On the KYP lemma, the hybrid Roesser models, and the matrix $\partial\mathcal{D}$-regularity

Olivier Bachelier[a], Olivier.Bachelier@univ-poitiers.fr
Driss Mehdi[a], Driss.Mehdi@univ-poitiers.fr

[a]Laii-Esip, Bâtiment de Mécanique, 40 avenue du recteur Pineau, 86022 Poitiers Cedex, France

Version 2

Abstract

In this paper, the multidimensional systems described by Roesser models that possibly involve continuous and discrete dynamics are considered through the lens of the famous Kalman-Yakubovich-Popov (KYP) lemma.

Index Terms

Hybrid Roesser nD-models, KYP lemma, S-procedure, Polynomial matrix $\partial\mathcal{D}$-rgularity, LMI.

1 Introduction

A large attention has been paid to the study of multidimensional systems (or nD-systems) in the past decades [1–4]. Many industrial domains can actually take benefit of such a theory [1,5–9]. Among the now large amount of contributions devoted to nD-systems, one can notice a strong focus on the so-called repetitive processes [10, 11] which can be handled as a special class of 2D-systems [12, 13].

As for conventional 1D-systems, an important matter in the realm of nD-systems is the stability analysis and the derivation of numerically tractable stability tests. A possible approach is to adapt Lyapunov's second method [14] or Stein's result [15] to nD-models. Such an extension has been investigated in [16–18] where some modified Lyapunov equalites are proposed. If inequalities are considered rather than equalities, the LMI machinery (LMI : Linear Matrix Inequality [19]), can be exploited as in [20–22] and is particularly useful in the uncertain context. It has to be mentioned that all the proposed conditions, when tractable, are conservative.

However, the most known conditions relevant to the stability of nD-models are expressed in terms of characterstic polynomial root-clustering [23–25]. In this framework, it becomes of interest to derive theoretical results on polynomial matrices [26, 27]. Once again, LMI have turned to be a good tool to tackle some problems induced by polynomial matrices, especially $\mathcal{D}$-stability analysis [28, 29]. The relation between, on the one hand, the polynomial approach, and, on the other hand, the LMI machinery, has also been studied in [30]. This paper lies in this framework.

The connection between the polynomial, or, in a more restrictive way, frequency approach, and the matrix inequalities is quite well established for 1D-linear models. The strongest result is the celebrated Kalman-Yakubovich-Popov (KYP) lemma [31–35] which gives equivalences between crucial frequency domain inequalities and LMI. Since a usual way of approaching the nD-models is to extend the tools that have proven their utility for 1D-models, a natural question is to wonder whether it is possible or not to provide a version of the KYP lemma which can be exploited in the realm of nD-systems. Few contributions exist and only special instances of the KYP lemma have been implicitely adressed for 2D-models [36, 37] but no real general result has been provided yet. The present contribution aims at filling this gap.

The paper is organized as follows. After this introduction, Section 2 states many prelimiaries. It recalls the Roesser models [38] and their various extensions. Those models are very popular to describe the behaviour of multidimensional systems. Besides, it extends some descriptions (encountered in the literature) of regions of the complex plane and defines the notion of $\partial\mathcal{D}$-regularity for multivariate matrix functions. Section 3 states the main result which is some extension of the the generalized KYP lemma. In Section 4, this lemma is applied to analyze the behaviour of the Roesser models introduced in Section 2, before the paper is concluded by section 5.

Notations

M' denotes the transpose conjugate of matrix M. Hence, λ' is the conjugate of complex number λ. $||M||_2$ is the matrix 2-norm (the maximum singular value) induced by the Euclidean vector norm. I_n is the identity matrix of dimension n and I (resp. 0) is the identity (resp. a null) matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Löwner *i.e.* > 0 (resp. < 0) means positive (resp. negative) definite and ≥ 0 (resp. ≤ 0) means positive (resp. negative) semi-definite. The notation $\mathcal{H}_n$ stands for the set of Hermitian matrices of dimension n. $\mathcal{H}_n^+ \subset \mathcal{H}_n$ is the the subset of positive definite matrices and $\mathcal{H}_n^- \subset \mathcal{H}_n$ that of negative definite matrices. Also let the following notations be defined:

$$\overset{k}{\underset{i=1}{\oplus}} M_i = \underset{i=1,\ldots k}{\text{diag}} \{M_i\} \tag{1}$$

For a given subset $\mathcal{S}$ of the set $\mathcal{X}$, the set $\mathcal{S}^C$ is the complementary set of $\mathcal{S}$ such that $\mathcal{S} \cup \mathcal{S}^C = \mathcal{X} \,\&\, \mathcal{S} \cap \mathcal{S}^C = \emptyset$. At last, symbol $\otimes$ denotes the matrix Kronecker product [39].

2 Preliminaries

This section introduces some preliminaries, some of them being borrowed from the literature. The first part is dedicated to the presentation of the Roesser models and to some associated stability conditions. The second provides the description of a class of regions of the "multidimensional" complex plane which will be used in the next section. The third part proposes the notion of $\partial\mathcal{D}$-regularity of a multivariate matrix function, inspired from the concept of matrix $\partial\mathcal{D}$-regularity.

2.1 Hybrid Roesser models

The Roesser models were originally introduced in [38] and provide a quite general description of the behaviour of discrete-time multidimensional systems. However, it is possible to adapt these models to the continuous-time case, and even to systems involving both continuous and discrete dynamics. Such *hybrid* models were first considered in [5]. The contribution [40] highlights the relevance of such 2D hybrid models to handle the so-called *differential repetitive processes*. But a quite general hybrid Roesser model was given in [41]. We here propose a slightly extended model.

$$\left\{\begin{array}{rcl} E\left[\begin{array}{c} \frac{\partial}{\partial t_1}x^1(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \vdots \\ \frac{\partial}{\partial t_r}x^r(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \hline x^{r+1}(t_1,\dots,t_r,j_{r+1}+1,\dots,j_k) \\ \vdots \\ x^k(t_1,\dots,t_r,j_{r+1},\dots,j_k+1) \end{array}\right] & = & \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c} x^1(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \vdots \\ x^r(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \hline x^{r+1}(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \vdots \\ x^k(t_1,\dots,t_r,j_{r+1},\dots,j_k) \end{array}\right] + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right]u(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ y(t_1,\dots,t_r,j_{r+1},\dots,j_k) & = & \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right]\left[\begin{array}{c} x^1(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \vdots \\ x^r(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \hline x^{r+1}(t_1,\dots,t_r,j_{r+1},\dots,j_k) \\ \vdots \\ x^k(t_1,\dots,t_r,j_{r+1},\dots,j_k) \end{array}\right] + Du(t_1,\dots,t_r,j_{r+1},\dots,j_k) \end{array}\right. \tag{2}$$

where $\sum_{i=1}^{k} n_i = n$.

