Robust Matrix Root-Clustering Analysis through Extended KYP Lemma

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Index Terms

Matrix $\mathcal{D}$-stability, $\partial\mathcal{D}$-regularity, KYP lemma, Generalized S-procedure.

Abstract

This paper is dedicated to robust matrix eigenvalue clustering in a subregion $\mathcal{D}$ of the complex plane. A norm-bounded uncertainty is considered. $\mathcal{D}$ is any intersection and/or union of half planes, discs, outsides of discs. This work aims at computing the associated complex $\mathcal{D}$-stability radius. This challenge was already tackled in a recent paper but the technique therein may fail or lead to conservative results when the subregions overlap. The present paper proposes to circumvent those limits thanks to an extended version of the Kalman-Yakubovich-Popov lemma which is deduced from the so-called generalized S-procedure.

1 Introduction

Matrix $\mathcal{D}$-stability appears to be a discerning tool to appreciate the transient performances of a linear model (a matrix is said to be $\mathcal{D}$-stable when its spectrum lies in a region $\mathcal{D}$ of the complex plane). Many contributions propose necessary and sufficient conditions for matrix $\mathcal{D}$-stability such as paper [1], its generalized Lyapunov equations and its Ω-regions, based on the pioneer work of [2], or [3] and its LMI-regions (LMI stands for linear matrix inequality: see [4]).

In the presence of uncertainties, the necessity of the abovementioned conditions cannot be preserved in most cases. Thus, it is important to check robust $\mathcal{D}$-stability through sufficient conditions. The Hurwitz-stability case was tackled in [5, 6, 7, 8]. Ω-regions were considered in [9, 10]. For polytopic uncertainty, LMI conditions given in [3, 11] prove to be very effective. The special case of norm-bounded uncertainty [12] is of interest since the complex $\mathcal{D}$-stability radius can be reached owing to the bounded real lemma when $\mathcal{D}$ is either a disc or a half plane [13, 14, 15] (the complex stability radius corresponds to the largest radius of the ball of complex uncertainties that preserve stability). For the real stability radius, see [16, 17].

Quite few results exist concerning non connected regions, especially combinations of several sub-regions, although it helps to handle the compromise between robustness and satisfactory transient behaviour. Since the first contribution [18], other works were presented in [19, 20, 21]. For structured uncertainty, see the work of [22] on the original good ride quality region particularly convenient for aircraft models. One of the most recent contributions dealing with norm-bounded uncertainty is [23] in which the reader can find details and discussions about the above-mentioned papers. Starting from the notion of $\partial\mathcal{D}$-regularity (non intersection between the matrix spectrum and some curve $\partial\mathcal{D}$), the approach presented in [23] is efficient and often leads to the complex $\mathcal{D}$-stability radius, except when the various subregions of $\mathcal{D}$ overlap. The present paper must be seen as an extension of [23] that circumvents the mentioned limits. It relies on an extended version of the Kalman-Yakubovich-Popov lemma deduced from the insightful paper [24] and its generalized S-procedure.

The present paper is organized as follows: Section 2 emphasizes the limits of [23] and states the problem under a new and more precise formulation. In section 3, once some background recalled an extended version of the KYP-lemma is proposed. section 4 is devoted to numerical examples. Section 5 concludes the paper.

Notation

M' denotes the transpose conjugate of matrix M. Hence, $s' \in \mathbb{C}$ is the conjugate of s. M^H equals matrix $M + M'$. The 2-norm of M (maximal singular value) is denoted by $||M||_2$. $\mathbf{I}$ and 0 are the identity matrix and the null matrix of appropriate dimensions respectively. In matrix inequalities, >0 and <0 mean positive definite and negative definite respectively ; ≥ 0 and ≤ 0 mean positive semi-definite and negative semi-definite respectively. Symbol $\mathbf{i}$ denotes the imaginary unit. $\lambda(M)$ and $\mathrm{tr}(M)$ are the spectrum and the trace of square matrix M respectively. Symbol $\otimes$ stands for the matrix Kronecker product. Finally, given a couple of complex matrices (X,Y) and a matrix

