Admissibility and State Feedback Admissibilization of Discrete Singular systems : An LMI Approach

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Abstract

This paper deals with the problems of robust admissibility and state feedback admissibilization of uncertain discrete descriptor systems. We propose a new and sufficient condition, which is necessary in terms of a strict Linear Matrix Inequality (LMI) for a nominal discrete descriptor system to be admissible (stable, regular and causal). Based on this, the state feedback admissibility problem is solved and the solution extended to the case of uncertain descriptor systems. Finally numerical examples are given to illustrate the results.

Index Terms

Singular systems, Stability, Causality, State feedback, Lyapunov methods.

1 Introduction

Singular systems, both continuous and discrete, have been of great interest in the specialized literature because they have many applications (see [1]) in electrical circuit networks, robotics and economics. Intuitively, a singular state space description of linear systems is more general than a conventional state space description. In particular, a descriptor form includes information about static as well as dynamic constraints.
Many classical concepts and results obtained for conventional systems have been extended to descriptor systems. Let us quote, for instance, controllability and observation, pole assignment, stability analysis [2, 3]) and stabilization techniques as well as results including robustness aspects [4, 5, 6].
The natural *Generalized Lyapunov Equation* (GLE) [2] has been improved for continuous singular systems as in [7, 4, 3] and for discrete singular systems as in [8, 6].

In the available literature on descriptor systems, there are two kinds of stabilization problems for singular discrete systems. One consists in designing a state feedback controller in such a way that the closed-loop system is regular, causal, and stable or equivalently admissible, the other is to design a state feedback controller in order to make the closed-loop system only regular and stable.
In a number of approaches, the system model is transformed into a special form and it is understandable

that this way of doing is not very appropriate in the presence of uncertainty.
Concerning the stability analysis and the stabilization problem, a number of approaches that assume or not the regularity of the descriptor system, have been proposed in the literature. Let us quote for instance [9, 10, 1, 11, 12, 13] among those assuming the regularity and [1, 11] those who did not.
Robust control of linear state space systems has been the focus of much attention during the past decades and various aspects and approaches for analysis and control design for linear uncertain systems have been investigated. See for instance [14]. In the available literature, we easily note that quadratic stability and stabilization approaches have taken a lion's share. The quadratic stability or stabilization is characterized by a determination of a unique so-called Lyapunov matrix, which gives the approach an inherent conservatism. Many results have been reported in quadratic stability analysis and/or stabilization, see for instance [15, 14, 16] and the reference therein.

Recently the Parameter Dependent Lyapunov (PDL) approach has been introduced to reduce the conservatism of the quadratic approach. The PDL approach consists in expressing the Lyapunov matrix as a function of the uncertainty and with the help of some slack additional variables, the approach yields a significant reduction of conservatism [17, 18].
In this paper, a Linear Matrix Inequality (LMI) formulation is adopted to express necessary and/or sufficient conditions for the admissibility of discrete descriptor systems. The proposed approach can be understood as the LMI-correspondent formulation of the proposed GLE in [8, 6] with some improvement. The non strict inequality in [8] is replaced by a strict inequality whereas the positive of the so called Lyapunov matrix is dropped. It is known that strict inequality conditions are tractable and reliable especially with the available LMI software solvers. The admissibility property includes the stability as well as the regularity and the causality.
The state feedback stabilization problem is solved by means of the admissibility of the closed loop. When the system contains uncertainties, the present method is extended to solve the robust state feedback admissibility problem through a PDL approach.
This paper is organized as follows. Section 2 gives the problem formulation and some preliminary definitions. Section 3 gives the result on admissibility of a discrete descriptor system. In section 4 the main result to solve the static feedback problems for nominal descriptor systems is given, whereas Section 5 presents the result for uncertain singular systems. Section 6 proposes illustrative examples. Section 7 concludes the paper.

Notation

In the rest of the paper the notation is standard unless specified. $L > 0$ ($L < 0$) means that the matrix L is a symmetric and positive definite matrix (a symmetric and negative definite). In the sequel, $\mathsf{Sym}\{\cdot\}$ is defined as $\mathsf{Sym}\{X\} = \left(X + X^\top\right)$ for any matrix X.

2 Problem formulation

Consider the following discrete descriptor system:

$$Ex(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R^m$ is the control input and $t \in I\!N$. The matrix E may be singular, we shall assume that $\mathsf{rank}(E) = r \leq n$. A and B are known real constant matrices with appropriate dimensions.

