Non-iterative pole placement technique: a step further

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Version 1

Abstract

This paper comes back to the hard problem of pole placement by static output feedback: let a triplet of matrices $\{A; B; C\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ be given, find a matrix $K \in \mathbb{R}^{m\times p}$ such that the spectrum of $A+BKC$ equals a specified set. More precisely, this article focuses on the derivation of non-iterative techniques, based upon the notion of eigenstructure assignment, to solve the problem, especially when Kimura's condition does not hold. Indeed, it is shown that such solutions can sometimes be found under assumptions detailed in the paper.

Index Terms

Pole placement, eigenstructure assignment, Kimura's condition.

1 Introduction

Since the contributions of Davison [1–3], the static ouput pole placement problem is one of the most investigated problems in the control community and is actually not yet completely solved. For interesting surveys about that problem, and more generally, about static output feedback, see [4–6]. The problem consists in considering matrices $\{A; B; C\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ and in finding a matrix $K \in \mathbb{R}^{m\times p}$ such that the spectrum of the closed-loop matrix $A_c = A + BKC$ equals a set of arbitrarily specified values $\{\lambda_i,\ i = 1, ..., n\}$. A first question is the existence of a solution. This question is in itself a hard topic. Various crucial contributions are recalled in [7]. In Brockett [8], it is proven that the generic condition for pole placement is $mp \geq n$ provided that the problem is solved in the field of complex matrices. In the field of real matrices (corresponding to the actual problem), a sufficient condition for generic pole assignability was proposed by Wang [9] with alternative proofs in [10–12]: $mp > n$ (only generic as illustrated by the counterexample of [13]). Although those results are fundamental to understand the present control problem, another challenge is to derive methods that exhibit a solution. The only one that is available for all the cases encompassed by $mp > n$ is presented in [9]. It relies on an optimisation process whose parameters may be difficult to tune. The basic reason might be that the static output feedback pole placement problem is NP-hard [14].

Regarding this result of NP-hardness, it becomes illusory to always expect to derive a direct technique (by "direct", the authors mean non-iterative) that solves the problem provided that Wang's condition holds. Nevertheless, it is important to list the various cases for which one can generically derive a suitable static feedback control law through direct methods. There are several approaches to solve the problem. Among those approaches, the authors focus on those based upon the notion of eigenstructure assignment, that is those inspired from the work of Moore [15] on the state feedback. When a static output feedback is to be computed, the best results are the so-called geometric approach [16], the parametric approach [17] and the approach based on two Sylvester equations coupled by an orthogonality condition [18]. All these methods require that Kimura's condition ($m+p > n$) [19] be satisfied. Indeed, this condition has been considered as the best sufficient condition for generic assignability for a long time [8, 19–21], until the work of Wang. From a practical point of view, it remains an interesting frontier between the systems for which a solution can be easily derived and those for which a solution exists but is not easily reached. A significant attempt to find a solution when Kimura's condition does not hold is [22]. In this contribution, an optimization procedure may allow to solve the case $m + p = n$ (but some convenient initialization has to be guessed). Interesting insights can also be found in [23]. Recently it has been shown that the case $m + p = n < mp$ could generically be solved by a direct method [24]. It must be mentionned that, as in [24], only the case of eigenvalues with multiplicty one is here addressed.

The present paper hinges upon [24] and can be considered as an extension. It aims at highlighting some other cases that allow to find a solution by a direct computation, even if Kimura's condition is not verified. For clarity, the next three sections recalls the content of [24]. Indeed, Section 2 states the problem and recalls some classical properties of the eigenstructure. In Section 3, the spectrum assignment is characterized by an "assignment equation". In section 4, assignment techniques are deduced for the above-mentioned cases : complete placement under Kimura's condition, partial placement under $mp > m + p$, complete placement under $m + p = n < mp$. Section 5 is the actual contribution of the present paper. It shows how the approach of [24] can be extended in some cases for which $m + p < n < mp$. Section 6 proposes some numerical illustration before the paper is concluded in Section 7.

2 Problem statement and Background

Before to state the problem, the next preliminary definition is introduced.

Definition 1 *let* $\mathbf{M}$ *be a set of complex matrices. If* $M \in \mathbf{M} \Rightarrow \bar{M} \in \mathbf{M}, \forall M \in \mathbf{M}$*, then the set* $\mathbf{M}$ *is said to be closed under conjugation.*

Of course, this definition is also valid for sets of complex vectors or scalars.

Problem 1 *Let* $\{A; B; C\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ *be a triplet of matrices satisfying* *(i)* (A, B, C) *is minimal;*

(ii) $\text{rank}(B) = m \leq n \quad \& \quad \text{rank}(C) = p \leq n$.