$$T = T' = \begin{bmatrix} t_{00} & t_{10} \\ t_{10}' & t_{11} \end{bmatrix} \in \mathbb{C}^{2\times 2},$$

the expressions $\mathcal{F}_T(X,Y)$ and $\mathcal{G}_T(X,Y)$ are defined by

$$\left\{ \begin{array}{l} \mathcal{F}_T(X,Y) = t_{00}XX' + (t_{10}YX')^H + t_{11}YY' \\ \\ \mathcal{G}_T(X,Y) = t_{00}X'X + (t_{10}X'Y)^H + t_{11}Y'Y. \end{array} \right.$$

2 Motivation

In this section, some results proposed in [23] are first recalled. However, the limits inherent to those techniques are highlighted. That is why the problem initially stated in [23] is presented under a new and more complete formulation.
Consider an uncertain matrix that reads:

$$A_c \;=\; A \,+\, B\Delta C \;\in \mathbb{C}^{n\times n}. \tag{1}$$

Matrix Δ is constant, unknown and assumed to belong to the ball $\mathcal{B}(\rho)$ of all matrices $\Delta \in \mathbb{C}^{q\times r}$ satisfying $||\Delta||_2 \leq \rho$. Matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$ and $C \in \mathbb{C}^{r\times n}$ are given. Such a description is referred to as norm-bounded uncertainty [12].
Now consider the next Hermitian matrices:

$$R_k = R_k' = \begin{bmatrix} r_{k_{00}} & r_{k_{10}} \\ r_{k_{10}}' & r_{k_{11}} \end{bmatrix} \in \mathbb{C}^{2\times 2}, \quad \forall k \in \{1,...,m\}, \tag{2}$$

from which the following geometric curves are defined:

$$\partial\mathcal{D}_k = \left\{ \; s \in \mathbb{C} | f_k(s) = \mathcal{F}_{R_k}(1,s) = \mathcal{G}_{R_k}(1,s) = 0 \; \right\} \quad \forall k \in \{1,...,m\}. \tag{3}$$

Relevant curves are lines or circles which can be nonsymmetric with respect to the real axis. Each curve $\partial\mathcal{D}_k$ enables to define an associated region:

$$\mathcal{D}_k \; = \; \{s \in \mathbb{C} \,|\, f_k(s) < 0\} \quad \forall k \in \{1,...,m\}. \tag{4}$$

Clearly, $\mathcal{D}_k$ denotes either one side or the other side of the boundary $\partial\mathcal{D}_k$. Then it can be a half plane, a disc or the outside of a disc. It is an open region (*i.e.* not including $\partial\mathcal{D}_k$) in order to encompass the concept of asymptotic stability for LTI systems.

Also define the region $\mathcal{D}$ as a combination, *i.e.* any union and/or intersection of the various subregions $\mathcal{D}_k$. Such a formulation of $\mathcal{D}$ clearly enables a very large choice of regions.

This contribution aims at computing the complex $\mathcal{D}$-stability radius $r_{\mathcal{D}}$ *i.e.* the largest value of ρ - the radius of $\mathcal{B}(\rho)$ - so that, A being $\mathcal{D}$-stable, A_c defined in (1) remains also $\mathcal{D}$-stable for any $\Delta \in \mathcal{B}(\rho)$. In [23] a technique to derive a lower bound $\rho^\star$ was proposed. This bound may equal the $\mathcal{D}$-stability radius, but it is sometimes conservative. The main result of [23] is now recalled:

Theorem 1 *[23] Let A_c, $\mathcal{D}$, $\partial\mathcal{D}$ be an uncertain matrix as defined in (1), a clustering region as defined above and its boundary respectively. Then A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exist m Hermitian matrices P_k, $k = 1,...,m$, such that*

$$\mathcal{Q}_k(P_k,\gamma) \; = \; \left[\begin{array}{cc} \mathcal{G}_{R_k}(P_k^{1/2}, P_k^{1/2}A) + C'C & r_{k_{10}}P_kB + r_{k_{11}}A'P_kB \\ r'_{k_{10}}B'P_k + r_{k_{11}}B'P_kA & r_{k_{11}}B'P_kB - \gamma\mathbf{I} \end{array} \right] < 0 \quad \forall k \in \{1,...,m\}, \tag{5}$$

where $\gamma = \rho^{-2}$.
In this event, A_c keeps the same number of eigenvalues inside $\mathcal{D}$ for any $\Delta \in \mathcal{B}(\rho)$.
Moreover, if $\partial\mathcal{D}$ can be identified with $\partial\mathcal{U} = \bigcup\limits_{k=1}^{m} \partial\mathcal{D}_k$, then condition (5) is also necessary.