Definition 1 *[1]*

1. *The pair (E, A) is said to be regular if $det(zE - A)$ is not identically zero.*

2. *The pair (E, A) is said to be causal if $deg(det(zE - A)) = \mathsf{rank}(E)$.*

It is worth noting that the stability property for conventional systems is not sufficient for singular systems but completed by the regularity and the causality, which lead us to introduce the notion of admissibility.

Definition 2 *[1] [5]*
The discrete singular system (1) is said to be admissible if it is regular, causal and stable.

In order to characterize the admissibility of a singular system, let us recall that, for a pair (E, A) it exists a transformation couple (U, V) such as

$$\bar{E} = UEV = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{A} = UAV = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}. \tag{2}$$

According to [1], the singular system is said to be admissible if it exits a symmetric and positive definite matrix $\bar{X}_{11}$ and a non singular matrix $\bar{Y}_{22}$ such as the following conditions

$$\begin{aligned} \left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)^{\top} \bar{X}_{11} \left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right) & \\ -\bar{X}_{11} < 0 & \qquad (3) \\ \mathsf{Sym}\left\{\bar{A}_{22}\bar{Y}_{22}\right\} < 0 & \qquad (4) \end{aligned}$$

hold.

Indeed, condition (4) means that matrix $\bar{A}_{22}$ is non singular whereas condition (3) states that matrix $(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21})$ is stable.

It is worth noting that conditions (3–4) are not tractable for an uncertain system and it is preferable to use the system matrices directly. The associated Lyapunov matrix is not required to be positive definite. The positive will be required only on a fraction of the Lyapunov matrix, that is $\bar{X}_{11} > 0$, in order to satisfy condition (3).

To solve the admissibility problem, we propose a Lyapunov-type admissibility condition, which is expressed by a strict LMI as given in Theorem 3 or Theorem 4.

The goal of this paper is to find a static state feedback controller $u(t) = Kx(t)$ such as the closed loop system $(E, A + BK)$ is admissible. This is defined, in this paper, as the static feedback admissibility problem of descriptor systems. The solvability of the above problems will be characterized by some LMI conditions. If the derived LMI conditions are feasible, the feedback gain matrix can be obtained. If the system contains Poly-topic uncertainties, the results are modified in order to find the static state feedback gain in a way that the closed loop system is admissible for any uncertainty.

3 Admissibility analysis

Consider the singular system described by the pair (E, A) and let $E^{\dagger}$ be defined as

$$E^{\dagger} = U^{-1}\left(I - UEV\right)U$$

where U and V are two non singular matrices satisfying (2).

Theorem 3 below is a reformulation of [8, Lemma 1] in which the non strict inequality is replaced by a strict inequality. Matrix X is not assumed to be positive definite as in [6, Theorem 1] because only the 11-block of $\bar{X} = V^{-1}XV^{-\top}$ is required to be positive definite and this is insured by condition (5).

Theorem 3 *The discrete singular system* (E, A) *is admissible if and only if there exist two matrices* $X = X^\top$ *and* Y *such as*

$$-EXE^\top + \mathsf{Sym}\left\{E^\dagger Y\right\} < 0 \tag{5}$$

and

$$AXA^\top - EXE^\top < 0 \tag{6}$$

Proof of Theorem 3

Sufficiency

First, we will show that the 11-block of $\bar{X} = V^{-1}XV^{-\top}$ is effectively positive definite thanks to condition (5). Indeed, if we pre and post-multiply condition (5) by U and $U^\top$, respectively, we get

$$\begin{aligned} & U\left(-EXE^\top + \mathsf{Sym}\left\{E^\dagger Y\right\}\right)U^\top \\ &= -\bar{E}\bar{X}\bar{E}^\top + \mathsf{Sym}\left\{\bar{E}^\dagger \bar{Y}\right\} \\ &= \begin{bmatrix} -\bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \mathsf{Sym}\left\{\begin{bmatrix} 0 & \bar{Y}_{12} \\ 0 & \bar{Y}_{22} \end{bmatrix}\right\} < 0 \end{aligned}$$

with $\bar{E} = UEV$, $\bar{E}^\dagger = UE^\dagger V$ and $\bar{Y} = UYU^\top$.

In other words, condition (5) implies that the 11-block of $\bar{X}$ is positive definite.

To show that system (1) is regular and causal we have to prove that $\bar{A}_{22}$ is invertible. To do so, let us pre and post-multiply condition (6) by U and $U^\top$ respectively to get

$$\bar{A}\bar{X}\bar{A}^\top - \bar{E}\bar{X}\bar{E}^\top < 0 \tag{7}$$

with $\bar{A}$ defined by (2).

