On matrix root-clustering in a combination of first order regions

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Index Terms

Robust matrix $\mathcal{D}$-stability, Matrix $\partial\mathcal{D}$-regularity, LMI.

Abstract

This paper focuses on the notion of robust matrix root-clustering in a combination $\mathcal{D}$ (i.e. union and/or intersection) of first order regions (half planes, discs, outsides of discs) that are possibly not symmetric. The concept exploited here is the $\partial\mathcal{D}$-regularity of a matrix (non intersection of its spectrum with a line or a circle $\partial\mathcal{D}$), that was introduced in [1, 2]. The purpose of the present work is the computation of the complex $\mathcal{D}$-stability radius with respect to LFT (Linear Fractional Transform) uncertainty through LMI (Linear Matrix Inequality) software. Some conservatism sometimes induced by the technique of [2] is here circumvented.

1 Introduction

The problem of robust matrix root-clustering in a region $\mathcal{D}$ of the complex plane, also referred to as matrix $\mathcal{D}$-stability problem, has been widely investigated in the last decades. Indeed, checking the robust $\mathcal{D}$-stability of the state matrix proves to be very useful to analyze the transient response of the linear uncertain model. It enables, for instance, to verify if some specified damping ratio and/or some settling time is satisfactory.

When nominal models are concerned, the well-known results on asymptotic stability can be extended to some stability regions different from the left complex plane or the unit disk. As a now classical example, let us quote the transformable Ω-regions and the associated clustering conditions based upon the resolution of a GLE (Generalized Lyapunov Equation: see the conjecture of [3]). More recently, some convex regions were handled with the so-called LMI description and the corresponding LMI condition [4]. Let us also mention the EMI-regions (Ellipsoidal Matrix Inequality [5]), quite similar to the LMI regions. Unfortunately, most of those regions are connected or even convex, which is rather

restrictive for some systems with separate dynamics or when the damping ratio is expected to lie is some desired range. For more sophisticated, such as unconnected regions, see for instance [6, 7].
But another problem, far more challenging, is the presence of uncertainty. If the uncertainty is mostly unstructured, a solution is to compute a bound on the uncertainty domain, called robust $\mathcal{D}$- stability bound, so that, assuming nominal $\mathcal{D}$-stability, robust $\mathcal{D}$-stability is preserved. The idea was introduced in [8] for stability, and extended for example to transformable Ω-regions in [9]. In the case of norm-bounded uncertainty, the notion of stability radius (ideal non conservative robust $\mathcal{D}$-stability bound) is well presented in [10]. Is is shown to be strongly related to the the notion of H_∞-norm of a realization (see [11] and the references therein). The extension to other convex regions $\mathcal{D}$ seems quite natural.
However, for non connected regions such as unions of several subregions, the computation of robust $\mathcal{D}$-stability bounds is not so straightforward. A first result is due to [12] where only the union of n disjoint discs is tackled (n being the dimension of the matrix). The so-called "good ride quality region of aircraft" is considered in [13] and, as an extension of the works on stability radius, see [14] that handles union of EMI-subregions described as an EEMI-region (Extended EMI).
Very recently, the computation of a robust $\mathcal{D}$-stability bound when $\mathcal{D}$ is any combination (union and/or combination) of first order regions was addressed in [2]. It was proved to equal the complex $\mathcal{D}$-stability radius, except for some special cases. That is exactly the point of this work. For these restrictive cases, an extension of [2] is now proposed. The main idea takes its origin in [15] and will be shown to be strongly connected to the so-called generalized KYP lemma (Kalman-Yakubovich-Popov) [16].
The paper is organized as follows: Section 2 is dedicated to the problem statement and to previously existing results, reminding the reader of the notion of $\partial\mathcal{D}$-regularity (non intersection of a matrix spectrum with some curve $\partial\mathcal{D}$) . It also highlights the limits encountered when applying these results. Section 3 proposes a new formulation of the problem including a new description of the boundary of $\mathcal{D}$. Section 4 states the results for both nominal and robust point of view. A connection with the generalized KYP lemma is given. Section 5 provides a simple illustrative example before the paper is concluded in section 6.