Let $\{\lambda_i,\ i = 1, ..., m+q\}$ *be a set of* $m+q$ *distinct complex eigenvalues* ($q \leq n-m$) *such that* $\{\lambda_i,\ i = 1, ...m\}$ *and* $\{\lambda_j,\ j = m+1, ...m+q\}$ *are closed under conjugation.*
Find a matrix $K \in \mathbb{R}^{m \times p}$ *such that* $\{\lambda_i,\ i = 1, ..., m+q\} \subset \lambda(A_c = A + BKC)$.
If $q < n-m$, *the pole placement is said partial; If* $q = n-m$, *it is said complete.*

Assumptions (i) and (ii) are actually classical. In practice, they mean that the considered model is controllable, observable and that sensors and actuators convey only meaningful information. Unlike in [24], there is no condition on the dimensions (*i.e.* n, m, p and q). In fact, they will be further detailed in the various tackled cases Also note that only eigenvalues with multiplicity 1 are considered for the sake of conciseness. An actual restriction is the separation of the whole desired spectrum into two self-conjugate subsets, which does not encompass all the possible problems. It is however a usual assumption too.

Even if the attention is focused on the eigenstructure (eigenvalues and eigenvectors) of A_c, the actual purpose is only eigenvalue assignment. Eigenstructure is a concept that reveals particularly useful to design pole placement through direct computation. The next reasoning, borrowed from [15], is a basis for this contribution. The spectrum of A_c is associated to right eigenvectors v_i by the relations

$$A_c v_i = \lambda_i v_i \Leftrightarrow (A - \lambda_i I_n) v_i + BKC v_i = 0 \quad \forall i \in \{1, ..., n\} \tag{1}$$

$$\Leftrightarrow (A - \lambda_i I_n) v_i + B w_i = 0 \quad \forall i \in \{1, ..., n\} \tag{2}$$

where the vectors $w_i = KCv_i$ are the input directions. It comes $v_i \in \mathcal{S}_i$ with

$$\mathcal{S}_i = \{v \in \mathbb{C}^n \,|\, (\exists w \in \mathbb{C}^m \,|\, (A - \lambda_i I_n) v + B w = 0)\} \quad \forall i \in \{1, ..., n\}. \tag{3}$$

This set is the so-called (A, B)-characteristic subspace [16] and is of dimension m when (A, B) is controllable. To derive v_i in $\mathcal{S}_i$, let the vectors π_i be defined by $\pi_i = \begin{bmatrix} v_i' & w_i' \end{bmatrix}'$. Then

$$T_i \pi_i = 0 \quad \text{with} \quad T_i = \begin{bmatrix} A - \lambda_i I_n & B \end{bmatrix} \Rightarrow \pi_i \in \text{Ker}(T_i) \quad \forall i \in \{1, ..., n\}. \tag{4}$$

Let R_i be a matrix whose columns span the right nullspace of T_i *i.e.*

$$\text{Span}(R_i) = \text{Ker}(T_i) \quad \text{with} \quad R_i = \begin{bmatrix} N_i \\ M_i \end{bmatrix},\ N_i \in \mathbb{C}^{n \times m},\ M_i \in \mathbb{C}^{m \times m} \quad \forall i \in \{1, ..., n\}, \tag{5}$$

which implies $T_i R_i = 0$. Each vector π_i is then characterized by a non-zero parameter vector $z_i \in \mathbb{C}^m$ such that

$$\forall i \in \{1, ..., n\}, \quad \pi_i = R_i z_i \Leftrightarrow \left\{ \begin{array}{l} v_i = N_i z_i, \\ w_i = M_i z_i. \end{array} \right. \tag{6}$$

z_i represents the dof available to assign v_i as a right eigenvector associated to λ_i. Since v_i (or w_i) can be scaled by any non-zero factor (one talks about eigendirection rather than eigenvector), each z_i provides $(m-1)$ degrees of freedom. When $C = I_n$ (state feedback), any choice of z_i, $i = 1, ..., n$, such that the v_i are linearly independent yields

$$K = WV^{-1} \quad \text{with} \quad \left\{ \begin{array}{l} V \; = \left[\begin{array}{ccc} v_1 & \ldots & v_n \end{array} \right] \\ W \; = \left[\begin{array}{ccc} w_1 & \ldots & w_n \end{array} \right] \end{array} \right. \tag{7}$$

that assigns the spectrum $\{\lambda_i,\, i = 1, ..., n\}$ for A_c [15]. Besides, in the case where the vectors Cv_i, $i = 1, ..., p$ are linearly independent, only the subset $\{\lambda_i,\, i = 1, ..., p\}$ is assigned by the static output feedback matrix

$$K = W_p V_p^{-1} \quad \text{with} \quad \left\{ \begin{array}{l} V_p \; = \left[\begin{array}{ccc} Cv_1 & \ldots & Cv_p \end{array} \right], \\ W_p \; = \left[\begin{array}{ccc} w_1 & \ldots & w_p \end{array} \right]. \end{array} \right. \tag{8}$$