The vectors $x'^i(t_1,\dots,t_r,j_{r+1},\dots,j_k) \in \mathbb{R}^{n_i}$, $i = 1,..,k$, $u(t_1,\dots,t_r,j_{r+1},\dots,j_k) \in \mathbb{R}^m$ and $y(t_1,\dots,t_r,j_{r+1},\dots,j_k) \in \mathbb{R}^p$ are the local state subvectors, the input vector and the ouput vector respectively. Matrices

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \in \mathbb{R}^{n\times n},\ B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \in R^{n\times m},$$

$$C = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] \in \mathbb{R}^{p\times n}, \quad \text{and} \quad D \in R^{p\times n}$$

are the state, control, observation and direct transfer matrices respectively. Matrix $E \in \mathbb{R}^{n\times n}$ is introduced to enable the description of some singularities in order to extend the notion of 1D-linear descriptor systems. It is quite clear that this model describes continuous dynamics along r dimensions and discrete dynamics along $(k-r)$ dimensions. If E equals I (no singularity in the dynamics), then the model of [41] is recovered. If, furthermore, $r = 0$, then it reduces to the classical Roesser model [38] whereas if $r = k$ then one gets its continuous counterpart. At last, if $r = 1$ and $k = 2$ (still with $E = I$), then the obtained 2D-model is particularly suitable to describe the *differential repetitive processes* [40]. Note that a "frequency" caracterization of this system, based upon the Laplace transform and the $\mathcal{Z}$-transform, is given by

$$Y(\lambda) = G(\lambda)U(\lambda), \tag{3}$$

where

$$G(\lambda) = C(EH(\lambda) - A)^{-1}B + D, \quad H(\lambda) := \overset{k}{\underset{i=1}{\oplus}} \lambda_i I_{n_i}. \tag{4}$$

Now let the function $c(\lambda, E, A)$ be defined by

$$c(\lambda, E, A) := \det(EH(\lambda) - A) \tag{5}$$

The reader is reminded of a stability condition in the next lemma, which is just a reformulation of a proposition in [41].

Lemma 1 *The system (2) with $E = I_n$ is asymptotically stable if and only if*

$$c(\lambda, I, A) \neq 0 \quad \forall \lambda \in \mathcal{S}^C, \tag{6}$$

where $c(.,.,.)$ is defined by (5) and where $\mathcal{S}^C$, the complementary set of the "asymptotic stability region", is defined by

$$\mathcal{S}^C = \left\{ \lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} \in \mathbb{C}^k : \mathrm{Re}(\lambda_i) \geq 0, i = 1, ..., r \ \& \ |\lambda_i| \geq 1, i = r+1, ..., k \right\} \tag{7}$$

If $r = 0$, then the condition for the stability of a discrete Roesser model proposed in [24,25] is recovered. If $r = k$, it reduces to the result of [23].

Remark 2 *The reader may have noticed that matrix E has not been taken into account in Lemma 1. The reason is that the stability of singular nD-systems has not yet been clearly connected to the notion of root-clustering altough some extended conditions could seem natural. Besides, the asymptotic stability of descriptor systems is not always sufficient to ensure some satisfactory behaviour.* Regularity and impulse-free responses *are also required [42]. Few papers address both multdimensional and singular systems. Let however reference [43] be quoted.*

In Lemma 1, the stability condition is expressed in terms of polynomial root-clustering. This is why in the next part, in order to embed such known results in a more general framework, a formulation of clustering region is proposed.

2.2 Point-clustering

Consider the following matrices :

$$\left\{ \begin{array}{l} R_i := \begin{bmatrix} R_{i_{11}} & R'_{i_{10}} \\ R_{i_{10}} & R_{i_{00}} \end{bmatrix} \in \mathbb{C}^{2d \times 2d}, \ R_{i_{00}} \in \mathbb{C}^{d \times d} \\ \\ \Phi_{i_l} := \begin{bmatrix} \phi_{i_{l_{11}}} & \phi'_{i_{l_{10}}} \\ \phi_{i_{l_{10}}} & \phi_{i_{l_{00}}} \end{bmatrix} \in \mathbb{C}^{2d \times 2d}, \ \phi_{i_{l_{00}}} \in \mathbb{C}^{d \times d} \, \forall l = 1, \ldots, \bar{l}_i \end{array} \right. \quad \forall i \in \{1, \ldots, k\} \tag{8}$$

Let the sets $\partial\mathcal{D}_i$ be described by

$$\partial\mathcal{D}_i := \{s \in \mathbb{C} \;:\; \mathcal{F}_{R_i}(s) = 0, \mathcal{G}_{\Phi_{i_l}} \geq 0 \; \forall l \in \{1,\ldots,\bar{l}_i\}\} \;\; \forall i \in \{1,\ldots,k\}\} \tag{9}$$

where the functions $\mathcal{F}_{R_i}(s)$ and $\mathcal{G}_{\Phi_{i_l}}$ are defined by

$$\left\{\begin{array}{l} \mathcal{F}_{R_i}(s) := \left[\begin{array}{c} sI \\ I \end{array}\right]' R_i \left[\begin{array}{c} sI \\ I \end{array}\right] \\ \\ \mathcal{G}\Phi_{i_l}(s) = \left[\begin{array}{c} sI \\ I \end{array}\right]' \Phi_{i_l} \left[\begin{array}{c} sI \\ I \end{array}\right] \; \forall l = 1,\ldots,\bar{l}_i \end{array}\right. \qquad \forall i \in \{1,\ldots,k\} \tag{10}$$

respectively. This is an extension of the sets considered in [44] to the case $d \geq 1$. When $d = 1$, the only equalities $\mathcal{F}_{R_i}(s) = 0$ that are meaningful correspond to lines or circles and the inequalities $\mathcal{G}_{\Phi_{i_l}} \geq 0$, describing discs or exteriors of discs, are introduced to select only parts of these lines and circles. When $d > 1$, the meaning of the equalities is far less obvious but the inequalities can characterize, for instance, Ellispoidal Matrix Inequality (EMI) regions as defined in [45], or exteriors of such regions. Now define the "k-region" $\partial\mathcal{D}$ as

$$\partial\mathcal{D} := \partial\mathcal{D}_1 \times \partial\mathcal{D}_2 \times \ldots \times \partial\mathcal{D}_k. \tag{11}$$

Without loss of generality, it is assumed that $\bar{l}_i = \bar{l}$, $\forall i \in \{1,\ldots,k\}$. Besides, to avoid heavy notations in the sequel, the sets of indices are now denoted as follows:

$$I(q) := \{1,\ldots,q\}, \;\; q \in \mathbb{N}. \tag{12}$$

Let λ be a complex vector that can be written

$$\lambda := \left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_k \end{array}\right] \in \mathbb{C}^k. \tag{13}$$

Also let the following matrices be defined:

$$\Lambda := \overset{k}{\underset{i=1}{\oplus}} \lambda_i; \tag{14}$$

$$\mathbf{R} := \left[\begin{array}{cc} \overset{k}{\underset{i=1}{\oplus}} R_{i_{11}} & \overset{k}{\underset{i=1}{\oplus}} R'_{i_{10}} \\ \overset{k}{\underset{i=1}{\oplus}} R_{i_{10}} & \overset{k}{\underset{i=1}{\oplus}} R_{i_{00}} \end{array}\right]; \tag{15}$$

$$\mathbf{\Phi}_l := \left[\begin{array}{cc} \overset{k}{\underset{i=1}{\oplus}} \phi_{i_{l_{11}}} & \overset{k}{\underset{i=1}{\oplus}} \phi'_{i_{l_{10}}} \\ \overset{k}{\underset{i=1}{\oplus}} \phi_{i_{l_{10}}} & \overset{k}{\underset{i=1}{\oplus}} \phi_{i_{l_{00}}} \end{array}\right] \;\; \forall l \in I(\bar{l}). \tag{16}$$

It is very easy to check that

$$\lambda \in \partial\mathcal{D} \Leftrightarrow \begin{cases} \left(\begin{bmatrix} \Lambda \\ I_k \end{bmatrix} \otimes I_d\right)' \mathbf{R} \left(\begin{bmatrix} \Lambda \\ I_k \end{bmatrix} \otimes I_d\right) = 0 \\ \left(\begin{bmatrix} \Lambda \\ I_k \end{bmatrix} \otimes I_d\right)' \mathbf{\Phi}_l \left(\begin{bmatrix} \Lambda \\ I_k \end{bmatrix} \otimes I_d\right) \geq 0 \quad \forall l \in I(\bar{l}). \end{cases} \tag{17}$$

The next lemma will be useful in the sequel.