Notations :

M' denotes the transpose conjugate of matrix M so s' is the conjugate of complex number s. The 2-norm of M induced by the Euclidean vector norm is denoted by $||M||_2$. I_n is the identity matrix of order n and 0 is a null matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Loëwner *i.e.* > 0 (resp. < 0) means positive (resp. negative) definite. $\otimes$ denotes the Kronecker product. At last, $M^H = M + M'$.

2 Motivation

In this section, the first part is dedicated to the problem statement. The second one recalls some solution of the literature. The third one points out some drawback of this solution.

2.1 Problem statement

In this subsection, the problem is stated as in paper [2] of which the present one can be considered as a sequel. Consider an uncertain matrix that reads

$$A_c(\Delta) = A + B\bar{\Delta}C \quad \text{with} \quad \bar{\Delta} = \Delta(I - D\Delta)^{-1}. \tag{1}$$

In the above expression, Δ is constant, unknown and belongs to $\mathcal{B}(\rho)$, the ball of all matrices $\Delta \in \mathbb{C}^{q\times r}$ satisfying $||\Delta||_2 \leq \rho$ and the existence of $\bar{\Delta}$ (well poseness). $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{r\times n}$ and $D \in \mathbb{C}^{r\times q}$ are known. This description is referred to as LFT (Linear Fractional Transform)-uncertainty. Now define the set of indices $\mathbf{k} = \{1, \ldots, m\}$ and consider the next Hermitian matrices:

$$R_k = R_k' = \begin{bmatrix} r_{k00} & r_{k01} \\ r'_{k10} & r_{k11} \end{bmatrix} \in \mathbb{C}^{2\times 2}, \quad \forall k \in \mathbf{k}, \tag{2}$$

from which the following curves are defined:

$$\partial\mathcal{D}_k = \left\{ \; s \in \mathbb{C} \,|\, f_k(s) = \begin{bmatrix} 1 & s' \end{bmatrix} R_k \begin{bmatrix} 1 \\ s \end{bmatrix} = 0 \; \right\} \quad \forall k \in \mathbf{k}. \tag{3}$$

Relevant curves are lines or circles which can be nonsymmetric with respect to the real axis. Each curve $\partial\mathcal{D}_k$ enables to define an associated region:

$$\mathcal{D}_k \; = \; \{s \in \mathbb{C} \mid f_k(s) < 0\} \quad \forall k \in \mathbf{k}. \tag{4}$$

Clearly, $\mathcal{D}_k$ denotes either one side or the other side of the boundary $\partial\mathcal{D}_k$. It can then be a half plane, a disc or the outside of a disc. It is an open region (*i.e.* not including $\partial\mathcal{D}_k$) in order to encompass the concept of asymptotic stability.

Also define the region $\mathcal{D}$ as a combination, *i.e.* any union and/or intersection of the various sub-regions $\mathcal{D}_k$. Such a formulation of $\mathcal{D}$ clearly enables a very large choice of clustering regions. This contribution aims at computing the complex $\mathcal{D}$-stability radius. More precisely, assume that A is $\mathcal{D}$-stable *i.e* $\lambda(A) \subset \mathcal{D}$. Define $r_{\mathcal{D}}$ as the largest value of ρ, the radius of $\mathcal{B}(\rho)$, such that A_c defined in (1) remains $\mathcal{D}$-stable for any $\Delta \in \mathcal{B}(\rho)$. Such a value is the so-called complex $\mathcal{D}$-stability radius. For this purpose, the concept of $\partial\mathcal{D}$-regularity was introduced in [2] in order to derive a lower bound $\rho^\star$, which often (but not always) equals $r_{\mathcal{D}}$. This former contribution is now summarized.

2.2 Some approach based on $\partial\mathcal{D}$-regularity

In [2], some LMI conditions were proposed in order to compute $\rho^\star$, a good lower bound of $r_{\mathcal{D}}$. This approach is based upon the concept of $\partial\mathcal{D}$-regularity.