To roughly sum up the above theorem and paper [23] in general, one can write that the uncertain matrix A_c defined in (1) is robustly $\mathcal{D}$-stable if the nominal matrix A is $\mathcal{D}$-stable and if A_c is robustly $\partial\mathcal{D}$-regular. The $\partial\mathcal{D}$-regularity of a matrix is simply the non intersection between its spectrum and the set $\partial\mathcal{D}$. In [23], there are also considerations on the relation between the inertias of matrices P_k and the dispatching of the eigenvalues of A or A_c with respect to the various $\mathcal{D}_k$. But these notions do not need to be developed here. Note that Theorem 1 is deduced from the KYP lemma [25, 26].

The value $\rho^\star$ of ρ, obtained by minimizing γ while satisfying (5), equals $\varrho_{\partial\mathcal{U}}$ (the so-called complex $\partial\mathcal{U}$-regularity radius) which is itself a lower bound of $\varrho_{\partial\mathcal{D}}$ (the complex $\partial\mathcal{D}$-regularity radius). It is shown in [23] that if A is $\mathcal{D}$-stable and if $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$, then $\varrho_{\partial\mathcal{U}}$ actually equals $r_{\mathcal{D}}$, the complex $\mathcal{D}$-stability radius. The $\mathcal{D}$-stability of A is obviously necessary to check the $\mathcal{D}$-stability of A_c but assumption $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$ is not always satisfied, hence the possible conservativeness of Theorem 1. To highlight it, Figure 1 shows both cases: the left side corresponds to $\partial\mathcal{U} \cap \mathcal{D} \neq \emptyset$ so $\varrho_{\partial\mathcal{U}} \leq r_{\mathcal{D}}$ whereas the right side corresponds to $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$ so $\varrho_{\partial\mathcal{U}} = r_{\mathcal{D}}$.
The drawback of Theorem 1 is that no rigorous description of the actual set $\partial\mathcal{D}$ is provided when $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$. For example, on the left side of Figure 1, the parts of the circles $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ that belong to $\mathcal{D}$ should not be taken into account. This selection of points in $\partial\mathcal{D}_k$ is not made. To remedy this lack, a new description of $\partial\mathcal{D}$ is now proposed.

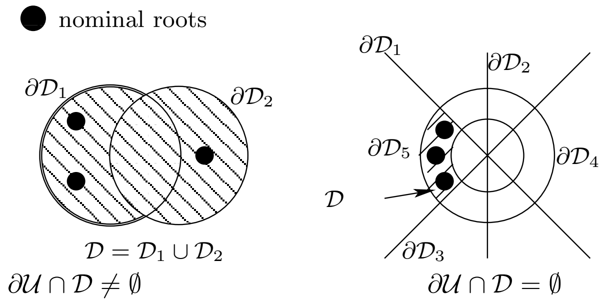

Figure 1: $\varrho_{\partial\mathcal{U}} = r_{\mathcal{D}}$?