A dual reasoning can be followed for the left eigenvectors $u_i \in \mathbb{C}^n$, defined by

$$u_i' A_c = \lambda_i u_i' \quad \forall i \in \{1, ..., n\} \tag{9}$$

and belonging to the (A', C')-characteristic subspace $\tilde{\mathcal{S}}_i$. They are associated to ouput directions $l_i = K'B'u_i$ and can be chosen through parameters $x_i \in \mathbb{C}^p$ such that

$$\forall i \in \{1, ..., n\}, \quad \left\{ \begin{array}{l} u_i' = x_i' \tilde{N}_i \\ l_i' = x_i' \tilde{M}_i, \end{array} \right. \tag{10}$$

where $\tilde{R}_i = \left[\begin{array}{cc} \tilde{N}_i & \tilde{M}_i \end{array} \right]$ is the orthogonal complement of

$$\tilde{T}_i = \left[\begin{array}{c} A - \lambda_i I \\ C \end{array} \right], \tag{11}$$

thus satisfying $\tilde{R}_i \tilde{T}_i = 0$. Only m eigenvalues $\{\lambda_i,\, i = 1, ..., m\}$ are assigned by the output feedback matrix

$$K = (U_m')^{-1} L_m' \quad \text{with} \quad \left\{ \begin{array}{l} U_m \; = \left[\begin{array}{ccc} B'u_1 & \ldots & B'u_m \end{array} \right] \\ L_m \; = \left[\begin{array}{ccc} l_m & \ldots & l_m \end{array} \right]. \end{array} \right. \tag{12}$$

When $B = I_n$, a complete pole placement is obtained owing to the output injection matrix

$$K = (U')^{-1} L' \quad \text{with} \quad \left\{ \begin{array}{l} U \; = \left[\begin{array}{ccc} u_1 & \ldots & u_n \end{array} \right] \\ L \; = \left[\begin{array}{ccc} l_1 & \ldots & l_n \end{array} \right]. \end{array} \right. \tag{13}$$

Note that both sets of eigenvectors are not independent of each other. These vectors can be scaled so that the modal matrices U and V satisfy the orthogonality condition

$$U'V = I_n. \tag{14}$$

Remark 1 *If the spectrum of A_c and the sets of associated "left" and "right" free parameter-vectors x_i and z_i, $i = 1, \ldots, n$, are denoted by Λ, $\mathbf{X}$ and $\mathbf{Z}$ respectively, it really matters that each "eigentriplet" satisfies*

$$\{\lambda, x, z\} \in \Lambda \times \mathbf{X} \times \mathbf{Z} \Rightarrow \{\bar{\lambda}, \bar{x}, \bar{z}\} \in \Lambda \times \mathbf{X} \times \mathbf{Z}, \tag{15}$$

otherwise the feedback matrix K may be complex. It means that the sets Λ, $\mathbf{X}$ and $\mathbf{Z}$ are closed under conjugation and therefore so are the sets of eigenvectors. See [15] for details about how to obtain a real K and to reduce numerical effects of imaginary parts.

3 Eigenstructure assignment equality

This section is still mainly borrowed from [24]. The eigenstructure assignment will now be characterized by a single equality that is the basis for the present approach. Although this equality could be referred to as ‘eigenvalue assignment equality”, the reader is reminded of the actual purpose which is only pole placement. For good insights about eigenvector assignment, see [25]. Matrix K must assign the m first eigenvalues and thus comply with $l_i = K'B'u_i$, $i = 1, \ldots, m$. It is also necessary that K satisfies the eigenstructure properties for the q remaining roots, *i.e.* $w_j = KCv_j$, $j = m+1, \ldots, m+q$. Hence, taking equations (6) and (10) into account yields

$$u_i' BKCv_j = l_i' Cv_j = x_i' \tilde{M}_i' CN_j z_j \quad \forall\{i, j\} \in \{1, ..., m\} \times \{m+1, ..., m+q\}, \tag{16}$$

which, still refering to equations (6) and (10), also writes

$$u_i' BKCv_j = u_i' Bw_j = x_i' \tilde{N}_i' BM_j z_j \quad \forall\{i, j\} \in \{1, ..., m\} \times \{m+1, ..., m+q\}. \tag{17}$$

