On Pole Placement via Eigenstructure Assignment Approach

Soumis à IEEE Transactions on Automatic Control

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Version 1

Résumé

This paper comes back to the hard problem of pole placement by static output feedback : let a triplet of matrices $\{A; B; C\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ be given, find a matrix $K \in \mathbb{R}^{m\times p}$ such that the spectrum of $A+BKC$ equals a specified set. Based upon an eigenstructure assignment approach, a simple technique that generically assigns $m+p$ roots is proposed when $mp > m+p$. It therefore enables to assign the whole of the desired spectrum when $m+p \le n < mp$.

Mots-clés

Pole placement, eigenstructure assignment, Kimura's condition.

1 Introduction

Due to its interest in practical control applications, pole placement of linear models by static output feedback has received a large amount of attention since the first contributions of Davison [1–3]. The associated problem consists in considering a triplet of matrices $\{A; B; C\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ and in finding a matrix $K \in \mathbb{R}^{m\times p}$ such that the spectrum of the closed-loop matrix $A_c = A + BKC$ equals a set of arbitrarily specified values $\{\lambda_i, i = 1, ..., n\}$. At first sight, there should be enough components in K to assign n poles if $mp \ge n$. However, due to the nonlinearity of the problem, such a simple reasoning cannot be seriously followed. Nevertheless, in [4, 5], it was proved that $mp \ge n$ is actually a necessary and sufficient condition for the existence of a **complex** gain K that solves the problem.

For a real feedback, this condition does not hold anymore as shown by counterexamples provided in [6]. In that case, a first sufficient condition for generic assignability was proposed in [7] : $m+p > n$. It is often referred to as Kimura's condition. The same condition was also obtained in [5]. More recently, a less restrictive sufficient condition for generic assignability was established in [8] : $mp > n$ (only generic as illustrated by the counterexample of [9]). It was also reached through a different approach in [10, 11].

But even when one can reasonably hope that some real solution K exists (*i.e.* when $mp > n$), it matters to have a technique to derive such a gain at one's disposal. The only one available to the authors' knowledge was presented in [8]. It requires to perform an optimization process based on Newton's method. If optimization is to be avoided and direct computation is preferred, the best results are those connected to the notion of eigenstructure assignment. Several techniques enable to address the static output feedback problem. They rely on the pioneer work of [12] on state feedback. Among the possible approaches, the most significant ones are the parametric approach [13], the elegant geometric approach [14] and the approach relying on two Sylvester equations coupled by an orthogonality conditions [15]. Let another less known contribution be quoted : [16]. The advantage of the geometric and parametric approaches is that the degrees of freedom are easily identified but the desired spectrum must comply with some specified partition. Note that the latter one is less restrictive than the former one since the spectra of A and A_c can intersect each other. The "Sylvester" approach is even less restrictive about the partition of the spectrum and is very tractable for implementation. It appears that Kimura's condition is generically necessary for those various techniques to be applied. In other words, as far as these methods are concerned (in their original presentations), Kimura's condition can nearly be seen as a necessary and sufficient condition for generic pole assignability by output feedback. That is why, from a practical point of view, it remains the most popular one.

However, since a better generically sufficient condition is $mp > n$, many authors aimed their efforts towards the case $m + p \leq n$. Recently, an explicit form of the assignment equations with respect to the entries of the feedback matrix was proposed in [17] through an algebraic approach.. It may enable to address the problem in some special cases among which the case $m = p = 2$, $n = 4$ is the most interesting. Concerning eigenstructure approach, discerning insights can be found in [18] but the most powerful result might be the one proposed in [19]. In this contribution, an optimization procedure may enable to solve the case $m+p = n$ (but some convenient initialization has to be guessed). The present paper must be connected to [19]. When $mp > m + p$, using the classical eigenstructure assignment approach, it aims at assigning $m+p$ eigenvalues of A_c by direct computation of K, while avoiding an optimization process. The complete spectrum assignment can therefore generically be tackled for the case $m + p = n < mp$.

Section 2 recalls some classical properties of the eigenstructure after the problem is presented. In Section 3, the main result is stated. It consists in characterizing the spectrum assignment by an "assignment equation". In section 4, assignment techniques are deduced from this equation for the above-mentioned cases. Section 5 proposes some numerical illustration before the paper is concluded in Section 6.