Lemma 3 *Let a k-region $\partial\mathcal{D}$ be defined as in (11) with (8-10) and $\lambda \in \mathbb{C}^k$ comply with (13). The two following statements are equivalent.*

i)

$$\lambda \in \partial\mathcal{D} \tag{18}$$

ii)

$$\left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \otimes I_d\right) \Upsilon(\mathcal{P},\mathcal{Q}) \left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix} \otimes I_d\right) \geq 0 \ \forall\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H} \tag{19}$$

where $\mathbb{H} = \{\mathcal{H}_{n_1} \times \ldots \times \mathcal{H}_{n_k}\} \times \{(\mathcal{H}^+_{n_1})^{\bar{l}} \times \ldots \times (\mathcal{H}^+_{n_k})^{\bar{l}}\}$, $H(\lambda)$ is defined in (4) and $\Upsilon(\mathcal{P},\mathcal{Q})$ stands for

$$\Upsilon(\mathcal{P},\mathcal{Q}) := \overline{\mathbf{R}}(\mathcal{P}) + \sum_{l=1}^{\bar{l}} \overline{\mathbf{\Phi}}_l(\mathcal{Q}), \tag{20}$$

with

$$\overline{\mathbf{R}}(\mathcal{P}) := \left[\begin{array}{c|c} \overset{k}{\underset{i=1}{\oplus}} (P_i \otimes R_{i_{11}}) & \overset{k}{\underset{i=1}{\oplus}} (P_i \otimes R'_{i_{10}}) \\ \hline \overset{k}{\underset{i=1}{\oplus}} (P_i \otimes R_{i_{10}}) & \overset{k}{\underset{i=1}{\oplus}} (P_i \otimes R_{i_{00}}) \end{array}\right], \tag{21}$$

$$\overline{\mathbf{\Phi}}_l(\mathcal{Q}) := \left[\begin{array}{c|c} \overset{k}{\underset{i=1}{\oplus}} (Q_{i_l} \otimes \Phi_{i_{l_{11}}}) & \overset{k}{\underset{i=1}{\oplus}} (Q_{i_l} \otimes \Phi'_{i_{l_{10}}}) \\ \hline \overset{k}{\underset{i=1}{\oplus}} (Q_{i_l} \otimes \Phi_{i_{l_{10}}}) & \overset{k}{\underset{i=1}{\oplus}} (Q_{i_l} \otimes \Phi_{i_{l_{00}}}) \end{array}\right] \ \forall l \in I(\bar{l}), \tag{22}$$

and with $P_{n_i} \in \mathcal{H}_{n_i}$ and $Q_{i_l} \in \mathcal{H}^+_{n_i}$, $\forall\{i;l\} \in I(k) \times I(\bar{l})$.

Proof. **i)** $\Rightarrow$ **ii)** : if $\lambda \in \partial\mathcal{D}$ then the equality in (17) holds or equivalently

$$\overset{k}{\underset{i=1}{\oplus}} \mathcal{F}_{R_i}(\lambda_i) = 0, \tag{23}$$

leading to

$$\overset{k}{\underset{i=1}{\oplus}} \mathcal{F}_{R_i}(\lambda_i) \otimes P_i = 0, \tag{24}$$

with considering any set $\mathcal{P}$ of k matrices $P_i \in \mathcal{H}_{n_i}$, $i \in I(k)$. By permutation on the rows ond the columns of (24), one gets

$$\left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \otimes I_d\right) \overline{\mathbf{R}}(\mathcal{P}) \left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix} \otimes I_d\right) = 0. \tag{25}$$

Now, consider the inequality in (17) or equivalently

$$\begin{bmatrix} H(\lambda) \\ I_{dn} \end{bmatrix} \overline{\mathbf{\Phi}}_l(\{I_{n_i}\}) \begin{bmatrix} H(\lambda) \\ I_{dn} \end{bmatrix} \geq 0 \ \forall l \in \mathbb{I}(\bar{l}). \tag{26}$$

Consider any set $\mathcal{Q}$ of matrices $Q_{i_l} \in \mathcal{H}^+_{n_i}$. Then the matrices

$$\overline{Q}_l := \overset{k}{\underset{i=1}{\oplus}} (Q_{i_l}^{1/2} \otimes I_d) \tag{27}$$

belong to $\mathcal{H}_{nd}$. Thus, left and right-multiply (26) by $\overline{Q}_l$, one gets

$$\left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix}' \otimes I_d \right) \overline{\mathbf{\Phi}}_l(\mathcal{Q}) \left(\begin{bmatrix} H(\lambda) \\ I \end{bmatrix} \otimes I_d \right) \geq 0, \quad \forall l \in I(\bar{l}). \tag{28}$$

Adding the equality in (25) and the inequalities in (28) gives (19).
ii) $\Rightarrow$ **i**): Assume that (19) holds. Let a matrix Z comply with

$$Z = \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} \otimes I_d \ \text{ with } \ z_i \in \mathbb{C}^{n_i}, \forall i \in I(k). \tag{29}$$

Left and right multiplying (19) by Z' and Z respectively yields

$$\sum_{i=1}^{k} ((z_i' P_i z_i) \mathcal{F}_{R_i}(\lambda_i)) + \sum_{i=1}^{k} \sum_{l=1}^{\bar{l}} (((z_i' Q_{i_l} z_i) \mathcal{G}_{\Phi i_l}(\lambda_i)) \geq 0 \tag{30}$$

The above inequalities hold for *any* set $\mathcal{P}$ and *any* set $\mathcal{Q}$.Thus, if each Q_{i_l} equals βI_{n_i} with β as small as possible, making the second term in (30) tend to zero and if, moreover, each P_i equals βI_{n_i} except one P_h equaling αI_{n_h} with α as large as possible, then, to keep (30) true whatever α and β may be, it is required that $\mathcal{F}_{R_h}(\lambda_h) \geq 0$. Considering the same set $\mathcal{Q}$ and the same set $\mathcal{P}$ except $P_h = -\alpha I_{n_h}$, then it leads to $\mathcal{F}_{R_h}(\lambda_h) \leq 0$. Hence, it comes that $\mathcal{F}_{R_h}(\lambda_h) = 0$. The reasoning followed for h can also be followed for any i in $I(k)$ and

$$\mathcal{F}_{R_i}(\lambda_i) = 0 \ \forall i \in I(k), \tag{31}$$

which makes the first term in (30) equal zero.
In the same way, consider matrices $Q_{i_l} = \beta I_{n_i}$ except for a couple of indices $\{h; j\} \in I(k) \times I(\bar{l})$ for which $Q_{h_j} = \alpha I_{n_h}$, then $\mathcal{G}_{\Phi_{h_j}}(\lambda_h) \geq 0$ has to hold otherwise it would contradict (30). This reasoning on $\{h; j\}$ can be followed for any couple $\{i, l\} \in I(k) \times I(\bar{l})$ so one gets

$$\mathcal{G}_{\Phi_{i_l}}(\lambda_i) \geq 0 \ \forall \{i, l\} \in I(k) \times I(\bar{l}). \tag{32}$$

From (31) and (32), it is clear that $\lambda \in \partial\mathcal{D}$. $\square$

2.3 Multivariate matrix $\partial\mathcal{D}$-regularity

To conclude this section, the definition of the $\partial\mathcal{D}$-regularity of a multivariate matrix function is proposed, accompanied by a short discussion.