These equalities lead to

$$\begin{bmatrix} x_1' \tilde{N}_1 \\ \vdots \\ x_m' \tilde{N}_m \end{bmatrix} BM_j z_j - \begin{bmatrix} x_1' \tilde{M}_1 \\ \vdots \\ x_m' \tilde{M}_m \end{bmatrix} CN_j z_j = 0 \quad \forall j \in \{m+1, ..., m+q\}. \tag{18}$$

which can be stacked into one single matrix equality:

$$XYZ = 0 \tag{19}$$

where

$$X = \underset{i=1,\ldots,m}{\text{blocdiag}} \{x_i'\} \in \mathbb{C}^{m \times mp}, \tag{20}$$

$$Z = \underset{j=m+1,\ldots,m+q}{\text{blocdiag}} \{z_j\} \in \mathbb{C}^{mq \times q}, \tag{21}$$

$$Y = \tilde{\mathcal{N}}' B \mathcal{M} - \tilde{\mathcal{M}}' C \mathcal{N}, \tag{22}$$

with

$$\left\{ \begin{array}{lcl} \tilde{\mathcal{N}} & = & \begin{bmatrix} \tilde{N}_1' & \ldots & \tilde{N}_m' \end{bmatrix} \\ \tilde{\mathcal{M}} & = & \begin{bmatrix} \tilde{M}_1' & \ldots & \tilde{M}_m' \end{bmatrix} \\ \mathcal{N} & = & \begin{bmatrix} N_{m+1} & \ldots & N_{m+q} \end{bmatrix} \\ \mathcal{M} & = & \begin{bmatrix} M_{m+1} & \ldots & M_{m+q} \end{bmatrix}. \end{array} \right. \tag{23}$$

This paragraph is summarized by the next proposition.

Proposition 1 *There exists a solution to Problem 1 if and only if there exist some non-zero vectors $x_i \in \mathbb{C}^p$, $i = 1, ..., m$, and $z_j \in \mathbb{C}^m$, $j = m+1, ..., n$, such that*

(i) x_i and z_j do not contradict the conjugation constraint (15);

(ii) The vectors $(x_i' \tilde{N}_i B)$ are linearly independent as well as the vectors v_j;

(iii) Equality (19) is satisfied.

A solution is then given by (12) or (8).

Condition (i) reduces the set of solutions to real feedback matrices. Condition (ii) ensures that K can be computed owing to (12). Condition (iii) corresponds to the actual assignment equality and encompasses $v_j \in \mathcal{S}_j$, $u_i \in \tilde{\mathcal{S}}_i$ and (14). The problem therefore appears to be essentially bilinear since X and Z have to be found so that to satisfy (19). The bilinearity of this formulation is not a surprise since another bilinear descritpion was proposed in [22]. A suitable solution to (19) may not be easy to reach. However, some special cases are considered in the sequel. In the next section, the authors recall the cases addressed in [24] but in section 5, some further investigations are led to tackle more restrictive cases.

4 Existing procedures

4.1 $m + p > n \Leftrightarrow p > r$ (Kimura's condition)

In this case, one considers $q = n - m$, that is complete pole placement. Spectrum assignment under Kimura's condition has been tackled in [16–18]. These references do not solve exactly equivalent problems since the desired spectrum is less constrained in [18]. The procedure proposed here (and also in [24]) can be considered as an alternative to the result of [18]. This subsection can therefore not be seen as an improvement compared with the background but it is necessary to recall the next procedure to clearly understand the forthcoming contribution.

Matrix Z defined in (21) is *arbitrarily* chosen. Matrix product $YZ \in \mathbb{C}^{mp \times q}$ is at most of rank q so if the product YZ is written as follows,

$$YZ = \begin{bmatrix} \tilde{Y}_1 \\ \vdots \\ \tilde{Y}_m \end{bmatrix} \quad \text{with} \quad \tilde{Y}_i \in \mathbb{C}^{p \times q} \quad \forall i \in \{1, ..., m\}, \tag{24}$$

it appears that the rank of each $\tilde{Y}_i$ cannot exceed $q < p$. Thus, a basis of the left nullspace of $\tilde{Y}_i$ can be computed. Let $\mathcal{X}_i$ be the left orthogonal complement of $\tilde{Y}_i$ such that

$$\mathcal{X}_i \tilde{Y}_i = 0 \quad \forall i \in \{1, ..., m\}. \tag{25}$$

The dimension of $\mathcal{X}_i$ is $(p - q) \times p$. Let $\beta_i \in \mathbb{R}^{p-q}$ be a non-zero parameter vector ($\forall i \in \{1, ..., m\}$). It comes that $\beta_i' \mathcal{X}_i \tilde{Y}_i = 0$ and then the choice $x_i = \mathcal{X}_i' \beta_i, , \forall i \in \{1, ..., m\}$ provides a solution to (19) from which K can be computed thanks to (10) and (12).