2 Problem statement and Background

The problem to be addressed is now stated.

Problem 1 *Let $\{A; B; C\} \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ be a triplet of matrices satisfying the following assumptions :*

(i) (A, B, C) *is minimal;*

(ii) $mp > n$ *(Wang's condition);*

(iii) $\mathrm{rank}(B) = m \leq n \quad \& \quad \mathrm{rank}(C) = p \leq n$*;*

(iv) $mp > m + p$*.*

Let $\{\lambda_i,\ i = 1, ..., n\}$ be a set of n complex distinct eigenvalues such that $\{\lambda_i,\ i = 1, ..., m\}$ and $\{\lambda_j,\ j = m+1, ..., n\}$ are closed under conjugation. Find a matrix $K \in \mathbb{R}^{m\times p}$ such that the spectrum of $A_c = A + BKC$ equals $\{\lambda_i,\ i = 1, ..., n\}$.

Condition (iv) means that Wang's condition is stronger than Kimura's one for the considered models. It will be justified when partial placement is handled. Also note that only distinct eigenvalues (*i.e.* with multiplicity 1) are here considered for the sake of conciseness. However, the case of multiple eigenvalues can also be tackled. An actual restriction is the separation of the whole spectrum between two self-conjugate subspectra which does not encompass all the possible pole placement problems. The attention is focused on the eigenstructure (eigenvalues and eigenvectors) of A_c. The reasoning is borrowed from [12] and is a basis for the results of Section 3. The spectrum of A_c is associated to right eigenvectors v_i, $i = 1, ..., n$ by the relations

$$A_c v_i = \lambda_i v_i \quad \forall i \in \{1, ..., n\} \tag{1}$$

$$\Leftrightarrow (A - \lambda_i I_n) v_i + BKC v_i = 0 \quad \forall i \in \{1, ..., n\} \tag{2}$$

$$\Leftrightarrow (A - \lambda_i I_n) v_i + B w_i = 0 \quad \forall i \in \{1, ..., n\} \tag{3}$$

where the vectors $w_i = KCv_i$ are known as the input directions. Consequently, v_i belongs to the set $\mathcal{S}_i$ defined by

$$\mathcal{S}_i = \{v \in \mathbb{C}^n \,|\, (\exists w \in \mathbb{C}^m \,|\, (A - \lambda_i I_n) v + Bw = 0)\} \quad \forall i \in \{1, ..., n\}. \tag{4}$$

This set is the so-called (A, B)-characteristic subspace [14] and is of dimension m when (A, B) is controllable. To derive v_i in $\mathcal{S}_i$, let the vectors π_i be defined by $\pi_i = \left[\begin{array}{cc} v_i' & w_i' \end{array}\right]'$. Then

$$T_i \pi_i = 0 \quad \text{with} \quad T_i = \left[\begin{array}{cc} A - \lambda_i I_n & B \end{array}\right] \quad \forall i \in \{1, ..., n\} \tag{5}$$

and thus

$$\pi_i \in \mathrm{Ker}(T_i). \tag{6}$$

Let R_i be a matrix the columns of which span the right nullspace of T_i *i.e.*

$$\mathrm{Span}(R_i) = \mathrm{Ker}(T_i) = \left[\begin{array}{c} N_i \\ M_i \end{array}\right] \quad \forall i \in \{1, ..., n\}, \tag{7}$$

where $N_i \in \mathbb{C}^{n\times m}$ and $M_i \in \mathbb{C}^{m\times m}$. Each vector π_i is then characterized by some non-zero parameter vector $z_i \in \mathbb{C}^m$ such that

$$\pi_i = R_i z_i \quad \forall i \in \{1, ..., n\}, \tag{8}$$

from which it can be deduced that

$$\left\{\begin{array}{l} v_i = N_i z_i \\ \\ w_i = M_i z_i \end{array}\right. \quad \forall i \in \{1,...,n\}. \tag{9}$$