Since $\bar{X}_{11}$ is invertible then $\bar{X}$ can be decomposed as

$$\begin{aligned} \bar{X} = \begin{bmatrix} I & 0 \\ \bar{X}_{21}\bar{X}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12} \end{bmatrix} \\ \times \begin{bmatrix} I & \bar{X}_{11}^{-1}\bar{X}_{12} \\ 0 & I \end{bmatrix}. \end{aligned}$$

Thus condition (7) can be written as follows

$$\begin{aligned} & \bar{A}\bar{X}\bar{A}^\top - \bar{E}\bar{X}\bar{E}^\top \\ &= \bar{A}\begin{bmatrix} I & 0 \\ \bar{X}_{21}\bar{X}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12} \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & \bar{X}_{11}^{-1}\bar{X}_{12} \\ 0 & I \end{bmatrix} \bar{A}^\top - \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} * & \bar{A}_{12} \\ \alpha & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12} \end{bmatrix} \\ &\quad \times \begin{bmatrix} * & \alpha^\top \\ \bar{A}_{12}^\top & \bar{A}_{22}^\top \end{bmatrix} - \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ * & \alpha\bar{X}_{11}\alpha^\top + \bar{A}_{22}\left(\bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12}\right)\bar{A}_{22}^\top \end{bmatrix} < 0 \end{aligned}$$

which implies that $\bar{A}_{22}$ is necessarily invertible since $\bar{X}_{11} > 0$ and this means that system (1) is regular and causal. The entries denoted by a *star* are not very relevant at this step of arguments.

In order to complete the sufficiency part, we have to prove the stability of matrix

$$\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right).$$

Thus, consider the two matrices

$$\Sigma = \begin{bmatrix} I & 0 \\ -\bar{A}_{22}^{-1}\bar{A}_{21} & I \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} I & -\bar{A}_{12}\bar{A}_{22}^{-1} \\ 0 & I \end{bmatrix} \tag{8}$$

that transform matrix $\bar{A}$ in a diagonal form as

$$\bar{\bar{A}} = \Gamma\bar{A}\Sigma = \begin{bmatrix} \bar{\bar{A}}_{11} & 0 \\ 0 & \bar{\bar{A}}_{22} \end{bmatrix} \tag{9}$$

with $\left\{ \begin{array}{lcl} \bar{\bar{A}}_{11} & = & \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}, \\ \bar{\bar{A}}_{22} & = & \bar{A}_{22}, \end{array} \right.$.

Thus, multiplying both sides of (7) respectively by Γ and its transpose yields

$$\bar{\bar{A}}\bar{\bar{X}}\bar{\bar{A}}^\top - \bar{\bar{E}}\bar{\bar{X}}\bar{\bar{E}}^\top < 0 \tag{10}$$

with

$$\begin{aligned} \bar{\bar{E}} &= \Gamma\bar{E}\Sigma = \bar{E}, \\ \bar{\bar{X}} &= \Sigma^{-1}\bar{X}\Sigma^{-\top} = \begin{bmatrix} \bar{X}_{11} & * \\ * & * \end{bmatrix}. \end{aligned}$$

Noting that the 11-block of (10) is in fact

$$\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)\bar{X}_{11}\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)^\top - \bar{X}_{11} < 0$$

confirming the stability of $\bar{\bar{A}}_{11}$ and completing the admissibility of the system.

Necessity
Assume that matrix $\bar{A}_{22}$ is invertible then there exists a negative definite matrix $\bar{\bar{X}}_{22}$ such that

$$\bar{A}_{22}^\top\bar{\bar{X}}_{22}\bar{A}_{22} < 0. \tag{11}$$

Moreover, if matrix $\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)$ is stable then there exists a positive definite matrix $\bar{X}_{11}$ such that (4) holds.
Combining (4) with (11) and denoting

$$\Psi = \left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)^\top\bar{X}_{11}\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)$$

we get

$$\begin{bmatrix} \Psi & 0 \\ 0 & \bar{A}_{22}^\top\bar{\bar{X}}_{22}\bar{A}_{22} \end{bmatrix} - \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} < 0$$

which is in fact condition (10) with

$$\begin{aligned} \bar{\bar{X}} &= \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} \end{bmatrix}, \quad \bar{\bar{E}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{\bar{A}} &= \begin{bmatrix} \left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right) & 0 \\ 0 & \bar{A}_{22} \end{bmatrix} \end{aligned}$$

Hence, proceeding backward from condition (10) to condition (7) as in the sufficiency part we can easily recover backward condition (6). Condition (5) is recovered easily since one can always find a matrix $\bar{Y}_{22}$ such that

$$\begin{bmatrix} -\bar{X}_{11} & 0 \\ 0 & \bar{Y}_{22} + \bar{Y}_{22}^\top \end{bmatrix} < 0$$

which can be written as

$$-\bar{E}\bar{X}\bar{E}^\top + \mathsf{Sym}\left\{\bar{E}^\dagger\bar{Y}\right\} < 0$$

from which one can easily recover condition (5) and this ends the necessity part.