Definition 4 *Let $\partial\mathcal{D}$ be a subset of $\mathbb{C}^k$ and $\lambda \in \mathbb{C}^k$ comply with (13). Also consider a kD-variate matrix function $\mathbf{A}(\lambda)$. Then, $\mathbf{A}(\lambda)$ is said to be*

- *$\partial\mathcal{D}$-regular if $\{\lambda \in \mathbb{C}^k : \det(\mathbf{A}(\lambda)) = 0\} \cap \partial\mathcal{D} = \emptyset$;*
- *$\partial\mathcal{D}$-singular if $\{\lambda \in \mathbb{C}^k : \det(\mathbf{A}(\lambda)) = 0\} \cap \partial\mathcal{D} \neq \emptyset$.*

In section 4, a particular attention will be paid to the kD-polynomial matrix of the form

$$\mathbf{A}(\lambda) := EH(\lambda) - A, \tag{33}$$

since, in this case, $\det(\mathbf{A}(\lambda)) = c(\lambda, E, A)$ corresponds to the characteristic polynomial of Roesser model (2). If $E = I$, from Lemma 1, it can be seen that the asymptotic stability of (2) is a special instance of the $\partial\mathcal{D}$-regularity of $\mathbf{A}(\lambda)$ as defined in (33). More precisely, for (2) with $E = I$ to be asymptotically stable, it is *necessary* that $\mathbf{A}(\lambda)$ be $\partial\mathcal{D}$-regular with the choice

$$\left\{\begin{array}{ll} R_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & \forall i \in \{1, \ldots, r\}, \\[2ex] R_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \forall i \in \{r+1, \ldots, k\}, \\[2ex] \Phi_{i_l} = 0 & \forall \{i, l\} \in I(k) \times I(\bar{l}). \end{array}\right. \tag{34}$$

In such a case, the difference between $\partial\mathcal{D}$-regularity and asymptotic stability lies in the distribution of the roots of $c(\lambda, I, A)$ with respect to $\partial\mathcal{D}$. More generally, $\mathcal{D}$-stability requires $\partial\mathcal{D}$-regularity when $\partial\mathcal{D}$ is the boundary of the region $\mathcal{D}$.

For the conventional one-dimensional case ($r = k = 1$ or $k = k - r = 1$), then $\mathbf{A}(\lambda)$ can be written $\mathbf{A}(\lambda) = (\lambda E - A)$ and, if $E = I$, Definition 4 becomes equivalent to the definition of the $\partial\mathcal{D}$-regularity of matrix A proposed in [44, 46]. If $E \neq I$, then the definition in [44, 46] can be directly extended to the notion of $\partial\mathcal{D}$-regularity of the pencil (E, A).

3 A version of the KYP lemma

This section is devoted to the derivation of the main result which is some sort of kD-version of the celebrated KYP-lemma. This version will only provide a "sufficient" condition but the necessity will be briefly discussed with the help of the so-called generalized S-procedure. A large reference is made to the seminal paper [35] where generalized KYP lemma for 1D-linear models and the generalized S-procedure are presented in a very elegant way.

Before stating the result, the following matrices and sets are introduced. Note that various notations are borrowed from [35]. Consider some matrix $F \in \mathbb{C}^{(n+m)\times(n+m)}$ and let another matrix Θ belong to $\mathcal{H}_{n+m}$.
Let the set $\mathcal{P}$ be made up by k matrices $P_i \in \mathcal{H}_{n_i}$, $\forall i \in I(k)$ and let the set $\mathcal{Q}$ be made up by $k\bar{l}$ matrices $Q_{i_l} \in \mathcal{H}^+_{n_i}$, $\forall \{i; l\} \in I(k) \times I(\bar{l})$. With each couple of sets $\{\mathcal{P}; \mathcal{Q}\}$ and with a set $\partial\mathcal{D}$ given by (11) with (8-10), one can associate a matrix $\Upsilon(\mathcal{P}, \mathcal{Q})$, as in Lemma 3, defined by (20) with (21-22).

The set $\mathbf{M}$ is the set of all complex matrices M associated with F and Υ:

$$\mathbf{M} := \{M \in \mathcal{H}_{d(n+m)} : M = (F \otimes I_d)'\Upsilon(\mathcal{P}, \mathcal{Q})(F \otimes I_d)\}. \tag{35}$$

From $\mathbf{M}$, the subset $\tilde{\mathbf{M}}$ can be described by

$$\tilde{\mathbf{M}} := \{M \in \mathbf{M} : (M + \Theta \otimes I_d) \in \mathcal{H}^-_{d(n+m)}\} \tag{36}$$

At last, for a complex vector λ defined as in (13), the matrices $\Gamma(\lambda)$ and $N(\lambda)$ are defined by

$$\Gamma(\lambda) := \left[\begin{array}{cc} I & -H(\lambda) \end{array}\right] \text{ and } \mathrm{Span}(N(\lambda)) := \mathrm{Ker}(\Gamma(\lambda)F) \tag{37}$$

respectively. Now, the main result, which can be considered as some extension of the generalized KYP lemma in its "strict inequality" version is stated.

Theorem 5 *With the assumptions detailed above, consider the two following statements:*

i)
$$N'(\lambda)\Theta N'(\lambda) \in \mathcal{H}^-_m \ \forall \lambda \in \partial\mathcal{D}; \tag{38}$$

ii)
$$\tilde{\mathbf{M}} \neq \emptyset. \tag{39}$$

Then **ii)** *is sufficient for* **i)***.*

Proof. Statement **ii)** means that there exist sets $\mathcal{P}$ and $\mathcal{Q}$ such that

$$M + \Theta \otimes I_d < 0. \tag{40}$$

Besides, from the definition of $N(\lambda)$, it comes that

$$\Gamma(\lambda)FN(\lambda) = 0, \ \ \forall \lambda \in \partial\mathcal{D}. \tag{41}$$

Inequality (40) implies that

$$(N'(\lambda) \otimes I_d)(M + \Theta \otimes I_d)(N(\lambda) \otimes I_d) < 0, \ \ \forall \lambda \in \partial\mathcal{D} \tag{42}$$

$$\Leftrightarrow (N'(\lambda)F' \otimes I_d)\Upsilon(\mathcal{P}, \mathcal{Q})(FN(\lambda) \otimes I_d) + (N'(\lambda)\Theta N(\Lambda)) \otimes I_d < 0, \ \ \forall \lambda \in \partial\mathcal{D}. \tag{43}$$

From (41), it can be seen that $\mathrm{Span}(FN(\lambda))$ is completely characterized by the relation

$$\mathrm{Span}(FN(\lambda)) = \mathrm{Span}\left(\left[\begin{array}{c} H(\Lambda) \\ I \end{array}\right] X\right), \tag{44}$$

where X is any full rank matrix. Thus, inequality (43) can be equivalently written

$$\Leftrightarrow (X' \otimes I_d)\left(\left[\begin{array}{c} H(\Lambda) \\ I \end{array}\right]' \otimes I_d\right)\Upsilon(\mathcal{P}, \mathcal{Q})\left(\left[\begin{array}{c} H(\Lambda) \\ I \end{array}\right] \otimes I_d\right)(X \otimes I_d) + (N'(\lambda)\Theta N(\Lambda)) \otimes I_d < 0, \ \ \forall \lambda \in \partial\mathcal{D}. \tag{45}$$

From Lemma 3, the first term of the left handside member of inequality (45) is positive semi-definite when $\lambda \in \partial\mathcal{D}$. Hence, the second term is negative definite which is equivalent to **i)**. $\square$

The 1D generalized KYP lemma is known to provide a necessary and sufficient condition whereas Theorem 5 only proposes a sufficient condition **ii)** for property **i)** to hold. More than proving Theorem 5, it is important to emphasize why condition **ii)** might not be necessary. The 1D-version of generalized KYP lemma [35] can be proven through the so-called S-procedure [47] in its generalized form (see [35, 48] and the references therein). In the remaining part of the section, Theorem 5 is studied through the lens of S-procedure, which is now recalled in its strict inequality version.