z_i represents the degrees of freedom (dof) available to assign v_i as a right eigenvector associated to λ_i. Since v_i (or w_i) can be scaled by any non-zero factor (one talks about eigendirection rather than eigenvector), each z_i provides $(m-1)$ dof. When $C = I_n$ (state feedback), any choice of z_i, $i = 1,...,n$, ($z_i \in \mathbb{R}^m$ if $\lambda_i \in \mathbb{R}$, and $z_i = \bar{z}_j$ if $\lambda_i = \bar{\lambda}_j$) such that the various v_i are linearly independent yields a matrix

$$K = WV^{-1} \quad \text{with} \quad \left\{\begin{array}{lcl} V & = & \left[\begin{array}{ccc} v_1 & \dots & v_n \end{array}\right] \\ W & = & \left[\begin{array}{ccc} w_1 & \dots & w_n \end{array}\right] \end{array}\right. \tag{10}$$

that assigns the spectrum $\{\lambda_i,\, i = 1,...,n\}$ for A_c [12]. Besides, if static output feedback is to be computed, only partial pole placement might be achieved. Indeed

$$K = W_p V_p^{-1} \quad \text{with} \quad \left\{\begin{array}{lcl} V_p & = & \left[\begin{array}{ccc} Cv_1 & \dots & Cv_p \end{array}\right] \\ W_p & = & \left[\begin{array}{ccc} w_1 & \dots & w_p \end{array}\right] \end{array}\right. \tag{11}$$

assigns only $\{\lambda_i,\, i = 1,...,p\}$ as partial spectrum. In this case, the vectors Cv_i, $i = 1,...,p$, must be linearly independent.

The same reasoning can be followed for the left eigenstructure. By duality, the left eigenvectors $u_i \in \mathbb{C}^n$ defined by

$$u_i' A_c = \lambda_i u_i' \quad \forall i \in \{1,...,n\} \tag{12}$$

belong to the (A', C')-characteristic subspace $\tilde{\mathcal{S}}_i$ and can be chosen through parameters $x_i \in \mathbb{C}^p$ such that

$$\left\{\begin{array}{l} u_i' = x_i' \tilde{N}_i' \\ \\ l_i' = x_i' \tilde{M}_i' \end{array}\right. \quad \begin{array}{c} \forall i \in \{1,...,n\} \\ \text{(output directions)}, \end{array} \tag{13}$$

where $\tilde{R}_i = \left[\begin{array}{cc} \tilde{N}_i & \tilde{M}_i \end{array}\right]$ span the left nullspace of

$$\tilde{T}_i = \left[\begin{array}{c} A - \lambda_i I \\ C \end{array}\right]. \tag{14}$$

and only m eigenvalues $\{\lambda_i,\, i = 1,...,m\}$ are assigned by the output feedback matrix

$$K = (U_m')^{-1} L_m' \quad \text{with} \quad \left\{\begin{array}{lcl} U_m & = & \left[\begin{array}{ccc} B'u_1 & \dots & B'u_m \end{array}\right] \\ L_m & = & \left[\begin{array}{ccc} l_m & \dots & l_m \end{array}\right]. \end{array}\right. \tag{15}$$

When $B = I_n$, a complete pole placement is obtained owing to the output injection matrix

$$K = (U')^{-1} L' \quad \text{with} \quad \left\{\begin{array}{lcl} U & = & \left[\begin{array}{ccc} u_1 & \dots & u_n \end{array}\right] \\ L & = & \left[\begin{array}{ccc} l_1 & \dots & l_n \end{array}\right]. \end{array}\right. \tag{16}$$

Note that both sets of eigenvectors are not independent on each other. These vectors can be scaled so that the modal matrices U and V satisfy the orthogonality condition

$$U'V = I_n. \tag{17}$$

Remarque 1 *It really matters that the sets* $\{\lambda_i\}$, $\{z_i\}$ *and* $\{x_i\}$ *are closed under conjugation and more precisely that* $\lambda_i \in \mathbb{R} \Rightarrow \{z_i; x_i\} \in \mathbb{R}^m \times \mathbb{R}^p$ *and* $\lambda_i = \bar{\lambda}_j \Rightarrow (z_i = \bar{z}_j \,\&\, x_i = \bar{x}_j)$, *otherwise the deduced matrix* K *that assigns the specified poles could be complex. See [12] for details about how to obtain a real* K *and to reduce numerical effects of imaginary parts.*

3 Eigenstructure assignment equality

In this paragraph, the eigenstructure assignment is characterized by a single equality that is the basis for the present approach.
Assume that K assigns the m first eigenvalues. Such a K can be defined by (15). If the purpose is to assign the whole spectrum, then it is necessary that K satisfies the eigenstructure properties for the $r = n - m$ remaining roots.