Definition 1 *Let $\partial\mathcal{D}$ be any curve in the complex plane, then matrix $A \in \mathbb{C}^{n\times n}$ is called:*

- *$\partial\mathcal{D}$-singular when $\lambda(A) \cap \partial\mathcal{D} \neq \emptyset$;*
- *$\partial\mathcal{D}$-regular when $\lambda(A) \cap \partial\mathcal{D} \; = \; \emptyset$.*

To sum up [2], roughly speaking, the uncertain matrix A_c defined in (1) is robustly $\mathcal{D}$-stable if it is robustly $\partial\mathcal{D}$-regular and if the nominal matrix A is $\mathcal{D}$-stable. It led to the following theorem [2], deduced from the KYP lemma [17].

Theorem 2 *[2] Let A_c (with D=0), $\mathcal{D}$ and $\partial\mathcal{D}$ be respectively an uncertain matrix as defined in (1), a region as defined in subsection 2.1 and its boundary. Then A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exist Hermitian matrices P_k, $k = 1, ..., m$, such that:*

$$\begin{bmatrix} r_{k_{00}} P_k + (r_{k_{10}} P A_k)^H + r_{k_{11}} A' P_k A & r_{k_{10}} PB + r_{k_{11}} A' PB \\ r'_{k_{10}} B'P + r_{k_{11}} B'PA & r_{k_{11}} B'PB \end{bmatrix} +$$
$$\begin{bmatrix} C'C & 0 \\ 0 & -\gamma I_n \end{bmatrix} < 0 \quad \forall k \in \mathbf{k}. \tag{5}$$

where $\gamma = \rho^{-2}$. In this event, A_c keeps the same number of eigenvalues inside $\mathcal{D}$ for any $\Delta \in \mathcal{B}(\rho)$. Moreover, if $\partial\mathcal{D}$ can be identified to $\partial\mathcal{U} = \bigcup_{k=1}^{m} \partial\mathcal{D}_k$, then condition (5) is also necessary.

Remark 3 *Note that the original version of the above theorem gives insights on the inertia of "Lyapunov matrices" which is part of the contribution of [2] but these considerations on inertias will not be followed here.*

2.3 Limits of the technique

The value $\rho^\star$ of ρ, obtained by minimizing γ while satisfying (5), equals $\varrho_{\partial\mathcal{U}}$, the so-called complex $\partial\mathcal{U}$-regularity radius, which is itself a lower bound of $\varrho_{\partial\mathcal{D}}$, the complex $\partial\mathcal{D}$-regularity radius. It is shown in [2] that if A is $\mathcal{D}$-stable and if $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$, then $\varrho_{\partial\mathcal{U}}$ actually equals $r_{\mathcal{D}}$, the complex $\mathcal{D}$-stability radius. The $\mathcal{D}$-stability of A is of course a necessary assumption to check the robust $\mathcal{D}$-stability of A_c but the assumption $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$ is not always satisfied. Hence, the result proposed in Theorem 2 can be conservative. Theorem 2 can even fail as highlighted by fig. 1 where $\partial\mathcal{U} \cap \mathcal{D} \neq \emptyset$. On this figure, $\varrho_{\partial\mathcal{U}}$ cannot be computed because the matrix is $(\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2)$-stable whereas it is both $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$-singular. If the radii of the disks are slightly increased, the LMI test does not fail (since nominal $\partial\mathcal{D}$-regularity holds) but the bound value is far weaker than the $\mathcal{D}$-stability radius due to the nearness of both the nominal eigenvalues and the boundaries of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$.

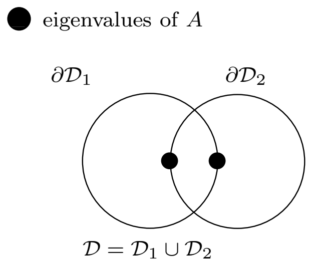

Figure 1: $\varrho_{\partial\mathcal{U}}$ cannot be computed

The present contribution aims at going round the previously presented obstacle. Moreover, LFT uncertainty is handled rather than norm-bounded one. The main drawback of the above statement is that there is no rigorous description of $\partial\mathcal{D}$, the boundary of region $\mathcal{D}$, when $\partial\mathcal{U} \cap \mathcal{D} \neq \emptyset$. Hence the necessity of reformulating the problem.