$\nabla\nabla\nabla$

Theorem 3 as stated here, has the following advantage: condition (5) states explicitly that $\bar{X}_{11}$ is positive definite in a strict LMI condition. However this result is not very tractable from a design point of view. To circumvent this design inconvenience, we introduce additional terms in condition (6) as in [6, Theorem 1].

The result stated by Theorem 4 below is an improvement of [6, Theorem 1]. First let $E^\perp$ be defined as

$$E^\perp = V\left(I - UEV\right)U$$

and

$$E^\ddagger = U^\top\left(I - UEV\right)U^{-\top}$$

Note that we have $EE^\perp = 0$ and $E^\top E^\ddagger = 0$.

Theorem 4 *The discrete singular system (E, A) is admissible for a given $\beta \in I\!R$, with $\beta > 1$, if and only if there exist three matrices X, Y and G such that*

$$\begin{aligned} M = &\begin{bmatrix} -EXE^\top & \left(E^\perp Y E^\ddagger\right)^\top \\ E^\perp Y E^\ddagger & X \end{bmatrix} \\ &+\mathsf{Sym}\left\{\begin{bmatrix} A \\ -I \end{bmatrix} G \begin{bmatrix} U^{-\top} & \beta V^\top \end{bmatrix}\right\} < 0 \end{aligned} \tag{12}$$

Proof of Theorem 4 :

By virtue of [19, Theorem 2.3.12] condition (12) is equivalent to

$$\begin{aligned} &\begin{bmatrix} I \\ A^\top \end{bmatrix}^\top \begin{bmatrix} -EXE^\top & \left(E^\perp Y E^\ddagger\right)^\top \\ E^\perp Y E^\ddagger & X \end{bmatrix} \begin{bmatrix} I \\ A^\top \end{bmatrix} \\ &= AXA^\top - EXE^\top + \mathsf{Sym}\left\{AE^\perp Y E^\ddagger\right\} < 0 \end{aligned} \tag{13}$$

$$\begin{aligned} &\begin{bmatrix} \beta U^\top \\ -V^{-\top} \end{bmatrix}^\top \begin{bmatrix} -EXE^\top & \left(E^\perp Y E^\ddagger\right)^\top \\ E^\perp Y E^\ddagger & X \end{bmatrix} \begin{bmatrix} \beta U^\top \\ -V^{-\top} \end{bmatrix} \\ &= -\beta^2\bar{E}\bar{X}\bar{E}^\top - \mathsf{Sym}\left\{\beta\bar{E}^\perp\bar{Y}\bar{E}^\ddagger\right\} + \bar{X} < 0 \end{aligned} \tag{14}$$

with

$$\begin{aligned} \bar{E}^\perp &= V^{-1}E^\perp U^{-1}, & \bar{E} &= UEV, \\ \bar{E}^\ddagger &= U^{-\top}E^\ddagger U^\top, & \bar{X} &= VXV^\top, \\ \bar{Y} &= UYU^\top. \end{aligned}$$

Condition (14) allows to assert that $\bar{X}_{11}$ is positive definite. Indeed, if we expand (14) we get

$$\begin{bmatrix} -\beta^2\bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} - \mathsf{Sym}\left\{\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}\right\} + \begin{bmatrix} \bar{X}_{11} & * \\ * & * \end{bmatrix}$$
$$= \begin{bmatrix} \left(1-\beta^2\right)\bar{X}_{11} & * \\ * & * \end{bmatrix} < 0$$

and this necessarily implies that $\left(1-\beta^2\right)\bar{X}_{11} < 0$ or in other words that $\bar{X}_{11}$ is positive definite since $\beta > 1$.

Condition (13) is similar to that obtained in [6] with the difference that matrix X was assumed to be positive definite. We will show that this sign restriction is not necessary. The regularity and causality are implied by the non singularity of matrix $\bar{A}_{22}$. To show that matrix $\bar{A}_{22}$ is non singular, let us transform condition (13) as

$$\begin{aligned} \bar{M}_0 &= \bar{A}\bar{X}\bar{A}^\top - \bar{E}\bar{X}\bar{E}^\top + \mathsf{Sym}\left\{\bar{A}\bar{E}^\perp\bar{Y}\bar{E}^\ddagger\right\} \\ &= \begin{bmatrix} * & * \\ * & \left(\bar{M}_0\right)_{22} \end{bmatrix} < 0. \end{aligned}$$

with

$$\begin{aligned} \left(\bar{M}_0\right)_{22} = \alpha\bar{X}_{11}\alpha^\top + \\ \bar{A}_{22}\left(\bar{X}_{22} - \bar{X}_{21}\bar{X}_{11}^{-1}\bar{X}_{12}\right)\bar{A}_{22}^\top + \mathsf{Sym}\left\{\bar{A}_{22}\bar{Y}_{22}\right\} \end{aligned}$$