Lemma 6 (S-procedure, from [35]) *Let* $\mathbf{\Theta}$ *be an Hermitian matrix and* $\mathbf{M}$ *be an arbitrary subset of* $\mathcal{H}_q$*. Moreover let* $\tilde{\mathbf{M}} \subset \mathbf{M}$ *be defined by*

$$\tilde{\mathbf{M}} := \{M \in \mathbf{M} : (M + \mathbf{\Theta}) \in \mathcal{H}_q^-\} \tag{46}$$

and the set $\mathbf{S}$ *be defined by*

$$\mathbf{S} := \{S \in \mathcal{H}_q : S \neq 0,\ \mathrm{rank}(S) = 1,\ S \geq 0, \mathrm{tr}(MS) \geq 0 \forall M \in \mathbf{M}\}. \tag{47}$$

Consider the two following statements:

a)

$$\mathrm{tr}(\Theta S) < 0 \forall S \in \mathbf{S}; \tag{48}$$

b)

$$\tilde{\mathbf{M}} \neq \emptyset. \tag{49}$$

Then **b)** *is sufficient for* **a)** *and if the set* $\mathbf{M}$ *is* rank-one separable *(see the definition in [35]) then* **b)** *is also necessary and the S-procedure is said* lossless*.*

A possible approach to understand why Theorem 5 only provides a sufficient condition unlike the classical KYP lemma is to try to see why S-procedure is useless or is simply not lossless when applied to the assumptions of Theorem 5. It is clear by the used notations that the idea is to compare the sets $\mathbf{M}$ and $\tilde{\mathbf{M}}$ defined, on the one hand, in Theorem 5 and, on the other hand, in Lemma 6 with $q = n + m$, and then to address two issues.

1. Is it possible to exhibit a set $\mathbf{S} = \mathbf{S}_a$ from the assumptions of Theorem 5 that could match the set $\mathbf{S} = \mathbf{S}_b$ in Lemma 6 (this would enable ones to apply the S-procedure, at least in the sense **b)** $\Rightarrow$ **a)**) ?
2. *If so*, is the S-procedure lossless (this would enable ones to apply the S-procedure in the sense **a)** $\Rightarrow$ **b)**)?

Fristly, have a look to statement **i)** in Theorem 5 which is equivalent to

$$y'N'(\lambda)\Theta N(\lambda)y < 0,\ \forall y \neq 0, \forall \lambda \in \partial\mathcal{D}. \tag{50}$$

With the substitution $\xi = N(\lambda)y$, (50) can be written

$$\mathrm{tr}(\Theta S) < 0,\ \forall S \in \mathbf{S}_a := \{S = \xi\xi' : \xi \neq 0,\ \Gamma(\lambda)F\xi = 0,\ \lambda \in \partial\mathcal{D}\}. \tag{51}$$

With the change $\eta = F\xi$, it is easy to see that η complies with

$$\eta = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ \ z \neq 0, \tag{52}$$

so the set $\mathbf{S}_a$ can be written

$$\mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0 : \eta = F\xi = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ \lambda \in \partial\mathcal{D}\right\}. \tag{53}$$

From Lemma 3, it can be deduced that

$$\mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0 : \eta = F\xi = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ \left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]' \otimes I_d\right) \Upsilon(\mathcal{P},\mathcal{Q}) \left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] \otimes I_d\right) \geq 0\ \forall\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H}\right\}. \tag{54}$$

$$\Leftrightarrow \mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0 : \eta = F\xi = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ q'\left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]' \otimes I_d\right) \Upsilon(\mathcal{P},\mathcal{Q}) \left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] \otimes I_d\right) q \geq 0\ \forall\{\mathcal{P};\mathcal{Q};q\} \in \mathbb{H} \times \mathbb{C}^{dn}\right\}. \tag{55}$$

so, it is possible to express **i)** as **a)** with $\boldsymbol{\Theta} = \Theta$ and with $\mathbf{S}$ matching $\mathbf{S}_a$ as described above. Secondly, the condition **ii)** in Theorem 5 can be expressed as **b)** in Lemma 6 with $\boldsymbol{\Theta} = \Theta \otimes I_d$ and with the set $\mathbf{M}$ as defined in (35). Apply Lemma 6 in the "easy" sense (**b)** $\Rightarrow$ **a)**) to get

$$\mathrm{tr}(\Theta S) < 0,\ \forall S \in \mathbf{S}_b := \{S = \xi\xi' : \xi \neq 0,\ \mathrm{tr}(M\xi\xi') \geq 0 \forall M \in \mathbf{M}\}, \tag{56}$$

where $\mathbf{S}_b$ can also be written

$$\Leftrightarrow \mathbf{S}_b = \{S = \xi\xi' : \xi \neq 0,\ \xi'(F' \otimes I_d)\Upsilon(\mathcal{P},\mathcal{Q})(F' \otimes I_d)\xi \geq 0 \forall\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H}\}. \tag{57}$$

Now, the issue 1 mentioned above is to compare $\mathbf{S}_a$ in (55) and $\mathbf{S}_b$ in (57). It is first noticeable that ξ is of dimension n when involved in the expression of $\mathbf{S}_a$ whereas it is of dimension dn in the definition of $\mathbf{S}_b$ so it is clear that when $d > 1$, S-procedure cannot be applied and the necessity does not hold. However, what can be said when $d = 1$? In this case, $\mathbf{S}_a$ can be written

$$\Leftrightarrow \mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0 : \eta = F\xi = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ q'\left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]'\right) \Upsilon(\mathcal{P},\mathcal{Q}) \left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]\right) q \geq 0\ \forall\{\mathcal{P};\mathcal{Q};q\} \in \mathbb{H} \times \mathbb{C}^{n}\right\}. \tag{58}$$

which, with $q = Xz$, also writes

$$\Leftrightarrow \mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0 : \eta = F\xi = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ z'X'\left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]'\right) \Upsilon(\mathcal{P},\mathcal{Q}) \left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]\right) Xz \geq 0\ \forall\{\mathcal{P};\mathcal{Q};X : \det(X) \neq 0\} \in \mathbb{H} \times \mathcal{H}_n\right\}. \tag{59}$$