3 New formulation of the problem

Let the m curves $\partial\mathcal{D}_k$ be defined as in (3). Also let other matrices be defined by

$$\Phi_{k,l} = \begin{bmatrix} \phi_{k,l_{00}} & \phi_{k,l_{10}} \\ \phi'_{k,l_{10}} & \phi_{k,l_{11}} \end{bmatrix} = \Phi'_{k,l} \in \mathbb{C}^{2\times 2} \quad \forall\{k,l\} \in \mathbf{k} \times \mathbf{l_k}, \tag{6}$$

where $\mathbf{l_k} = \{1, ..., \bar{l}_k\}$. The sets $\mathcal{E}_k$ are defined by

$$\mathcal{E}_k = \{s \in \mathbb{C} \,|\, g_{k,l}(s) = \\ \begin{bmatrix} 1 & s' \end{bmatrix} \Phi_{k,l} \begin{bmatrix} 1 \\ s \end{bmatrix} \geq 0 \; \forall l \in \mathbf{l}_k\} \; \forall k \in \mathbf{k}. \tag{7}$$

Let now the sets $\partial\Gamma_k$ be defined by

$$\partial\Gamma_k = \partial\mathcal{D}_k \cap \mathcal{E}_k = \{s \in \mathbb{C} \,|\, f_k(s) = 0 \;\&\; g_{k,l}(s) \geq 0 \\ \forall l \in \mathbf{l}_k\} \; \forall k \in \mathbf{k}. \tag{8}$$

Each set $\partial\Gamma_k$ is the intersection of a line or a circle with $\bar{l}_k$ closed half planes, discs or outsides of discs, possibly not symmetric. In other words, it is a line or a circle from which only some points are selected through the choice of $\Phi_{k,l}$.
At last, from these considerations, one defines the following class of curves:

$$\partial\mathcal{D} = \bigcup_{k=1}^{m} \partial\Gamma_k, \tag{9}$$

where the sets $\partial\Gamma_k$ obeys (8) with (3) and (6-7).
Once this new class of curves is defined, the stability region $\mathcal{D}$ is described as any region whose boundary complies with (9). Assuming that A is $\mathcal{D}$-stable, computing a robust $\mathcal{D}$-stability bound is the same as computing a robust $\partial\mathcal{D}$-regularity bound.

Remark 4 *As an example of the utility of $l_k > 1$, it is possible to approximate a complicated region by the combination of several regions $\mathcal{D}_k$. For instance, the ellipse, which is a second order EMI-region for which the complex $\mathcal{D}$-stability radius cannot be computed, can be approximated by the union of three disks, the middle one intersecting with the two other ones. Besides, considering the equivalence between computing the complex stability radius and the $\mathcal{H}_\infty$-norm of a linear model, it becomes possible to exclude* several *peaks from this $\mathcal{H}_\infty$ analysis while preserving only one LMI condition.*

4 Complex $\mathcal{D}$-stability radius

This section provides a solution to the above-stated problem. First, a condition for nominal $\partial\mathcal{D}$-regularity is given. Then it is used to derive a technique for the computation of a robust $\mathcal{D}$-stability bound. A connection with the generalized KYP lemma is also emphasized.

Theorem 5 *Consider a matrix $A \in \mathbb{C}^{n\times n}$ and let $\partial\mathcal{D}$ be a set as defined in (9). Matrix A is $\partial\mathcal{D}$-regular if and only if there exist complex matrices $P_k = P_k'$, $\forall k \in \mathbf{k}$ and complex matrices $Q_{k,l} = Q_{k,l}' > 0$, $\forall k \in \mathbf{k}$, $\forall l \in \mathbf{l_k}$ solutions to*

$$\begin{array}{l} \mathcal{W}_k(A, P_k, \{Q_{k,l}\}) = \\ \left[\begin{array}{c} I \\ A \end{array}\right]' \left(R_k \otimes P_k + \sum_{l=1}^{\bar{l}_k} (\Phi_{k,l} \otimes Q_{k,l}) \right) \left[\begin{array}{c} I \\ A \end{array}\right] < 0, \forall k \in \mathbf{k}. \end{array} \tag{10}$$

Proof : First note that (10) can be written

$$\mathcal{W}_k(A, P_k, \{Q_{k,l}\}) = F(A, R_k, P_k) + \sum_{l=1}^{\bar{l}_k} F(A, \Phi_{k,l}, Q_{k,l}) < 0 \tag{11}$$

with the generic notation

$$F(A, \Psi, X) = \left[\begin{array}{c} I \\ A \end{array}\right]' (\Psi \otimes X) \left[\begin{array}{c} I \\ A \end{array}\right]. \tag{12}$$