The remaining eigenvalues λ_j, $j = m+1, ..., n$, are such that the associated eigenvectors v_j and input directions w_j comply with

$$w_j = KCv_j \quad \forall j \in \{m+1, ..., n\} \tag{18}$$

$$\Leftrightarrow\; M_j z_j = KCN_j z_j \quad \forall j \in \{m+1, ..., n\}. \tag{19}$$

From (15), one gets

$$M_j z_j = (U'_m)^{-1} L'_m C N_j z_j \quad \forall j \in \{m+1, ..., n\} \tag{20}$$

$$\Leftrightarrow\; U'_m M_j z_j - L'_m C N_j z_j = 0 \quad \forall j \in \{m+1, ..., n\} \tag{21}$$

$$\Leftrightarrow\; \begin{bmatrix} x'_1 \tilde{N}_1 \\ \vdots \\ x'_m \tilde{N}_m \end{bmatrix} B M_j z_j - \begin{bmatrix} x'_1 \tilde{M}_1 \\ \vdots \\ x'_m \tilde{M}_m \end{bmatrix} C N_j z_j = 0 \quad \forall j \in \{m+1, ..., n\}. \tag{22}$$

These r equalities are summarized by the following matrix equality :

$$XYZ = 0 \tag{23}$$

where

$$X = \text{blocdiag}\{x'_i,\, i = 1, ..., m\} \in \mathbb{C}^{m \times mp}, \tag{24}$$

$$Z = \text{blocdiag}\{z_j,\, j = m+1, ..., n\} \in \mathbb{C}^{mr \times m}, \tag{25}$$

$$Y = \tilde{\mathcal{N}}' B \mathcal{M} - \tilde{\mathcal{M}}' C \mathcal{N}, \tag{26}$$

with

$$\left\{ \begin{array}{rcl} \tilde{\mathcal{N}} & = & \left[\begin{array}{ccc} \tilde{N}_1 & \ldots & \tilde{N}_m \end{array} \right] \\ \tilde{\mathcal{M}} & = & \left[\begin{array}{ccc} \tilde{M}_1 & \ldots & \tilde{M}_m \end{array} \right] \\ \mathcal{N} & = & \left[\begin{array}{ccc} N_{m+1} & \ldots & N_n \end{array} \right] \\ \mathcal{M} & = & \left[\begin{array}{ccc} M_{m+1} & \ldots & M_n \end{array} \right]. \end{array} \right. \qquad (27)$$

This paragraph is summarized by the next proposition.

Proposition 1 *There exists a solution to Problem 1 if and only if there exists some non zero vectors $x_i \in \mathbb{C}^p$, $i = 1, ..., m$, and $z_j \in \mathbb{C}^m$, $j = m+1, ..., n$, such that*

(i) $x_k \in \mathbb{R}^p$ if $\lambda_k \in \mathbb{R}$, $x_h = \bar{x}_l$ if $\lambda_h = \bar{\lambda}_l$, $z_r \in \mathbb{R}^m$ if $\lambda_r \in \mathbb{R}$ and $z_s = \bar{z}_t$ if $\lambda_s = \bar{\lambda}_t$;

(ii) The vectors $(x_i'\tilde{M}_i'B)$ are linearly independent ;

(iii) Equality (23) is satisfied.

A solution is then given by (15).

Condition (i) reduces the set of solutions to real feedback matrices. Condition (ii) ensures that K can be computed owing to (15). Condition (iii) corresponds to the actual assignment equality and encompasses $v_j \in \mathcal{S}_j$, $u_i \in \tilde{\mathcal{S}}_i$ and (17). The problem therefore appears to be essentially bilinear since X and Z have to be found so that to check (23). A suitable solution may not be easy to reach. However, some special cases are considered in the sequel.