Remark 2 *With the expression "*arbitrary *choice of Z", the reader could think that the vectors z_j are completely free. Actually, they are a bit constrained since they have to comply with (15) and to lead to independent eigenvectors $v_i = N_j z_j$, which is almost always true.*

4.2 $q < r$ and $mp > m + p$ (partial placement)

In this part, only partial placement is achieved. The procedure described above can be easily applied when $q < p$. It confirms, as in [21], that $\min(n, m + p - 1)$ eigenvalues can be assigned.

But when $q = p$, the above procedure fails. Indeed, in this case, it becomes impossible to derive the orthogonal complements $\mathcal{X}_i$ (they are reduced to 0). The reason is that $\tilde{Y}_i$ is generically of rank $q = p$. The idea is then to determine Z such that each $\tilde{Y}_i$ becomes rank deficient, *i.e.* satisfies $\text{rank}(\tilde{Y}_i) \leq p-1$. Among such admissible matrices Z, some can be derived. Let $\mathcal{Z} \in \mathbb{C}^{mp\times(mp-m-p)}$ be such that

$$\text{Span}(\mathcal{Z}) = \text{Ker}(Y). \tag{26}$$

It is at most of rank $(m+p)$. Any non-zero $\gamma \in \mathbb{R}^{mp-m-p}$ such that $g = \mathcal{Z}\gamma$ complies with

$$Yg = Y \begin{bmatrix} z_{m+1} \\ \vdots \\ z_{m+p} \end{bmatrix} = 0 \quad \text{with} \quad z_j \in \mathbb{C}^m \quad \forall j \in \{m+1, ..., m+p\}. \tag{27}$$

One can write Y as follows:

$$Y = \begin{bmatrix} Y_{m+1} & \dots & Y_{m+p} \end{bmatrix} \quad \text{with} \quad Y_j \in \mathbb{C}^{mp\times m} \quad \forall j \in \{m+1, ..., m+p\}. \tag{28}$$

From (27) and (21), it can be deduced that the columns of YZ are linearly dependent so $\text{rank}(\tilde{Y}_i) \leq p-1$. At this stage, it suffices to proceed as in subsection 4.1 by computing matrices $\mathcal{X}_i$. Unless additional rank deficiency on $\tilde{Y}_i$, each matrix $\mathcal{X}_i$ is reduced to a vector $x_i = \mathcal{X}_i$. From these vectors, equations (10) and (12) lead to a solution K. In the case of an additional rank deficiency on $\tilde{Y}_i$, $\mathcal{X}_i$ is a matrix and some additional dof appears and can be exploited under the form of vectors β_i as defined in subsection 4.1.

The reader may have noticed that another condition was assumed: $mp > m+p$. It is actually related to the existence of $\mathcal{Z}$. It is easy to see that $\mathcal{Z}$ is non-zero if and only if $mp > m+p$. If this condition is not satisfied, a classical partial pole placement can be performed to assign only $\max(m,p)$ poles.

4.3 $mp > m+p = n$ (Kimura's non-strict condition)

The above condition implies $mp > n$ and which is Wang's condition, indicating that complete pole placement is generically possible. The equality $q = p$ can still be considered so that assigning $m+p$ poles is equivalent to assign the whole spectrum. Thus, this complete pole placement becomes a special case of the procedure described in the previous subsection. For most of the cases, Kimura's strict condition $(m+p > n)$ is no longer a limit for the complete pole assignment through modal approach. A better sufficient condition for pole placement becomes $m+p \geq n < mp$.

Remark 3 *The reader is invited to have a glimpse to [24] where some further comments on the remaining flexibility available on K are provided: only one degree of freedom is not exploited when placing the poles. Other details about the way to handle complex desired eigenvalues are given in [24]. They are not repeated here to avoid redundancy. Besides, the linear independence of the vectors $(x_i'\tilde{N}_iB)$ (item (ii) in Proposition 1) is not ensured by the procedure. It is not a drastic drawback and few models lead to dependent vectors. When it occurs, there are however some techniques that can be used to try to remedy the problem [24].*

The advantage of the above procedures are [24]:

- $\min(n, m+p)$ poles can be assigned rather than $\min(n, m+p-1)$;
- The procedure is *non-iterative*;
- The complete pole placement is easily performed when $m+p=n<mp$.

5 $m+p<n$: A step further

This section contains the actual contribution of the paper. It consists in an attempt to deal with the case $m+p<n$. The authors do not pretend to solve such a NP-hard problem, especially by a direct technique, but they want to emphasize the fact that some few cases can be easily solved, even when $m+p<n$.