Sufficiency : Assume that (11) holds. For any eigenvalue of A, denoted by λ, there exists a non zero vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$. By congruence, it follows that

$$\begin{array}{c} v'\mathcal{W}_k(A, P_k, \{Q_{k,l}\})v < 0 \\ \Leftrightarrow f_k(\lambda)v'Pv + \displaystyle\sum_{l=1}^{\bar{l}_k} g_{k,l}(\lambda)v'Q_{k,l}v < 0. \end{array}$$

Thus, if $\lambda \in \partial\Gamma_k$, then the first term above is zero (since $\lambda \in \partial\mathcal{D}_k \Leftrightarrow f_k(\lambda) = 0$) and the second one is positive or zero (since $v'Q_{k,l}v > 0 \forall v$ & $\lambda \in \mathcal{E}_k \Leftrightarrow g_{k,l}(\lambda) \geq 0 \forall \{k,l\}$). Such an assumption contradicts the inequality. Hence, it is false and $\lambda \notin \partial\Gamma_k, \forall k \in \mathbf{k}$ so $\lambda \notin \partial\mathcal{D}$.

Necessity : If A is $\partial\mathcal{D}$-regular, then it is $\partial\Gamma_k$-regular for any $k \in \mathbf{k}$. It suffices to consider one curve $\partial\Gamma_k$. The n eigenvalues of A, denoted by λ_i, verify one of the three following inequalities :

(i) $f_k(\lambda_i) > 0$;

(ii) $f_k(\lambda_i) < 0$;

(iii) $f_k(\lambda_i) = 0 \quad \& \quad g_{k,l}(\lambda_i) < 0 \quad \forall l \in \mathbf{l}_k$.

Let the matrix Λ be defined as follows :

$$\Lambda \;=\; \underset{i=1,\ldots,n}{\mathrm{diag}}\{\lambda_i\} \;=\; \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{bmatrix}.$$

$\Lambda_1 \in \mathbb{C}^{n_{k-}}$ contains the λ_i satisfying condition (i). $\Lambda_2 \in \mathbb{C}^{n_{k+}}$ contains the λ_i satisfying condition (ii). $\Lambda_3 \in \mathbb{C}^{n_{k0}}$ contains the λ_i satisfying condition (iii). It is possible to find two scalar numbers $\alpha > 0$ and $\beta > 0$ such that

$$\begin{cases} \alpha F(\Lambda_1, R_k, -I_{n_-}) + \beta \displaystyle\sum_{l=1}^{\bar{l}_k} G_{k,l}(\Lambda_1, \Phi_{k,l}, I_{n_{k_-}}) < 0 \\ \alpha F(\Lambda_2, R_k, I_{n_+}) + \beta \displaystyle\sum_{l=1}^{\bar{l}_k} G_{k,l}(\Lambda_2, \Phi_{k,l}, I_{n_{k_+}}) < 0 \\ \alpha F(\Lambda_3, R_k, 0) + \beta \displaystyle\sum_{l=1}^{\bar{l}_k} G_{k,l}(\Lambda_3, \Phi_{k,l}, I_{n_{k_0}}) < 0 \end{cases}$$

where these three inequalities are deduced from (i),(ii) and (iii) respectively. Indeed, since the matrices involved in the inequalities are diagonal, it suffices to choose α large enough compared with β to ensure that inequality holds for each diagonal entry. With $\mathcal{I}$ defined by

$$\mathcal{I} = \alpha \begin{bmatrix} -I_{n_{k_-}} & 0 & 0 \\ 0 & I_{n_{k_+}} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

it follows that

$$\mathcal{W}_k(\Lambda, \mathcal{I}, \{\beta I\}) \;=\; \underbrace{F_k(\Lambda, \mathcal{I})}_{<0} \;+\; \sum_{l=1}^{\bar{l}_k} G_{k,l}(\Lambda, \beta I) \;<\; 0. \tag{13}$$