4 Eigenvalue assignment procedures

4.1 $m + p > n \Leftrightarrow p > r$ (Kimura's condition)

In this case, condition (iv) of Problem 1 can be omitted. Matrix Z defined in (25) is arbitrarily chosen. Matrix product $YZ \in \mathbb{C}^{mp \times r}$ is at most of rank r so if YZ is written as follows

$$YZ = \left[\begin{array}{c} \tilde{Y}_1 \\ \vdots \\ \tilde{Y}_m \end{array} \right] \quad \text{with} \quad \tilde{Y}_i \in \mathbb{C}^{p \times r} \quad \forall i \in \{1, ..., m\}, \qquad (28)$$

it appears that the generic rank of $\tilde{Y}_i$ cannot exceed $r < p$. Thus, a basis of the left nullspace of $\tilde{Y}_i$ can be computed. Let $\mathcal{X}_i'$ be the left orthogonal complement of $\tilde{Y}_i$ such that

$$\mathcal{X}_i'\tilde{Y}_i = 0 \quad \forall i \in \{1, ..., m\}. \qquad (29)$$

The generic dimension of $\mathcal{X}_i$ is $p \times (p - r)$. Let $\beta_i \in \mathbb{R}^{p-r}$ be any non-zero parameter vector $\forall i \in \{1, ..., m\}$. It comes that $\beta_i'\mathcal{X}_i'\tilde{Y}_i = 0$ and then the choice $x_i = \mathcal{X}_i\beta_i, , \forall i \in \{1, ..., m\}$ provides a solution to (23) from which some matrix K can be computed thanks to (15).

Remarque 2 *This solution is real provided Remark 1 is respected, which is not really restrictive since Z is free. If Z is suitably chosen* i.e. *if $\{z_j\}$ is closed under conjugation, so is $\{x_i\}$.*

Remarque 3 *Each vector z_j provides $(m - 1)$* dof *and each vector β_i provides $(p - r - 1)$* dof *so $(r(m - 1) + m(p - r - 1)) = mp - n$ among the mp offered by K are used to assign the eigenvectors. The n remaining* dof *in K correspond to the choice of the desired spectrum. Thus, all the flexibility brought by K can be exploited. For good insights about eigenvector assignment, see [20].*

Remarque 4 *This technique provides results which are equivalent to those induced by the technique in [15]. However, the basic idea may seem closer to the one developed in [13].*

4.2 $m + p = n \Leftrightarrow p = r$ (Kimura's non strict condition)

Following the same reasoning as in the previous paragraph fails because $\mathcal{X}_i$ cannot be computed. Indeed, $\tilde{Y}_i$ is generically of full rank p. The idea is to compute Z such that $\tilde{Y}_i$ becomes rank deficient *i.e.* $\mathrm{rank}(\tilde{Y}_i) = p - 1$. There exist many suitable matrices Z among which some can be picked up. Define $\mathcal{Z} \in \mathbb{C}^{mp\times(mp-n)}$, a matrix whose columns span the right nullspace of Y. Thus, any vector $\gamma \in \mathbb{R}^{mp-n}$ is such that $g = \mathcal{Z}\gamma$ belongs to the nullspace of Y and checks

$$Yg = Y \begin{bmatrix} z_{m+1} \\ \vdots \\ z_n \end{bmatrix} = 0 \quad \text{with} \quad z_j \in \mathbb{C}^m \quad \forall j \in \{m+1, ..., n\}. \tag{30}$$

Also write Y as follows :

$$Y = \begin{bmatrix} Y_{m+1} & \dots & Y_n \end{bmatrix} \quad \text{with} \quad Y_j \in \mathbb{C}^{mp\times m} \quad \forall j \in \{m+1, ..., n\}. \tag{31}$$

From (30) and (25), the columns of the product YZ are linearly dependent so each matrix $\tilde{Y}_i$ is of generic rank $p-1$. At this stage, one can proceed as in the previous paragraph by computing matrices $\mathcal{X}_i$. Except additional rank deficiency in $\tilde{Y}_i$, each matrix $\mathcal{X}_i$ is reduced to a vector $x_i = \mathcal{X}_i$. From these vectors, equation (15) leads to a solution K.