It is obvious that from the condition above we have

$$\left(\bar{M}_0\right)_{22} < 0. \tag{15}$$

Now, assume that $\bar{A}_{22}$ is singular then there exists a non null vector ξ such that $\bar{A}_{22}\xi = 0$. If we multiply both sides of (15) by ξ and its transpose we get $\xi^\top\alpha\bar{X}_{11}\alpha^\top\xi < 0$. This leads to a contradiction since matrix $\bar{X}_{11}$ was proved to be positive definite so matrix $\bar{A}_{22}$ is necessarily invertible.

The stability of the singular system is achieved if matrix $\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)$ is stable, to show this let us consider the product

$$\begin{aligned} \Gamma\bar{M}_0\Gamma^\top &= \bar{\bar{A}}\bar{\bar{X}}\bar{\bar{A}}^\top - \bar{\bar{E}}\bar{\bar{X}}\bar{\bar{E}}^\top + \mathsf{Sym}\left\{\bar{\bar{A}}\bar{\bar{E}}^\perp\bar{\bar{Y}}\bar{\bar{E}}^\ddagger\right\} \\ &= \begin{bmatrix} \hat{A}\bar{X}_{11}\hat{A}^\top - \bar{X}_{11} & * \\ * & * \end{bmatrix} < 0 \end{aligned} \tag{16}$$

with $\hat{A} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$, $\bar{\bar{X}} = \Sigma^{-1}\bar{X}\Sigma^{-\top}$, $\bar{\bar{E}} = \Gamma\bar{E}\Sigma$, $\bar{\bar{E}}^\perp = \Sigma^{-1}\bar{E}^\perp\Sigma^{-\top}$, $\bar{\bar{E}}^\ddagger = \Sigma\bar{E}^\ddagger\Gamma^\top$, $\bar{\bar{Y}} = \Sigma^{-\top}\bar{Y}\Sigma^{-1}$. Matrices Σ, Γ and $\bar{\bar{A}}$ are defined as in (8) and (9) respectively.

It comes then that (16) implies that

$$\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)\bar{X}_{11}\left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right)^\top - \bar{X}_{11} < 0$$

for a positive definite matrix $\bar{X}_{11}$ which, in fact, states that the singular system is stable. Consequentely the singular system is then admissibile.

The converse is true since, if the system is admissible, one can always find X and Y such as (13)-(14) are satisfied which yields (12).

Indeed, assume that $\bar{A}_{22}$ is invertible and $\hat{A} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$ is asymptotically stable, then there exist two invertible matrices $\bar{U}$ and $\bar{V}$ such that [1]

$$E = \bar{U}^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\bar{V}^{-1}, \quad A = \bar{U}^{-1}\begin{bmatrix} \hat{A} & 0 \\ 0 & I \end{bmatrix}\bar{V}^{-1}.$$

Let also $E^{\perp}$ and $E^{\ddagger}$ be as

$$E^{\perp} = \bar{V}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\bar{U}, \quad E^{\ddagger} = \bar{U}^{\top}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\bar{U}^{-\top}.$$

It comes then that for

$$X = \bar{V}^{-1}\begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} \end{bmatrix}\bar{V}^{-\top},$$

and

$$Y = \bar{U}^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \bar{Y}_{22} \end{bmatrix}\bar{U}^{-\top}$$

we have

$$\begin{aligned}
& AXA^{\top} - EXE^{\top} + \mathsf{Sym}\left\{AE^{\perp}YE^{\ddagger}\right\} \\
= \ & \bar{U}^{-1}\left(\begin{bmatrix} \hat{A} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} \end{bmatrix}\begin{bmatrix} \hat{A}^{\top} & 0 \\ 0 & I \end{bmatrix}\right. \\
& \left. - \begin{bmatrix} \bar{X}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \mathsf{Sym}\left\{\begin{bmatrix} \hat{A} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \bar{Y}_{22} \end{bmatrix}\right\}\right)\bar{U}^{-\top} \\
= \ & \bar{U}^{-1}\left(\begin{bmatrix} \hat{A}\bar{X}_{11}\hat{A}^{\top} - \bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} + \bar{Y}_{22} + \bar{Y}_{22}^{\top} \end{bmatrix}\right)\bar{U}^{-\top}
\end{aligned}$$