Let $\mathcal{J}$ be the Jordan canonical form of A. There exists a sequence of nonsingular matrices T_h such that [4]:

$$\lim_{h\to\infty} (T_h \mathcal{J} T_h^{-1}) \;=\; \Lambda \tag{14}$$

(for example, if $\mathcal{J}$ is a simple Jordan block $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, T_h equals $\begin{bmatrix} \frac{1}{h} & 0 \\ 0 & 1 \end{bmatrix}$). Since $\mathcal{W}_k$ is continuous with respect to its first argument, it follows that

$$\lim_{h\to\infty} \mathcal{W}_k(T_h \mathcal{J} T_h^{-1}, \mathcal{I}, \{\beta I\}) \;\leq\; 0. \tag{15}$$

Let $\mathcal{T}$ be an instance of T_h for a sufficiently large $h < \infty$ (ensuring that $\mathcal{T}^{-1}$ exists). One gets

$$\mathcal{W}_k(\mathcal{T}\mathcal{J}\mathcal{T}^{-1}, \mathcal{I}, \{\beta I\}) \;<\; 0.$$

$$\Rightarrow \mathcal{T}'\mathcal{W}_k(\mathcal{T}\mathcal{J}\mathcal{T}^{-1}, \mathcal{I}, \{\beta I\})\mathcal{T} = \mathcal{W}_k(\mathcal{J}, \mathcal{T}'\mathcal{I}\mathcal{T}, \{\beta \mathcal{T}'\mathcal{T}\}) \;<\; 0.$$

Let V be a modal matrix such that $\mathcal{J} = VAV^{-1}$. Another congruence enables to recover (11) with

$$P_k = V'\mathcal{T}'\mathcal{I}\mathcal{T}V; \quad Q_{k,l} = \beta V'\mathcal{T}'\mathcal{T}V > 0 \quad \forall l \in \mathbf{l}_k. \square \tag{16}$$

Based upon Theorem 5, a robust $\partial\mathcal{D}$-regularity bound can be deduced from the next theorem.

Theorem 6 *Let A_c, $\mathcal{D}$ and $\partial\mathcal{D}$ be an uncertain matrix defined as in (1), a clustering region, and its boundary complying with (9). Matrix A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$* **if** *there exist a set $\mathcal{P}$ of matrices $P_k = P_k'$, $\forall k \in \mathbf{k}$ and m sets of matrices $Q_{k,l} = Q_{k,l}' > 0$ such that, $\forall k \in \{1, ..., m\}$,*

$$\mathbf{W}_U(M, \mathcal{P}, \{Q_{k,l}\}) =$$

$$\begin{bmatrix} I_n & 0 & 0 \\ A & B & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_r \end{bmatrix}' \left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{R_k \otimes P_k + \sum_{l=1}^{\bar{l}_k} \Phi_{k,l} \otimes Q_{k,l}} & 0 & \begin{array}{c} C' \\ \hline 0 \end{array} \\ \hline \multicolumn{2}{c|}{0} & -I_q & D' \\ \hline C & 0 & D & -\gamma I_r \end{array}\right] \times$$
$$\begin{bmatrix} I_n & 0 & 0 \\ A & B & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_r \end{bmatrix} < 0, \tag{17}$$

with $\gamma = \rho^{-2}$.

Proof : Applying the following congruence,

$$\begin{bmatrix} I & 0 \\ \bar{\Delta}C & 0 \\ 0 & I \end{bmatrix}' \mathbf{W}_U(M, \mathcal{P}, \{Q_{k,l}\}) \begin{bmatrix} I & 0 \\ \bar{\Delta}C & 0 \\ 0 & I \end{bmatrix} < 0,$$

leads to

$$\begin{bmatrix} \mathcal{W}_k(A_c, \{P_k\}, \{Q_{k,l}\}) - C'\bar{\Delta}'\bar{\Delta}C & C'(I + \bar{\Delta}'D' \\ (I + D\bar{\Delta})C & -\gamma I \end{bmatrix} < 0.$$

By vitue of Schur's lemma, it follows that

$$\begin{array}{c} \mathcal{W}_k(A_c, \{P_k\}, \{Q_{k,l}\}) < \\ C'\bar{\Delta}'\bar{\Delta}C - \rho^2 C'(I + \bar{\Delta}'D')(I + D\bar{\Delta})C \\ \Leftrightarrow \mathcal{W}_k(A_c, \{P_k\}, \{Q_{k,l}\}) < \\ C'(\bar{\Delta}'\bar{\Delta} - \rho^2(I + \bar{\Delta}'D')(I + D\bar{\Delta}))C. \end{array} \tag{18}$$