The idea remains the same as in the previous sections. More precisely, the purpose is to manage to compute an admissible matrix Z, that is a matrix Z which induces a suitable rank deficiency in the various matrices $\tilde{Y}_i$. Considering the structure of Y given by (22), Y can be written

$$Y = \left[\begin{array}{cc} \mathcal{N}'B & -\mathcal{M}' \end{array}\right] \left[\begin{array}{c} \mathcal{M} \\ C\mathcal{N} \end{array}\right]. \tag{29}$$

Hence, rather than inducing rank deficiencies in the blocks of YZ, it is easier to do the same on HZ where

$$H = \left[\begin{array}{c} \mathcal{M} \\ C\mathcal{N} \end{array}\right] = \left[\begin{array}{ccccc} H_{m+1} & \dots & H_j & \dots & H_{m+q} \end{array}\right], \quad \text{with} \quad H_j \in \mathbb{C}^{(m+p)\times m}. \tag{30}$$

Therefore, assume that Z can be found so that the matrix product HZ has column rank less than p. Therefore, matrix product

$$YZ = \left[\begin{array}{cc} \mathcal{N}'B & -\mathcal{M}' \end{array}\right] HZ \tag{31}$$

is also of rank less than p and the procedure of section 4 allows us to compute matrices $\mathcal{X}_i$ that make XYZ equal zero. So the challenge consists in finding a suitable Z such that HZ is of column rank less than p, *i.e.* $p-1$. This is a hard challenge but some cases for which such a calculation is possible will now be emphasized.

To proceed, let the matrix $\mathbf{H}$ be defined by

$$\mathbf{H} = \left[\begin{array}{ccc|ccc} H_{m+1} & \dots & H_{m+p-h} & H_{m+p-h+1} & 0 & 0 \\ H_{m+1} & \dots & H_{m+p-h} & 0 & \ddots & 0 \\ H_{m+1} & \dots & H_{m+p-h} & 0 & 0 & H_{m+q} \end{array}\right] \in \mathbb{C}^{(m+p)(q-p+h-1)\times mq}, \tag{32}$$

where h is a positive integer. The idea is to compute, if possible, a matrix $\mathcal{Z}$ whose columns span the right nullspace of $\mathbf{H}$. As in subsection 4.2, let γ be any non-zero vector such that $g=\mathcal{Z}\gamma$ complies with

$$\mathbf{H}g = \mathbf{H}\left[\begin{array}{c} z_{m+1} \\ \vdots \\ z_{m+q} \end{array}\right] = 0 \quad \text{with} \quad z_j \in \mathbb{C}^m \quad \forall j \in \{m+1, ..., m+q\}. \tag{33}$$

If such a computation is possible, *i.e.* if $\mathcal{Z}$ is not reduced to zero, then the matrix product HZ is of generic rank $(p-h) < p$ Thus, from (31), matrix product

$$YZ = \begin{bmatrix} \tilde{Y}_1 \\ \vdots \\ \tilde{Y}_m \end{bmatrix}, \quad \text{with} \quad \tilde{Y}_i \in \mathbb{C}^{p\times(p-h)}\, \forall i \in \{1, ..., m\} \tag{34}$$

is such that each block $\tilde{Y}_i$ is of rank $(p-h)$ at most, allowing the computation of suitable vectors x_i as in the previous section.

But satisfying (19) is not sufficient to derive a static ouput feedback control law. Two other constraints have to be satisfied. The vectors v_i and $(x_i'\tilde{N}_iB)$ have to be linearly independent and of course, cannot be reduced to zero. It is classical to verify the independence *a posteriori*. But it is absolutely necessary to ensure that eigenvectors do not equal zero, that is, in the above procedure, to avoid that $z_j = 0$ for some j. To summarize the problem from a technical point of view, there must exist some h such that two conditions are satisfied:

1. $\mathcal{Z}$ is not reduced to zero;
2. All the vectors z_i are non zero.

The former condition means, in the generic case, that matrix $\mathbf{H}$ must have more columns than rows. In terms of model dimensions, it can be written

$$mq > (m+p)(q-p+h+1). \tag{35}$$

The later condition means that $\mathcal{Z} = \mathrm{Ker}(\mathbf{H})$ should not equal $\mathrm{Ker}(\hat{\mathbf{H}})$ where

$$\hat{\mathbf{H}} = \left[\begin{array}{ccc} H_{m+1} & \dots & H_{m+p-h} \end{array}\right] \in \mathbb{C}^{(m+p)\times m(p+p-h)}. \tag{36}$$

In terms of dimensions, this yields

$$m(p-h) < m+p. \tag{37}$$

Thus, the problem can generically be solved if some positive *integer* h satisfying (35) and (37) can be found.