Remarque 5 *The flexibility available is that offered by γ, hence $(mp - n - 1)$* dof*. Since K offers mp* dof *with n being used for the choice of the desired spectrum, it means that only one* dof *is not exploited.*

A particular attention has to be paid to the case of complex desired eigenvalues. In the previous paragraph, as Z was free, it was easy to respect Remark 1. However, Z is here no longer free and is deduced from $\mathcal{Z}$. If $\mathcal{Z}$ is written as follows,

$$\mathcal{Z} = \begin{bmatrix} Z_{m+1} \\ \vdots \\ Z_n \end{bmatrix} \quad \text{with} \quad Z_j \in \mathbb{C}^{m\times(mp-n)} \quad \forall j \in \{m+1, ..., n\}, \tag{32}$$

analyzing the structure of Y given by (26) enables to conclude that a possibility to compute $\mathcal{Z}$ is to satisfy

$$\begin{bmatrix} \mathcal{M} \\ C\mathcal{N} \end{bmatrix} \mathcal{Z} = 0. \tag{33}$$

Note that $M_k \in \mathbb{R}^{m\times m}$ and $N_k \in \mathbb{R}^{n\times m}$ if $\lambda_k \in \mathbb{R}$ and $M_h = \bar{M}_l$ and $N_h = \bar{N}_l$ if $\lambda_h = \bar{\lambda}_l$. Denote $\hat{\mathcal{M}}$ the matrix obtained from $\mathcal{M}$ by replacing each block $[M_h \ M_l]$ by $[2\mathrm{Re}(M_h) - 2\mathrm{Im}(M_h)]$. Achieve the same procedure to get $\hat{\mathcal{N}}$ from $\mathcal{N}$. Compute matrix

$$\hat{\mathcal{Z}} = \begin{bmatrix} \hat{Z}_{m+1} \\ \vdots \\ \hat{Z}_n \end{bmatrix} \in \mathbb{R}^{mp\times(mp-n)} \quad \text{with} \quad \hat{Z}_j \in \mathbb{R}^{m\times(mp-n)} \quad \forall j \in \{m+1, ..., n\} \tag{34}$$

such that

$$\begin{bmatrix} \hat{\mathcal{M}} \\ C\hat{\mathcal{N}} \end{bmatrix} \hat{\mathcal{Z}} = 0. \tag{35}$$

A matrix $\mathcal{Z}$ can be deduced by choosing

$$\begin{cases} Z_k = \hat{Z}_k & \text{if} \quad \lambda_k \in \mathbb{R}, \\ \begin{bmatrix} Z_h \\ Z_l \end{bmatrix} = \begin{bmatrix} \hat{Z}_h + \mathbf{i}\hat{Z}_l \\ \hat{Z}_h - \mathbf{i}\hat{Z}_l \end{bmatrix} & \text{if} \quad \lambda_h = \bar{\lambda}_l \in \mathbb{C}\,. \end{cases} \tag{36}$$

At this stage, the matrices $\mathcal{X}_i \in \mathbb{C}^{(p-r)\times p}$ can be directly computed. Nevertheless, it is possible to avoid the calculation of complex nullspaces by proceeding as follows. For each pair of conjugate eigenvalues $\{\lambda_h; \lambda_l\}$, replace matrix

$$\begin{bmatrix} \tilde{Y}_h \\ \tilde{Y}_l \end{bmatrix} \quad \text{by} \quad \hat{Y}_h = \begin{bmatrix} \mathrm{Re}(\tilde{Y}_h) & \mathrm{Im}(\tilde{Y}_h) \\ -\mathrm{Im}(\tilde{Y}_h) & \mathrm{Re}(\tilde{Y}_h) \end{bmatrix}, \tag{37}$$

and compute the matrix

$$\begin{bmatrix} \hat{\mathcal{X}}_h & \hat{\mathcal{X}}_l \end{bmatrix}, \quad \text{with} \quad \{\hat{\mathcal{X}}_h; \hat{\mathcal{X}}_l\} \in \{\mathbb{R}^{2\times 2p}\}^2. \tag{38}$$

such that

$$\begin{bmatrix} \hat{\mathcal{X}}_h & \hat{\mathcal{X}}_l \end{bmatrix} \hat{Y}_h = 0. \tag{39}$$