The left-hand side member is negative definite if and only if the right-hand side member is negative semi-definite *i.e.*

$$\bar{\Delta}'\bar{\Delta} \leq \rho^2(I + \bar{\Delta}'D')(I + D\bar{\Delta}).$$

If the LFT uncertainty is well-posed, it can be deduced that

$$\bar{\Delta} = \Delta(I - D\Delta)^{-1} = (I - \Delta D)^{-1}\Delta = \Delta(I + D\bar{\Delta}),$$

and one gets

$$(I+\bar{\Delta}'D')\Delta'\Delta(I+D\bar{\Delta}) \leq \rho^2(I+\bar{\Delta}'D')(I+D\bar{\Delta}).$$

which is always satisfied when $\Delta \in \mathcal{B}(\rho)$. Hence, the left-hand side member of (18) is negative definite so, by virtue of Theorem 5, A_c is $\partial\Gamma_k$-regular $\forall k$ and thus, $\partial\mathcal{D}$-regular. □

Minimizing γ to $\gamma^\star$ under LMI constraints (6) enables to derive $\rho^\star = {\gamma^\star}^{-1/2}$, a robust $\partial\mathcal{D}$-regularity bound. If A is assumed $\mathcal{D}$-stable, by continuity of the spectrum of A_c with respect to Δ, A_c remains $\mathcal{D}$-stable over $\mathcal{B}(\rho^\star)$. Therefore, $\rho^\star$ becomes a robust $\mathcal{D}$-stability bound. Now, the question of conservatism naturally comes. Is this bound conservative ?

Conjecture 1 *Condition (17) in Theorem 6 is also necessary.*

As a consequence, $\rho^\star$ is the complex $\partial\mathcal{D}$-regularity radius $\varrho_{\partial\mathcal{D}}$ and, if A is $\mathcal{D}$-stable, also equals $r_{\mathcal{D}}$, the complex $\mathcal{D}$-stability. This conjecture could actually be proven. Indeed, when $\bar{l}_k = 1$, (17) falls out as a special case of the so-called generalized KYP lemma deduced from the generalized S-procedure [15, 16]. It would suffice to adapt the arguments of [16] in order to include $\bar{l}_k > 1$ but it is not achieved here since it would require many borrowings from [16]. The reader is invited to refer to this very discerning paper to complete the connection.

5 Short illustration

This very simple example highlights the improvement of the present work compared to [2]. Consider equation (1) with

$$A = \begin{bmatrix} -1.1388 & 1.8993 \\ -2.5882 & -2.8612 \end{bmatrix},$$

$$B = C = I_2, \quad D = 0.$$

The nominal spectrum is $\lambda(A) = \{-2.5; -1.5\}$. The clustering region $\mathcal{D}$ is chosen as the union of two intersecting open discs $\mathcal{D}_1$ and $\mathcal{D}_2$ of respective boundaries $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$, centred around the eigenvalues of A and both of radius 1. Hence, A is $\mathcal{D}$-stable. Since A is $\partial\mathcal{D}_1$-singular and $\partial\mathcal{D}_2$-singular, the LMI condition of Theorem 2 cannot be applied and is simply found infeasible. The reason is that the parts of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ that belong to $\mathcal{D}$ can not be excluded from the $\partial\mathcal{D}$-regularity test. Even a nominal test would fail. In the present work, it suffices to apply Theorem 6 with considering first $\partial\mathcal{D}_1$ intersecting the outside of $\mathcal{D}_2$ and second $\partial\mathcal{D}_2$ intersecting the outside of $\mathcal{D}_1$. It leads to $\rho^\star = \varrho_{\partial\mathcal{D}} = r_{\mathcal{D}} = 0.39212$. which is the complex $\mathcal{D}$-stability radius.
Of course, it is a quite trivial example but it highlights the limits of the results provided in [2] and shows the relevance of the present contribution when Theorem 2 fails or brings too much conservatism.