Since the partial pole placement is of weak interest in practice, some tests have been performed to find which dimensions could allow complete pole placement by a direct procedure, considering that, regarding the above discussions, the procedure can be applied in three cases cases:

- $m+p \geq n$ (Kimura's non strict condition);
- state feedback and, by duality, input injection;
- $m+p < n$, (35) and (37) and, by duality, the same conditions by substituting m with p and the other way around.

For instance, with $n = 12$, table 1, where '1' denotes a successful test and '0' denotes a failure, is obtained.

$m \setminus p$	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	0	1	1	1	1	1
3	0	0	0	0	0	0	0	0	1	1	1	1
4	0	0	0	0	0	0	0	1	1	1	1	1
5	0	0	0	0	0	0	1	1	1	1	1	1
6	0	0	0	0	0	1	1	1	1	1	1	1
7	0	0	0	0	1	1	1	1	1	1	1	1
8	0	1	0	1	1	1	1	1	1	1	1	1
9	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
11	0	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1	1	1	1	1	1

Table 1: test results for $n = 12$

$m \setminus p$	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	0	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	0	1	1	1	1	1
3	0	0	0	0	0	0	0	1	1	1	1	1
4	0	0	0	0	0	0	0	1	1	1	1	1
5	0	0	0	0	0	0	1	1	1	1	1	1
6	0	0	0	0	0	1	1	1	1	1	1	1
7	0	0	0	0	1	1	1	1	1	1	1	1
8	0	1	1	1	1	1	1	1	1	1	1	1
9	0	1	1	1	1	1	1	1	1	1	1	1
10	0	1	1	1	1	1	1	1	1	1	1	1
11	0	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	1	1	1	1	1	1	1	1	1

Table 2: refreshed test results for $n = 12$

Many comments can be made thanks to Table 1. First, this table is symmetric. This is due to duality. If duality was not taken into account, then this table would not be symmetric. If one focuses on entry $(m = 2, p = 8)$, one can see that the complete pole placement is possible throught direct computation although Kimura's condition is far from being satisfied. Indeed, $m + p = 10 < n = 12$.

Another striking entry is $(m = 3, p = 8)$. For this case, one cannot find any positive integer h that complies with (35) and (37). It is a very surprising fact since it suffices to ignore one ouput to make the problem be solvable. Perhaps it means that some further investigations can be done to explain this paradox by fullfilling the authors' technique. Whatever this contradiction is, since it is always possible to ignore one ouput, Table 1 can be actualized into Table 2.

To conclude about this technique, it has been proven that Kimura's condition (even non strict) is not always necessary for a complete pole placement problem to be solved *by a direct procedure*.

6 Numerical illustration

To illustrate the previous the efficiency of the previously presented technique, an example with $n = 12$, $m = 2$, $p = 8$ is considered. Kimura'condition is then not satisified at all.

$$A = \begin{bmatrix} 0.0147 & 0.8289 & 0.8518 & 0.7134 & 0.1310 & 0.3722 & 0.5979 & 0.4984 & 0.0174 & 0.6595 & 0.3073 & 0.0743 \\ 0.6641 & 0.1663 & 0.7595 & 0.2280 & 0.9408 & 0.0737 & 0.9492 & 0.2905 & 0.8194 & 0.1834 & 0.9267 & 0.1932 \\ 0.7241 & 0.3939 & 0.9498 & 0.4496 & 0.7019 & 0.1998 & 0.2888 & 0.6728 & 0.6211 & 0.6365 & 0.6787 & 0.3796 \\ 0.2816 & 0.5208 & 0.5579 & 0.1722 & 0.8477 & 0.0495 & 0.8888 & 0.9580 & 0.5602 & 0.1703 & 0.0743 & 0.2764 \\ 0.2618 & 0.7181 & 0.0142 & 0.9688 & 0.2093 & 0.5667 & 0.1016 & 0.7666 & 0.2440 & 0.5396 & 0.0707 & 0.7709 \\ 0.7085 & 0.5692 & 0.5962 & 0.3557 & 0.4551 & 0.1219 & 0.0653 & 0.6661 & 0.8220 & 0.6234 & 0.0119 & 0.3139 \\ 0.7839 & 0.4608 & 0.8162 & 0.0490 & 0.0811 & 0.5221 & 0.2343 & 0.1309 & 0.2632 & 0.6859 & 0.2272 & 0.6382 \\ 0.9862 & 0.4453 & 0.9771 & 0.7553 & 0.8511 & 0.1171 & 0.9331 & 0.0954 & 0.7536 & 0.6773 & 0.5163 & 0.9866 \\ 0.4733 & 0.0877 & 0.2219 & 0.8948 & 0.5620 & 0.7699 & 0.0631 & 0.0149 & 0.6596 & 0.8768 & 0.4582 & 0.5029 \\ 0.9028 & 0.4435 & 0.7037 & 0.2861 & 0.3193 & 0.3751 & 0.2642 & 0.2882 & 0.2141 & 0.0129 & 0.7032 & 0.9477 \\ 0.4511 & 0.3663 & 0.5221 & 0.2512 & 0.3749 & 0.8234 & 0.9995 & 0.8167 & 0.6021 & 0.3104 & 0.5825 & 0.8280 \\ 0.8045 & 0.3025 & 0.9329 & 0.9327 & 0.8678 & 0.0466 & 0.2120 & 0.9855 & 0.6049 & 0.7791 & 0.5092 & 0.9176 \end{bmatrix};$$