Now, assume that $k = 1$ which corresponds to the 1D-case, then , with no loss of generality, the matrices P_1 and Q_{1_l} can be substituted with $X^{-1}P_1X^{-1}$ and $X^{-1}Q_{1_l}X^{-1}$ and $\mathbf{S}_a$ can then be described by

$$\Leftrightarrow \mathbf{S}_a = \left\{\xi\xi' : \xi \neq 0 : \eta = F\xi = \left[\begin{array}{c} H(\lambda) \\ I \end{array}\right] z,\ z'\left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]'\right) \Upsilon(\mathcal{P},\mathcal{Q}) \left(\left[\begin{array}{c} H(\lambda) \\ I \end{array}\right]\right) z \geq 0\ \forall\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H}\right\}. \tag{60}$$

which is simplified in

$$\Leftrightarrow \mathbf{S}_a = \left\{ \xi\xi' : \xi \neq 0 : \eta = F\xi = \begin{bmatrix} H(\lambda) \\ I \end{bmatrix} z,\ \eta'\Upsilon(\mathcal{P},\mathcal{Q})\eta \geq 0\ \forall\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H} \right\}. \tag{61}$$

or equivalently in $\mathbf{S}_b$. But this equality $\mathbf{S}_a = \mathbf{S}_b$ holds when $k = d = 1$. It means that the S-procedure can be directly applied from **ii)** to **i)** to prove Theorem 5 when the set $\partial\mathcal{D}$ only involves intersection of lines and circles with half planes, discs and exteriors of discs, and when, furthermore, one comes back to the 1D-case. In this case only, shall one ask the question of necessity *i.e.* of the losslessness of the S-procedure. So it is lossless if the set $\mathbf{M}$ defined in (35) is rank-one separable. Actually, if $\bar{l} = 1$, this set is the one studied in [35] so the answer is yes, it is rank-one separable. For the case $\bar{l} > 1$, the proof of losslessness is available in [44], based upon arguments borrowed from [48].

As a summary of the above discussion, here are four points:

- Theorem 5 provides a sufficient condition that is generally not necessary.
- When $d > 1$, the S-procedure cannot be applied to prove Theorem 5 because $\mathbf{S}$ would not preserve its dimension whether statement **a)** or **b)** is considered.
- When $d = 1$, the dimension of $\mathbf{S}$ remains the same (or equivalently the size of the elements of $\mathbf{S}_a$ and $\mathbf{S}_b$ is the same). However, in this case, $\mathbf{S}_a \subset \mathbf{S}_b$, so **b**) is clearly more constraining than **a**); Thus, **b**) $\Rightarrow$ **a**). But it is not a direct application of the S-procedure. However **ii**) $\Leftrightarrow$ **b**) with $\mathbf{S} = \mathbf{S}_b \Rightarrow$ **b**) with $\mathbf{S} = \mathbf{S}_a \overset{\text{S-procedure}}{\Rightarrow}$ **a**) with $\mathbf{S} = \mathbf{S}_a \Leftrightarrow$ **i**). Hence, **ii**) $\Rightarrow$ **i**).
- At last, when $k = d = 1$ (simple regions and 1D-case), then $\mathbf{S} = \mathbf{S}_a = \mathbf{S}_b$ which enables ones to directly apply the S-procedure. Moreover, it has been proven that it is lossless in this case so **ii**) $\Leftrightarrow$ **i**).

4 Application to the analysis of Roesser models

In this section, the relevance of Theorem 5 for the study of multidimensionnal hybrid state-space Roesser model (2) is emphasized. The idea is simply to make a suitable choice of matrices F and θ. Matrix F is chosen as follows:

$$F := \begin{bmatrix} A & B \\ E & 0 \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{62}$$

where the matrices A, B and E are of course those involved in (2). It can be noticed that matrix F is here real which corresponds to a special case of Theorem 5. Condition **i)** in Theorem 5 must be considered as a performance level or a property of (2) to be tested. In the sequel, three properties are considered, depending on the choice of Θ.

4.1 Robust $\partial\mathcal{D}$-regularity against an LFT-based uncertainty

This part is devoted to the analysis of the $\partial\mathcal{D}$-regularity of the uncertain k-variate polynomial matrix described by

$$\mathbb{A}(\lambda) = \mathbf{A}(\lambda) - \bar{A} = EH(\lambda) - A_c, \tag{63}$$

where $\mathbf{A}(\lambda)$ is given by (33), where A, B, C, and E have the same dimensions as those involved in (2), where $A_c = A + \bar{A}$ and where $\bar{A}$, given by

$$\bar{A} = B\bar{\Delta}C, \tag{64}$$

is an uncertain matrix complying with the so-called LFT (Linear Fractional Transform)-based structure:

$$\bar{\Delta} = \Delta(I_p - D\Delta)^{-1}, \quad \det(I_p - D\Delta) \neq 0. \tag{65}$$

The above full rank assumption is referred to as the well posedness of the uncertainty. Matrix Δ is the actual uncertainty matrix which belongs to $\mathcal{B}(\rho)$, the ball of complex matrices Δ verifying $||\Delta||_2 \leq \rho$.

4.1.1 Complex $\partial\mathcal{D}$-regularity radius

Nominal matrix $\mathbf{A}(\lambda)$ is assumed to be $\partial\mathcal{D}$-regular with $\partial\mathcal{D}$ as defined by (11) with (8-10). The purpose is here to derive what can be called the complex $\partial\mathcal{D}$-regularity radius $\varrho_{\partial\mathcal{D}}$, which is the largest value of the radius ρ such that the uncertain polynomial matrix $\mathbb{A}(\lambda)$ defined by (63) remains $\partial\mathcal{D}$-regular over $\mathcal{B}(\rho)$. To reach such a goal, Θ is chosen as follows:

$$\Theta := \begin{bmatrix} C'C & C'D \\ D'C & D'D - \gamma I \end{bmatrix}, \quad \gamma = \rho^{-1/2}. \tag{66}$$

Corollary 7 *Let an uncertain k-variate matrix $\mathbb{A}(\lambda)$ be defined as in (63) and a set $\partial\mathcal{D}$ be defined by (11) with (8-10). $\mathbb{A}(\lambda)$ is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exists a set $\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H}$ such that (40) holds with M defined as in (35), $\Upsilon(\mathcal{P},\mathcal{Q})$ defined by (20) with (21-22), F given by (62) and Θ given by (66).*

Proof. With the choice (66), the property **i**) in Theorem 5 can be expressed

$$||G(\lambda)||_2 < \sqrt{\gamma} \;\; \forall \lambda \in \partial\mathcal{D}, \tag{67}$$

(with $G(\lambda)$ given by (4)) by noting that the choice of F proposed in (62) leads to

$$\mathrm{Span}(N(\lambda)) = \mathrm{Span}\left(\begin{bmatrix} (EH(\lambda) - A)^{-1}B \\ I \end{bmatrix} \right). \tag{68}$$

Then, inequality (67) can be written

$$\{ \sup_{\lambda \in \partial\mathcal{D}} \bar{\sigma}(G(\lambda))\}^{-1} > \rho. \tag{69}$$

Simple arguments on singular values show that

$$\bar{\sigma}(G(\lambda)) = \mu_{\mathbb{C}}(G(\lambda)) \tag{70}$$

where $\mu_{\mathbb{C}}(.)$ denotes the celebrated complex structured singular value introduced in [49]. Then it comes

$$\{ \sup_{\lambda \in \partial\mathcal{D}} \mu_{\mathbb{C}}(G(\lambda))\}^{-1} > \rho. \tag{71}$$

Taking the fact that

$$\mu_{\mathbb{C}}\left(G(\lambda)\right) = \left[\inf_{\Delta}\{\bar{\sigma}(\Delta) : \det(I - G(\lambda)\Delta) = 0\}\right]^{-1} \tag{72}$$

into account, one gets

$$\inf_{\lambda\in\partial\mathcal{D}}\left\{\inf_{\Delta}\{\bar{\sigma}(\Delta) : \det(I - G(\lambda)\Delta) = 0\}\right\} > \rho. \tag{73}$$