6 Conclusion

In this paper, a tool was proposed to analyze the robustness of matrix root-location when this matrix is subject to an LFT uncertainty. The originality lies in the fact that the clustering region is any combination (union and/or intersection) of first order regions (half planes, discs, outsides of discs).The results hinge upon the notion of $\partial\mathcal{D}$-regularity of a matrix (non intersection of the matrix spectrum with a curve $\partial\mathcal{D}$) and improve former results of the literature. The robustness bound computed owing to the presented technique enables to derive the so-called complex $\mathcal{D}$-stability radius. Although it is

based on classical matrix arguments related to basic results on root-clustering [18, 3, 4], it was shown to be strongly connected to the generalized KYP lemma [16].
As further investigations, a possible extension to polytopic LFT uncertainty would be probably possible, but with conservatism. Another challenge would be to address the problem of real $\mathcal{D}$-stability radius, for instance with exploiting the idea of [11].

References

[1] O. Bachelier and B. Pradin, "∂D-regularity for robsut matrix root-clustering," in *Robust Control Design, ROCOND*, Milan, Italie, 2003.

[2] O. Bachelier, D. Henrion, B. Pradin, and D. Mehdi, "Robust Matrix Root-Clustering of a Matrix in Intersections or Unions of Subregions," *SIAM Journal of Control and Optimization*, vol. 43, no. 3, pp. 1078–1093, 2004.

[3] S. Gutman and E. I. Jury, "A general theory for matrix root-clustering in subregions of the complex plane," *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 853–863, August 1981.

[4] M. Chilali and P. Gahinet, "H_∞ Design with Pole Placement Constraints : An LMI Approach," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 358–367, March 1996.

[5] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust D-stability condition for real convex polytopic uncertainty," *Systems & Control Letters*, vol. 40, no. 1, pp. 21–30, May 2000.

[6] M. Chilali, "Méthode LMI pour la synthèse multi-critère," Université Paris IX-Dauphine, U.F.R. Mathématiques de la Décision, Paris-France, Thse de doctorat, 1996.

[7] O. Bachelier and D. Mehdi, "Robust D_U-stability analysis," *International Journal of Robust and Nonlinear Control*, vol. 13, no. 6, pp. 533–558, May 2003.

[8] R. V. Patel and M. Toda, "Quantitative measures of robustness for multivariable systems," in *Joint Automatic Control Conference*, TP-8A, San Francisco, CA, 1980.

[9] R. K. Yedavalli, "Robust root clustering for linear uncertain systems using generalized Lyapunov theory," *Automatica*, vol. 29, no. 1, pp. 237–240, January 1993.

[10] D. Hinrichsen and A. J. Pritchard, "Stability radii of linear systems," *Systems & Control Letters*, vol. 7, no. 1, pp. 1–10, February 1986.

[11] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. Doyle, "A formula for computation of the real stability radius," *Automatica*, vol. 31, no. 6, pp. 879–890, 1995.

[12] C. R. Ashokkumar and R. K. Yedavalli, "Eigenstructure perturbation analysis in disjointed domains for linear uncertain systems," *International Journal of Control*, vol. 67, no. 6, pp. 887–899, 1997.

[13] S. Wang, "Robust pole clustering in a good ride quality region of aircraft for matrices with structured uncertainties," *Automatica*, vol. 39, no. 3, pp. 525–532, March 2003.

[14] J. Bosche, O. Bachelier, and D. Mehdi, "An approach for robust matrix root-clustering analysis in a union of regions," *To appear in IMA Journal of Mathematical Control and Information*, 2004.

[15] T. Iwasaki, G. Meinsma, and M. Fu, "Generalized S−procedure and finite frequency KYP lemma," *Mathematical Problems in Engineering*, vol. 6, pp. 305–320, 2000.

[16] T. Iwasaki and S. Hara, "Generalized KYP lemma : unified frequency domain inequalities with design applications," *IEEE Transactions on Automatic Control*, vol. 50, no. 1, pp. 41–59, 2005.

[17] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Systems* & *Control Letters*, vol. 28, pp. 7–10, 1996.

[18] J. L. Howland, "Matrix equations and the separation of matrix eigenvalues," *Journal of Mathematical Analysis and Applications*, vol. 33, pp. 683–691, 1971.