$$B = \begin{bmatrix} 0.6390 & 0.6690 \\ 0.7721 & 0.3798 \\ 0.4416 & 0.4831 \\ 0.6081 & 0.1760 \\ 0.0020 & 0.7902 \\ 0.5136 & 0.2132 \\ 0.1034 & 0.1573 \\ 0.4075 & 0.4078 \\ 0.0527 & 0.9418 \\ 0.1500 & 0.3844 \\ 0.3111 & 0.1685 \\ 0.8966 & 0.3227 \end{bmatrix}; \; C = \begin{bmatrix} 0.1131 & 0.4194 & 0.4881 & 0.5651 & 0.8952 & 0.8686 & 0.0693 & 0.5695 \\ 0.8121 & 0.2130 & 0.9926 & 0.9692 & 0.9424 & 0.6264 & 0.8529 & 0.1593 \\ 0.9083 & 0.0356 & 0.3733 & 0.0237 & 0.3351 & 0.2412 & 0.1803 & 0.5944 \\ 0.1564 & 0.0812 & 0.5314 & 0.8702 & 0.4374 & 0.9781 & 0.0324 & 0.3311 \\ 0.1221 & 0.8506 & 0.1813 & 0.0269 & 0.4712 & 0.6405 & 0.7339 & 0.6586 \\ 0.7627 & 0.3402 & 0.5019 & 0.5195 & 0.1493 & 0.2298 & 0.5365 & 0.8636 \\ 0.7218 & 0.4662 & 0.4222 & 0.1923 & 0.1359 & 0.6813 & 0.2760 & 0.5676 \\ 0.6516 & 0.9138 & 0.6604 & 0.7157 & 0.5325 & 0.6658 & 0.3685 & 0.9805 \\ 0.7540 & 0.2286 & 0.6737 & 0.2507 & 0.7258 & 0.1347 & 0.0129 & 0.7918 \\ 0.6632 & 0.8620 & 0.9573 & 0.9339 & 0.3987 & 0.0225 & 0.8892 & 0.1526 \\ 0.8835 & 0.6566 & 0.1919 & 0.1372 & 0.3584 & 0.2622 & 0.8660 & 0.8330 \\ 0.2722 & 0.8912 & 0.1112 & 0.5216 & 0.2853 & 0.1165 & 0.2542 & 0.1919 \end{bmatrix}'. \tag{38}$$

The spectrum to be assigned is the following vector:

$$\Lambda = \begin{bmatrix} -0.1 & -0.2 & -0.3 & -0.4 & -0.5 & -0.6 & -0.7 & -0.8 & -0.9 & -1 & -1.1 & -1.2 \end{bmatrix}. \tag{39}$$

Applying the procedure described in section 5 on the dual model enables the designer to eventually obtain the static output feedback matrix

$$K = \begin{bmatrix} -5.0815 & -8.2561 & -1.2112 & 2.4683 & 2.6506 & -0.8361 & 3.0586 & 5.3662 \\ 1.9190 & -6.3295 & 0.1042 & -0.1048 & 1.0469 & -1.3748 & -0.7923 & 1.6888 \end{bmatrix}. \tag{40}$$

This matrix assigns the desired spectrum. It may occur that some desired eigenvalues cannot be precisely obtained. It is due to either weak controllability or sometimes to inaccurate computation of nullspaces for high dimensional models.

7 Conclusion

In this paper, an extension of [24] has been proposed. A technique to design a static output feedback pole placement control law has been presented. It relies on the notion of eigenstructure. Only direct (*i.e.* non-iterative) computation has been considered so that the technique remains very tractable and very easy to implement. Other methods may be used to solve the problem, such as [9, 22], but when Kimura's strict condition does not hold, they require iterative processes for which some parameters might be difficult to tune or some suitable initial points have to be guessed. In this work, not only, as in [24], Kimura's condition can be non strict but it has been shwon that even some cases for which Kimura's condition does not hold can be easily handled.

As further investigations, this method could be extended to the case of multiple desired closed-loop eigenvalues. But the most interesting challenge is rather to search for some other cases for which a direct computation can be performed to design a solution to this hard problem.

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