The corresponding matrices $\mathcal{X}_h$ and $\mathcal{X}_l$ are given by :

$$\begin{cases} \mathcal{X}_h = \hat{\mathcal{X}}_h + \mathbf{i}\hat{\mathcal{X}}_l \\ \mathcal{X}_l = \hat{\mathcal{X}}_h - \mathbf{i}\hat{\mathcal{X}}_l. \end{cases} \tag{40}$$

Remarque 6 *It is amazing to note that the technique succeeds when $\mathcal{Z}$ can be computed* i.e. *if $mr > m + p \Leftrightarrow mp > n$, which is Wang's condition for generic arbitrary assignment [8] (condition (ii) in Problem 1). In this case condition (ii) implies condition (iv).*

4.3 $m + p < n$ (partial pole placement)

Since only partial pole placement must be achieved, condition (ii) in Problem 1 is now omitted. The reasoning is quite close to that of the previous paragraph. Compute matrices $\hat{\mathcal{M}}$ and $\hat{\mathcal{N}}$. Matrix $\hat{\mathcal{Z}} \in \mathbb{R}^{mp\times(mr-m-p)}$ can be computed so that its upper part makes I_{mr-m-p} appear *i.e.*

$$\hat{\mathcal{Z}} = \begin{bmatrix} I_{mr-m-p} \\ H \end{bmatrix} = \begin{bmatrix} \xi_1 & \dots & \xi_{mr-m-p} \end{bmatrix}, \tag{41}$$

with $H \in \mathbb{R}^{(m+p)\times(mr-m-p)}$. Define the matrix $\check{\mathcal{Z}}$ from $\mathcal{Z}$ by

$$\check{\mathcal{Z}} = \left[\begin{array}{ccc} \xi_{m(r-p)+1} & \dots & \xi_{mr-m-p} \end{array}\right] = \left[\begin{array}{c} \check{Z}_{m+1} \\ \vdots \\ \check{Z}_n \end{array}\right] \in \mathbb{R}^{m\times(mp-m-p)}, \quad \text{with} \quad \check{Z}_k \in \mathbb{R}^{m\times(mp-m-p)}. \tag{42}$$

Now, compute $\mathcal{Z}$ (defined by the equality in (32)) by choosing

$$\left\{\begin{array}{rcll} Z_k & = & \check{Z}_k \in \mathbb{R}^{m\times(mp-m-p)} & \text{if} \quad \lambda_k \in \mathbb{R}, \\ \left[\begin{array}{c} Z_h \\ Z_l \end{array}\right] & = & \left[\begin{array}{c} \check{Z}_h + \mathbf{i}\check{Z}_l \\ \check{Z}_h - \mathbf{i}\check{Z}_l \end{array}\right] \in \mathbb{C}^{m\times(mp-m-p)} & \text{if} \quad \lambda_h = \bar{\lambda}_l \in \mathbb{C}. \end{array}\right. \tag{43}$$

In this case, the columns of $\mathcal{Z}$ only span a subset of the right nullspace of Y. Because of the structure of $\hat{\mathcal{Z}}$ and $\check{\mathcal{Z}}$ given by (41) and (42), the matrices Z_j comply to :

$$\left\{\begin{array}{rcll} Z_j & = & 0 & \forall j \in \{m+1,...,m+r-p\} \\ Z_j & \neq & 0 & \forall j \in \{m+r-p+1,...,m+r\}. \end{array}\right. \tag{44}$$

Let $\delta \in \mathbb{R}^{mp-m-p}$ be a free parameter vector and define vectors z_j by

$$z_j = Z_j\delta \quad \forall j \in \{m+1,...,n\}. \tag{45}$$

Thus, there are p non zero vectors z_j and $(r-p)$ zero vectors z_j. Moreover, the vector $g = \mathcal{Z}\delta \in \mathbb{C}^{mr}$ is such that $Yg = 0$ as in (30). Consequently, the product YZ is of generic rank $(p-1)$ and so are the matrices $\tilde{Y}_i$. Each vector x_i can therefore be computed and relation (15) gives a matrix K. Even if equality (23) is satisfied, only $(m+p)$ eigenvalues are assigned. Indeed, since $z_j = 0$ for $j = m+1,...,m+r-p$, the corresponding $(r-p)$ eigenvalues are not placed. For numerical reason, it is better to choose them real. Of course, it is necessary to choose $\{\lambda_j, j \in \{m+r-p+1,...,m+r\}\}$ that is closed under conjugation.

