000 \\ -0.1972 & -1.0384 & 0.1964 \\ -2.0000 & 0.0000 & 0.0000 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.2561 & 0.3302 & 0.1578 \\ 0.0000 & 1.0000 & 0.0000 \\ -0.7439 & -0.6698 & 2.1578 \end{bmatrix}.$$

The norm-bounded uncertainty is described by

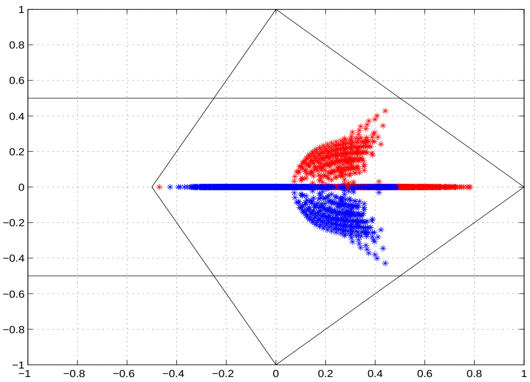
$$[\, J_1 \quad J_2 \quad J_3 \,] = \left[\begin{array}{c|c|c} 0.1376 & 0.7 & 0.4 \\ -0.0264 & -0.1 & 0.2 \\ -0.0021 & 0.1 & 0.1 \end{array}\right] \quad \text{and}$$

$$[\, L_1^\top \quad L_2^\top \quad L_3^\top \,] = \left[\begin{array}{c|c|c} 0.1 & 0.2 & 0.4 \\ 0.1 & 0.0 & 0.3 \\ 1.0 & 0.5 & 0.8 \end{array}\right]$$

The chosen EMI-region is defined by the intersection of three regions, respectively a horizontal strip and two conic sector. By applying theorem 4.1, the LMI is solvable and using the LMI toolbox of MATLAB we get:

$$\rho = 0.340.$$

The Figure shows the pole migration plotted for several values of Δ such that $||\Delta||_2 \leq \rho$



Finite eigenvalues location of the uncertain closed loop system

On this figure, one can notice that the pole migration nearly reaches the boundary of region which highlights the weak conservatism induced by our condition. Actually it is possible to find a matrix Δ such that $||\Delta||_2 = 0.341$ which makes the uncertain system loose its $\mathcal{D}_R$-admissibility. It proves this weak conservatism for the example.

6 Conclusion

In this paper, a new method to compute robustness bounds against polytopic norm-bounded has been proposed. A strict LMI condition for checking the $\mathcal{D}_\mathcal{R}$-admissibility for descriptor systems is given. An LMI technique to compute robust $\mathcal{D}_R$-admissibility bounds was presented, where $\mathcal{D}_R$ is a generic name for an EMI region. It enables ones to get a large choice of clustering regions including many convex regions or intersections of such convex-subregions (preserving convexity). Besides of the unstructured additive uncertainty, another convex polytopic one can be taken into account. In this case, the bound is obtained through the implicit computation of a parameter-dependent Lyapunov function which allows a strong reduction of conservatism. No expression of bound is proposed since the bound value is reached by maximizing a linear criterion under LMI constraints. Hence, powerful numerical tools can be used for computation and the proposed technique turns to be not only efficient but also tractable.

It would be interesting, in a future work, to consider more sophisticated regions such as the unions of several disjoint EMI-subregions.

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