Besides,

$$\det(I - G(\lambda)\Delta) = 0 \tag{74}$$

$$\Leftrightarrow \det(I - C(EH(\lambda) - A)^{-1}B\Delta - D\Delta) = 0 \tag{75}$$

$$\Leftrightarrow \det(I - C(EH(\lambda) - A)^{-1}B\Delta(I_p - D\Delta)^{-1})\det(I_p - D\Delta) = 0. \tag{76}$$

From the well posedness assumption, $\det(I_p - D\Delta) \neq 0$, thus the above equality is equivalent to

$$\det(I - C(EH(\lambda) - A)^{-1}B\bar{\Delta}) = 0 \tag{77}$$

$$\Leftrightarrow \det(I - (EH(\lambda) - A)^{-1}B\bar{\Delta}C) = 0 \tag{78}$$

$$\Leftrightarrow \det((EH(\lambda) - A)^{-1})\det(EH(\lambda) - A - B\bar{\Delta}C) = 0. \tag{79}$$

Since, for $\Delta = 0$, $\mathbb{A}(\lambda) = \mathbf{A}(\lambda)$ is implicitely assumed to be $\partial\mathcal{D}$-regular (otherwise why testing robust $\partial\mathcal{D}$-regularity?), the left factor of the left handside member is non zero so it follows that

$$\det(EH(\lambda) - A - \bar{A}) = 0. \tag{80}$$

From (73) and (80), it can be deduced that

$$\inf_{\lambda\in\partial\mathcal{D}}\left\{\inf_{\Delta}\{\bar{\sigma}(\Delta) : \det(\mathbb{A}(\lambda)) = 0\}\right\} > \rho, \tag{81}$$

which proves that $\mathbb{A}(\lambda)$ remains $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$. $\square$.

It is clear that $\rho^\star$, the maximum value of $\rho = \gamma^{-1/2}$ obtained by computing $\mathcal{P}$ and $\mathcal{Q}$ proving (39), is a lower bound of $\varrho_{\partial\mathcal{D}}$. If $k = d = 1$, then $\rho^\star = \varrho_{\partial\mathcal{D}}$. Note however that matrix Δ is here assumed to be complex. In practice, it would be interesting to take its possible realness into account in order to derive the real $\partial\mathcal{D}$-regularity radius but this is much more complicated. Maybe some insights to investigate this issue can be found in [50].

4.1.2 Application to asymptotic stability analysis of hybrid Roesser model

In this part, the model (2) is considered in association with the uncertainty structure defined by the feedback equation

$$u(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k) = \Delta y(t_1, \ldots, t_r, j_{r+1}, \ldots, j_k),\ \Delta \in \mathcal{B}(\rho) \cap \mathbb{R}^{m\times p}, \tag{82}$$

corresponding to the LFT formalism mentioned in the above subsection. Indeed, the closed-loop state matrix becomes:

$$A_c = A + \bar{A}, \tag{83}$$

where $\bar{A}$ is given by (64-65).
Before to study various performance levels, the main property is considered, namely asymptotic stability. As mentioned in Remark 2, the stability of descriptor nD-models is rather still an open problem so this discussion on stability will be led with assuming that $E = I$. In this case, Corollary 7 enables ones to test, with conservatism, if matrix $\mathbb{A}(\lambda) = H(\lambda) - A - \bar{A}$ is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$. As mentioned at the end of subsection 2.3, asymptotic stability of Roesser model (2) with $E = I$ depends on the root-clustering of polynomial $c(\lambda, I, A)$. Thus, robust stability of the uncertain hybrid Roesser model described by (2) with $E = I$ and (82) against $\mathcal{B}(\rho)$ depends on the root-clustering of $c(\lambda, I, A_c) = \det(\mathbb{A}(\lambda))$. More precisely, for $c(\lambda, I, A_c)$ to match the robust stabililty conditions induced by Lemma 1, it is necessary that $\mathbb{A}(\lambda)$ be at least $\partial\mathcal{D}$-regular with the choice (34). But the difference between $\partial\mathcal{D}$-regularity and asymptotic stability lies in the distribution of the roots of $c(\lambda, I, A_c)$ with respect to $\partial\mathcal{D}$. However, if nominal k-variate polynomial matrix is "stable", meaning that $c(\lambda, I, A)$ fills the conditions given by Lemma 1 or equivalently that (2) with $E = I$ is asymptotically stable, then this property is obvioulsy preserved when Δ describes $\mathcal{B}(\rho)$ if $\mathbb{A}(\lambda)$ remains $\partial\mathcal{D}$-regular. From this reasoning, the following theorem is deduced.

Corollary 8 *Let an uncertain multidimensional hybrid system be described by Roesser model (2) with $E = I$ and (82). It is robustly asymptotically stable against $\mathcal{B}(\rho)$ if there exists a set $\mathcal{P}$ of k matrices $P_i \in \mathcal{H}^+_{n_i}$, $i = 1, \ldots, k$, such that the following LMI holds,*

$$F'\overline{\mathbf{R}}(\mathcal{P})F + \Theta < 0 \tag{84}$$

where F is given by (62), $\overline{\mathbf{R}}(\mathcal{P})$ is given by (21) with (34), and Θ is given by (66).

Proof. (84) implies that $\mathbb{A}(\lambda)$ is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ with $\partial\mathcal{D}$ characterized by (34), by virtue of Corollary 7. Note that such a choice of region does not require to involve matrices Q_{i_l} and imposes that $d = 1$. Besides, since the matrices P_i are assumed to be positive definite, the first block in (84), which is

$$\begin{bmatrix} A \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} A \\ I \end{bmatrix} + C'C < 0 \Rightarrow \begin{bmatrix} A \\ I \end{bmatrix}' \overline{\mathbf{R}}(\mathcal{P}) \begin{bmatrix} A \\ I \end{bmatrix} < 0, \tag{85}$$

proves the nominal asymptotic stability of (2) with $E = I$ by virtue of [30, Theorem 1]. Then $c(\lambda, I, A) = \det(\mathbf{A}(\lambda))$ satisfies the condition of Lemma 1. It is clear that $c(\lambda, I, Ac) = \det(\mathbb{A}(\lambda))$ also satisfies the condition of Lemma 1 otherwise, by continuity of λ with respect to Δ, $\mathbb{A}(\lambda)$ would becomes $\partial\mathcal{D}$-singular. Hence, (2) with $E = I$ and (82) is robustly asymptotically stable against $\mathcal{B}(\rho)$. □

It comes that $\rho^\star$, the maximum value of ρ while satisfying LMI (84), is a lower bound of the complex stability radius. but it has to be mentioned that this value might be conservative for two reasons. First, Corollary 7 and thus Corollary 8 only offer sufficient conditions when $k > 1$. Second, even if some Δ leads to the $\partial\mathcal{D}$-singularity of $\mathbb{A}(\lambda)$, it does not necessarily imply instability. This is exactly why [30, Theorem 1] offers only a sufficient condition. However, if $k = 1$, one recovers the well known KYP lemma and $\rho^\star$ is exactly the complex stability radius. Nevertheless, whatever the value of k may be, the complex stability radius remains equal or lower than the real stability radius which is the ideal robustness criterion.