Remarque 7 *$(mp-m-p-1)$ dof are available on δ. Among the mp entries of K, $(m+p)$ are used to assign the partial spectrum (the choice of non assigned λ_j does not provide additional flexibility since the corresponding z_j equal zero). It remains only 1 non exploited dof.*

Condition (iv) in Problem 1 is now justified. It is actually related to the existence of $\check{\mathcal{Z}}$. From (42) and (41), it can be seen that $\check{\mathcal{Z}}$ exists if and only if $mp > m+p$. If this condition does not hold, any other choice of columns of $\hat{\mathcal{Z}}$ to build $\check{\mathcal{Z}}$ induces more than p non zero vectors z_j and the suitable rank deficiency of YZ is not obtained. For such cases, a classical partial spectrum assignment can be achieved to place $\max(m,p)$ poles.

5 Numerical illustration

Since the case $m+p = n < mp$ can actually be presented as a special case of partial pole placement, only one example with $m+p = 3+2 = 5 = n$ is proposed. The matrices A and B are borrowed from [21] :

$$A = \begin{bmatrix} -0.3400 & 0.0517 & 0.0010 & -0.9970 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 \\ -2.6900 & 0 & -1.1500 & 0.7380 & 0 \\ 5.9100 & 0 & 0.1380 & -0.5060 & 0 \\ -0.3400 & 0.0517 & 0.0010 & 0.0031 & 0 \end{bmatrix} ; \ B = \begin{bmatrix} 0.0755 & 0 & 0.0246 \\ 0 & 0 & 0 \\ 4.4800 & 5.2200 & -0.7420 \\ -5.0300 & 0.0998 & 0.9840 \\ 0.0755 & 0 & 0.0246 \end{bmatrix}.$$

In [21] a state feedback control law was derived to assign the spectrum $\{-3 \pm 2\mathbf{i}; -2 \pm 2\mathbf{i}; -0.5\}$. For the same purpose, it is here assumed that only two state components are measured so that

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The technique presented in Subsection 4.2 yields the following output feedback matrix (the only one since no dof can be exploited) :

$$K = \begin{bmatrix} -12.7499 & -24.4928 \\ 569.7537 & 781.0785 \\ -86.0455 & -145.3437 \end{bmatrix}$$

which assigns the closed-loop spectrum at the right location.

6 Conclusions

In this paper, a pole placement technique was proposed for linear models of order n with m inputs and p outputs. It is based on the notion of eigenstructure assignment. Like the other techniques relying on that notion, it enables to assign the whole desired spectrum when $m+p > n$ (Kimura's condition). Besides, if $m+p < mp$, it allows to assign $m+p$ poles. Therefore, the complete pole placement can be achieved when $m+p = n < mp$ (Kimura's non strict condition).

From a numerical point of view, the derivation of the feedback matrix requires the computation of the bases of several nullspaces. Nevertheless, no optimization process is needed (such as some Newton-based process for example). Numerous random and practical examples led to very satisfying results up to $n = 8$. For higher values of n, some inaccurate computations may occur, especially because the entries of the feedback matrix can turn to be very large (in that case, the computation of the obtained eigenvalues is itself inaccurate).

As further investigations, it could be interesting to avoid the separation of the desired spectrum into two subsets. Indeed, because of this acceptable restriction, if $n = 8$, $m = 5$ and $p = 3$ for instance, it is impossible to assign 8 eigenvalues whose imaginary parts are all non zero. But most of all, the main challenge is to assign the whole of the spectrum even if $m+p < n$. However, Equation (23) makes this problem appear as a bilinear (thus probably difficult) one.

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