which asserts that condition (13) holds if

$$\bar{X}_{22} + \bar{Y}_{22} + \bar{Y}_{22}^{\top} < 0 \tag{17}$$

Moreover, for the same X and Y we have

$$\begin{aligned}
& -\beta^2\bar{E}\bar{X}\bar{E}^{\top} - \beta\mathsf{Sym}\left\{\bar{E}^{\perp}\bar{Y}\bar{E}^{\ddagger}\right\} + \bar{X} \\
& = \begin{bmatrix} (1-\beta^2)\bar{X}_{11} & 0 \\ 0 & \bar{X}_{22} - \beta\left(\bar{Y}_{22} + \bar{Y}_{22}^{\top}\right) \end{bmatrix}
\end{aligned}$$

which states that condition (14) holds if

$$\bar{X}_{22} - \beta\left(\bar{Y}_{22} + \bar{Y}_{22}^{\top}\right) < 0 \tag{18}$$

Taking $\bar{X}_{22} = -2I$ and $\bar{Y}_{22} + \bar{Y}_{22}^{\top} = I$ condition (17) becomes

$$\bar{X}_{22} + \bar{Y}_{22} + \bar{Y}_{22}^{\top} = -I < 0$$

and for a given $\beta > 1$ condition (18) becomes

$$\bar{X}_{22} - \beta\left(\bar{Y}_{22} + \bar{Y}_{22}^{\top}\right) = -(2+\beta)I < 0.$$

which means that conditions (13) and (14) also hold. This implies that condition (12) is satisfied. $\nabla\nabla\nabla$

4 Stabilization by state feedback

In this section we address the problem of stabilization by state feedback for the singular system given by

$$Ex(t+1) = Ax(t) + Bu(t) \tag{19}$$

where $x(t) \in I\!R^n$ is the state, $u(t) \in I\!R^m$ is the control input and $t \in I\!N$.

The control law given by a state feedback is then

$$u(t) = Kx(t) \tag{20}$$

where the gain K, of appropriate dimension is computed in a way that the singular closed loop system is admissible.

Theorem 5 *The closed loop singular system is admissible with the state feedback*

$$K = RG^{-1} \tag{21}$$

for a given $\beta \in I\!R$, with $\beta > 1$, if and only if the LMI *problem*

$$\begin{bmatrix} -EXE^\top & \left(E^\perp Y E^\ddagger\right)^\top \\ E^\perp Y E^\ddagger & X \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} AG + BR \\ -G \end{bmatrix} [\,U^{-\top} \quad \beta V^\top\,] \right\} < 0 \tag{22}$$

is feasible in the variables X, Y, R and G.

Proof of Theorem 5

The closed loop singular system is admissible if and only if three matrices exist X, Y, G, and a state feedback K, satisfying

$$\begin{bmatrix} -EXE^\top & \left(E^\perp Y E^\ddagger\right)^\top \\ E^\perp Y E^\ddagger & X \end{bmatrix} + \mathsf{Sym}\left\{ \begin{bmatrix} A + BK \\ -I \end{bmatrix} G\,[\,U^{-\top} \quad \beta V^\top\,] \right\} < 0 \tag{23}$$

for a given $\beta > 1$. Thus, it becomes clear that a change of variable $R = KG$ yields condition (22). $\nabla\nabla\nabla$

From Theorem 5 we can easily be aware that this approach has the advantage of being nicely applicable in the case the system is uncertain with a poly-topic uncertainty description, which is the subject of the next section.

5 Robust Stabilisation

In this section, we consider the feedback stabilization problem for systems containing uncertainties. The descriptor system is characterized by the pair $(E,\ A(\bar\alpha), B(\bar\alpha))$ where matrices $A(\bar\alpha)$ and $B(\bar\alpha)$ belong to a poly-tope, that is

$$[\,A(\bar\alpha) \quad B(\bar\alpha)\,] = \sum_{i=1}^{p} \alpha_i\,[\,A_i \quad B_i\,]$$

with

$$\alpha_i \geq 0, \; i = 1, \; \dots, \; p$$
$$\sum_{i=1}^{p} \alpha_i = 1 \qquad \text{and} \qquad \bar{\alpha} = [\, \alpha_1 \quad \dots \quad \alpha_p \,]$$

The problem for a system corrupted by uncertainty is to preserve its performances for all admissible uncertainties or in other words for every instance of matrices $A(\bar{\alpha})$ and $B(\bar{\alpha})$.