Remark 9 *It is possible to consider other sets $\partial\mathcal{D}$ in some cases of system analysis. For instance, some sector boundaries can be chosen to handle fractional continuous models [51]. As another example, it is possible, when $k = d = 1$, to choose the various matrices R_i and Φ_{i_l} so that $\partial\mathcal{D}$ is the boundary of a so-called first order EEMI (Extended Ellipsoidal Matrix Inequality)-region [52] in order to recover [52, Theorem 4.1 with $d = 1$].*

4.2 Positive realness of hyrid Roesser model

Since the work of Popov on hyperstability, positive realness [53] of transfer matrices have been widely investigated, especially for conventional 1D-systems, noting that, for such models, positive realness means passivity and enables the designer to include sector-bounded nonlinearities. For multidimensional linear models, few contributions exist. However, the 2D-discrete Roesser models are considered from the viewpoint of positive realness in [36]. Adapted from this article, the next definition is stated for hybrid Roesser models.

Definition 10 *Let the set $\partial\mathcal{D}$ be defined by (11) with (8-10). The model (2) is said to be strictly positive real over $\partial\mathcal{D}$ if*

$$G(\lambda) + G'(\lambda) > 0 \quad \forall\lambda \in \partial\mathcal{D} \tag{86}$$

Following this idea, the next corollary is proposed.

Corollary 11 *Let the hybrid Roesser model and the set $\partial\mathcal{D}$ be described by (2) and (11) with (8-10) respectively. Model (2) is strictly positive real over $\partial\mathcal{D}$ if there exists a set $\{\mathcal{P};\mathcal{Q}\} \in \mathbb{H}$ such that (40) holds with M defined as in (35), $\Upsilon(\mathcal{P},\mathcal{Q})$ defined as by (20), F given by (62) and Θ given by*

$$\Theta := \left[\begin{array}{cc} 0 & -C' \\ -C & -D-D' \end{array}\right]. \tag{87}$$

Proof. It is a simple application of Theorem 5, noting that with (87), property **i)** corresponds to (86). □

But the notion of strict positive realness over $\partial\mathcal{D}$ might not be very useful and one can come back to the more classical notion of strict positive realness.

Definition 12 *The model (2) is strictly positive real if $G(\lambda)$ is analytic on $\mathcal{S}^C$ defined by (7) and if inequality*

$$G(\lambda) + G'(\lambda) > 0 \tag{88}$$

hold on $\mathcal{S}^C$.

Corollary 13 *Let a multidimensional hybrid system be described by Roesser model (2) with $E = I$. It is strictly positive real if there exists a set $\mathcal{P}$ of k matrices $P_i \in \mathcal{H}_{n_i}^+$, $i = 1,\ldots,k$, such that the following LMI (84) holds, where F is given by (62), $\overline{\mathbf{R}}(\mathcal{P})$ is given by (21) with (34) and Θ is given by (87).*

Proof. It is a simple application of Corollary 11 considering that matrices P_i are positive definite to ensure asympotic stability. $\square$.

Corollary 13 might then be seen as an extension of the LMI version of the classical positive real lemma (see [19] for example).
Besides, it has to be mentioned that in the reference [36] quoted above, the control design is also investigated.

4.3 Bounded realness of hybrid Roesser model

The idea is about the same as in the above subsection but the performance to be considered is the so-called $\mathcal{H}_\infty$ level. As some extension of the $\mathcal{H}_\infty$-norm proposed in [54] for the 2D-discrete Roesser models, the following definition introduces the $\mathcal{L}_\infty$-norm of model (2) with respect to $\partial\mathcal{D}$.

Definition 14 *Let a hybrid Roesser model and the set $\partial\mathcal{D}$ be described by (2) and (11) with (8-10) respectively. The $\mathcal{L}_\infty$-norm of (2) with respect to $\partial\mathcal{D}$ is defined by*

$$||G||_\infty = \sup_{\lambda\in\partial\mathcal{D}} ||G(\lambda)||_2. \tag{89}$$

From this definition and from Theorem 5, the next corollary is stated.

Corollary 15 *Let the hybrid Roesser model and the set $\partial\mathcal{D}$ be described by (2) and (11) with (8-10) respectively. The $\mathcal{L}_\infty$-norm of (2) with respect to $\partial\mathcal{D}$ is strictly lower than $\sqrt{\gamma}$ if there exists a set $\{\mathcal{P};\mathcal{Q}\}\in\mathbb{H}$ such that (40) holds with M defined as in (35), $\Upsilon(\mathcal{P},\mathcal{Q})$ defined as by (20) with (21-22), F given by (62) and Θ given by (66).*

Proof. Direct from the discussion in subsection 4.1.1 and especially inequality (67). $\square$

A particular attention can be paid to the case where $E = I$ and where $\partial\mathcal{D}$ is reduced to the choice (34). Assuming that (2) is asymptotically stable (*i.e.* $G(\lambda)$ is analytic on $\mathcal{S}^C$), then the $\mathcal{L}_\infty$-norm becomes the so-called $\mathcal{H}_\infty$-norm.

Definition 16 *Let the hybrid Roesser model (2) with $E = I$ comply with Lemma 1. Its $\mathcal{H}_\infty$-norm is defined by (89) with (11) and (8-10) together with (34).*

This $\mathcal{H}_\infty$-norm actually equals the so-called $\mathcal{L}_2$-gain but this connection is not detailed here for the sake of conciseness.
From this definition, the next corollary is formulated.

Corollary 17 *Let the hybrid Roesser model described by (2) with $E = I$. It is asymptotically stable and its $\mathcal{H}_\infty$-norm is strictly lower than $\sqrt{\gamma}$ if there exists a set $\mathcal{P}$ of k matrices $P_i\in\mathcal{H}^+_{n_i}$, $i = 1,\dots,k$, such that LMI (84) holds, where F is given by (62), $\overline{\mathbf{R}}(\mathcal{P})$ is given by (21) with (34) and Θ is given by (66).*

Proof. Simple application of Corollary 15 with (34). Note that, once again, the matrices P_i are positive definite to ensure asymptotic stability. □

Corollary 17 can be seen as an extension of classical LMI version of the Bounded real lemma [55] to hybrid Roesser models. It has to be mentioned that the $\mathcal{H}_\infty$ control design is considered in [37] through an equivalent approach.

Remark 18 *Corollaries 7 and 15 actually involve the same conditions. The same for Corollaries 8 and 17 In Corollary 7 (resp. 8), robust $\partial\mathcal{D}$-regularity (resp. robust stability) is concerned whereas in Corollary 15 (resp.17), $\mathcal{L}_\infty$-norm on $\partial\mathcal{D}$ (resp. $\mathcal{H}_\infty$ performance level) is addressed. This connection between the $\mathcal{L}_\infty$ or $\mathcal{H}_\infty$-norm of a system and the quadratic stability against an LFT-based uncertainty is already well known in the 1D-case [56, 57].*

5 Conclusion

In this paper, a KYP-like theorem has been proposed with a purpose of application to the very general hybrid state-space Roesser models. As a special instance of the result, the KYP lemma for conventional 1D-linear models is recovered. Unlike for this 1D-case, for multidimensional models, some conservatism appears. This is due to the nature of Roesser models themselves. The origin of this conservatism has been highlighted.
Some possibilities of exploiting this new theorem in terms of robust stability, positive realness and $\mathcal{H}_\infty$ analysis of hybrid Roesser models have been emphasized. The authors hope that this point of view will be seen as an attempt to unify numerous contributions dealing with LMI approach applied to nD-models.

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