Definition 6 *The uncertain singular system is robustly admissible if and only if it is admissible for every instance of the uncertainty $\bar{\alpha}$.*

Thus, the uncertain system is admissible if and only if it is regular, causal and the eigenvalues of pencil matrix $(E, A(\bar{\alpha}))$ lie in the open unity disc centered at the origin of the complex plan for all admissible uncertainty $\bar{\alpha}$. In other words, for every instance of the uncertainty $\bar{\alpha}$ there exist $\bar{X}_{11}(\bar{\alpha})$ and $\bar{Y}_{22}(\bar{\alpha})$

$$\begin{array}{l} \left(\bar{A}_{11}(\bar{\alpha}) - \bar{A}_{12}(\bar{\alpha})\bar{A}_{22}^{-1}(\bar{\alpha})\bar{A}_{21}(\bar{\alpha})\right)\bar{X}_{11}(\bar{\alpha}) \\ \qquad \times \left(\bar{A}_{11}(\bar{\alpha}) - \bar{A}_{12}(\bar{\alpha})\bar{A}_{22}^{-1}(\bar{\alpha})\bar{A}_{21}(\bar{\alpha})\right)^{\top} - \bar{X}_{11}(\bar{\alpha}) < 0 \\ \qquad \mathsf{Sym}\left\{\bar{A}_{22}(\bar{\alpha})\bar{Y}_{22}(\bar{\alpha})\right\} < 0 \end{array} \tag{24}$$

and the system will be termed as robustly admissible. However if the matrices $\bar{X}_{11}(\bar{\alpha})$ and $\bar{Y}_{22}(\bar{\alpha})$ are unique over the set of the uncertainty or equivalently do not depend on the uncertainty, that is to say if we have

$$\begin{array}{l} \left(\bar{A}_{11}(\bar{\alpha}) - \bar{A}_{12}(\bar{\alpha})\bar{A}_{22}^{-1}(\bar{\alpha})\bar{A}_{21}(\bar{\alpha})\right)\bar{X}_{11} \\ \qquad \times \left(\bar{A}_{11}(\bar{\alpha}) - \bar{A}_{12}(\bar{\alpha})\bar{A}_{22}^{-1}(\bar{\alpha})\bar{A}_{21}(\bar{\alpha})\right)^{\top} - \bar{X}_{11} < 0 \\ \qquad \mathsf{Sym}\left\{\bar{A}_{22}(\bar{\alpha})\bar{Y}_{22}\right\} < 0, \end{array} \tag{25}$$

then the system will be termed as quadratic admissible.

Hence, the quadratic admissibility implies robust admissibility but in general the converse is false. Despite the inherent conservativeness of the quadratic admissibility with respect to the robust admissibility, solving condition (25) can be carried out more easily than (24). In this paper the use of the PDL approach allows to circumvent this difficulty and to give a nice solution to the robust admissibility problem. This is stated by Theorem 7 below.

Theorem 7 *The uncertain singular system is robustly admissible for a given $\beta \in \mathbb{R}$, with $\beta > 1$, if there exist matrices X_i, Y_i, for $i = 1, \; \dots, \; p$, and two matrices G and R such that the* LMI *problem*

$$\begin{array}{l} \begin{bmatrix} -EX_iE^{\top} & \left(E^{\perp}Y_iE^{\ddagger}\right)^{\top} \\ E^{\perp}Y_iE^{\ddagger} & X_i \end{bmatrix} \\ \qquad + \mathsf{Sym}\left\{ \begin{bmatrix} A_iG + B_iR \\ -G \end{bmatrix} [\, U^{-\top} \quad \beta V^{\top} \,] \right\} < 0 \end{array} \tag{26}$$

is feasible in the variables X_i, Y_i, G and R then the admissible feedback gain is given by

$$K = RG^{-1}$$

Proof of Theorem 7 :

Let matrices X and Y be defined as

$$[\, X \quad Y \,] \;=\; \sum_{i=1}^{p} \alpha_i \, [\, X_i \quad Y_i \,]$$

then multiplying conditions (26) by α_i and summing up from 1 to p one gets the same conditions as in Theorem 5. $\nabla\nabla\nabla$

6 Illustrative examples

Consider the same data as in [20, Example 3] :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -3.2000 & -1.6000 & 0 \\ 0 & 0 & -1.0000 \\ 0.8000 & 2.4000 & 0 \end{bmatrix}$$

The finite poles are located at 2.9612 and -2.1612, which implies that the open loop is not admissible.

Using the Matlab LMI solver for Theorem 5 with $\beta = 5$ we obtain the feedback gain :

$$K = RG^{-1} = \begin{bmatrix} 2.5043 & 1.3380 & 0.1945 \\ -0.5231 & -2.0536 & -0.1075 \end{bmatrix}$$

that renders the resulting closed loop system admissible with the finite closed loop poles located at -0.0766 and 0.1544 and the $\Big(U(A + BK)V\Big)_{22} = -0.8055$, which confirms the admissibility of the closed loop system.

Moreover, the eigenvalues of X are 0.3613×10^8, 0.3613×10^8 and -9.7480×10^8, which comfirms the non necessity of the positivity of X. In addition, note that the 11-block of $\bar{X}$ is given by

$$\bar{X}_{11} = \begin{bmatrix} 3.6129 & -0.0000 \\ -0.0000 & 3.6129 \end{bmatrix} \times 10^7$$

and we can easily check that $\bar{X}_{11}$ is positive definite.

In this example, we consider the data from [20, Example 3] but with a slight modification. Matrix A is given by:

$$A = \begin{bmatrix} -3.2 & -1.6 & 0.0 \\ 0.0 & -1.0 & 0.0 \\ 0.8 & 2.4 & 0.0 \end{bmatrix}$$

and the system is no longer regular nor causal since the 22-block of UAV is no longer invertible.

Application of Theorem 5 gives the state feedback gain

$$K = \begin{bmatrix} 2.5545 & 1.4846 & -0.0261 \\ 1.4129 & -2.0274 & 0.1310 \end{bmatrix}$$

and

$$\bar{A}_c = U\left(A + BK\right)V = \begin{bmatrix} -0.6455 & -0.1154 & -0.0261 \\ 2.2129 & 0.3726 & 0.1310 \\ 2.5545 & 0.4846 & -0.0261 \end{bmatrix}$$

Note that the 22-block of $U(A + BK)V$ is invertible and the finite poles are -0.1987 and -0.0732, which states the admissibility of the closed loop.

Consider the uncertain system defined by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with matrices A and B belonging to a polytope whose vertices are given by :
the state matrices

$$\begin{aligned} &[\,A_1 \quad A_2 \quad A_3\,] \\ &= \left[\begin{array}{rrr|rrr|rrr} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$[\,B_1 \quad B_2 \quad B_3\,] = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

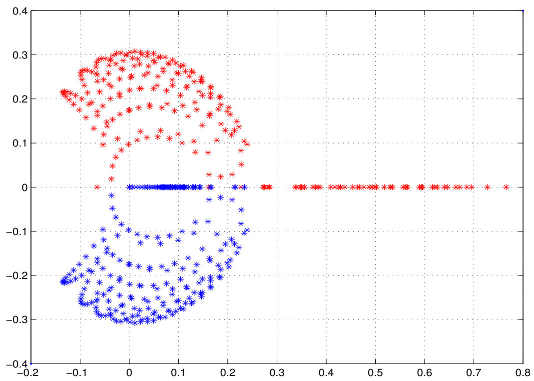
Applying Theorem 7 with $\beta = 5$ and

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

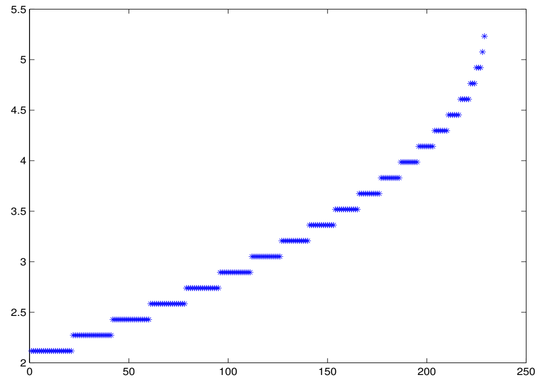
and $V = I$, we get the robust state feedback gain

$$K = RG^{-1} = [\,0.0695 \quad 0.8593 \quad -2.1160\,]$$

that renders the uncertain closed loop system robustly admissible.



Finite given values location of the uncertain closed loop system over the considered poly-tope



$\det\{(\overline{A+BK})_{22}\}$ of the uncertain closed loop system over the considered polytope

7 Conclusion

The problem of admissibility and state feedback admissibilisation for discrete descriptor systems, have been addressed. In terms of a strict LMI, a necessary and sufficient condition for discrete descriptor systems to be admissible has been proposed. This condition is a LMI twin formulation of the well-known, and improved generalized Lyapunov's equation. LMI conditions are obtained to ensure the admissibility of the closed loop via a state feedback control law. The use of slack additional variables allows us to give a nice LMI condition to design an admissible state feedback gain. A robust admissible state feedback control law is proposed for polytopic uncertain discrete descriptor systems. Numerical examples are given to illustrate the usefulness of the proposed methods.

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