Let m curves $\partial\mathcal{D}_k$ be described as in (3). Let the matrices $\Phi_{k,l}$ be defined by

$$\Phi_{k,l} = \begin{bmatrix} \phi_{k,l_{00}} & \phi_{k,l_{10}} \\ \phi'_{k,l_{10}} & \phi_{k,l_{11}} \end{bmatrix} = \Phi'_{k,l} \in \mathbb{C}^{2\times 2}, \quad k = 1, ..., m, \; l = 1, ..., p_k. \tag{6}$$

Define m subregions $\bar{\mathcal{E}}_k$ by

$$\bar{\mathcal{E}}_k = \left\{ s \in \mathbb{C} | \mathcal{F}_{\Phi_{k,l}}(1, s) = \mathcal{G}_{\Phi_{k,l}}(1, s) \geq 0 \; \forall l \in \{1, ..., p_k\} \right\} \;\; \forall k \in \{1, ..., m\}. \tag{7}$$

Each region $\bar{\mathcal{E}}_k$ is actually the intersection of various closed subregions $\bar{\mathcal{E}}_{k,l}$ characterized by inequalities $\mathcal{G}_{\Phi_{k,l}}(1, s) \geq 0$. Subregions $\bar{\mathcal{E}}_{k,l}$ are half planes, discs or outsides of discs, not necessarily symmetric. Consider the sets $\partial\Gamma_k$ defined by

$$\partial\Gamma_k = \partial\mathcal{D}_k \cap \bar{\mathcal{E}}_k = \left\{ \;\; s \in \mathbb{C} | \mathcal{G}_{R_k}(1, s) = 0, \mathcal{F}_{\Phi_{k,l}}(1, s) \geq 0 \quad \forall l \in \{1, ..., p_k\} \;\; \right\} \forall k \in \{1, ..., m\}. \tag{8}$$

Each set $\partial\Gamma_k$ is the intersection of a line or a circle with p_k closed half planes, discs or outsides of discs. In other words, it is a line or a circle from which only some points are selected through the choice of various $\Phi_{k,l}$. Furthermore, define the set $\partial\mathcal{D}$ as

$$\partial\mathcal{D} = \bigcup_{k=1}^{m} \partial\Gamma_k. \tag{9}$$

If one considers any combination (intersection and/or union) of half planes, discs, outsides of discs, then its boundary can read formulation (9). For $\mathcal{D}$ as pictured on Figure 1 (left side), the circles $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ can be restricted to the subsets that are not in $\mathcal{D}$, thanks to an appropriate choice of $\Phi_{1,1}$ and $\Phi_{2,1}$ (the first example in section 4 will highlight this assertion). With the description given in (9), the computation of the complex $\mathcal{D}$-stability radius $r_{\mathcal{D}}$ becomes equivalent to that of the complex $\partial\mathcal{D}$-regularity radius $\varrho_{\partial\mathcal{D}}$.

3 Extended KYP lemma and complex $\mathcal{D}$-stability radius

This section is devoted to the computation of $r_{\mathcal{D}}$ through an LMI approach. First, some technical preliminaries are needed. Then, some sort of extended KYP lemma is deduced. Finally, it is shown that this lemma can be directly exploited to solve the stated problem.

3.1 Background and preliminary lemmas

A key point in the present reasoning is that each LMI condition of Theorem 1 enables to test the robust $\partial\mathcal{D}_k$-regularity of a matrix whereas it should test the robust $\partial\Gamma_k$-regularity ($\partial\Gamma_k \subset \partial\mathcal{D}_k$, $\forall k$). It is similar to the problem encountered while applying the finite frequency KYP lemma. Such a lemma is proposed in [24] and is proven with the help of the so-called generalized S-procedure. The present paper largely exploits the results provided in [24]. The reader is strongly invited to refer to this paper from which Definition 2 is extracted:

Definition 2 *[24] A subset $\mathcal{S}$ of $n \times n$ Hermitian matrices is said lossless if*

(a) $\mathcal{S}$ is convex;

(b) $S \in \mathcal{S} \Rightarrow \tau S \in \mathcal{S}$ $\forall \tau > 0$;

(c) For each nonzero matrix $H \in \mathbb{C}^{n\times n}$ of rank r such that

$$H = H' \geq 0, \ \mathrm{tr}(SH) \leq 0, \ \forall S \in \mathcal{S},$$

there exist vectors $\xi_i \in \mathbb{C}^n$, $i = 1, ..., r$, such that

$$H = \sum_{i=1}^{r} \xi_i \xi_i' \quad \text{and} \quad \xi_i' S \xi_i \leq 0 \ \forall S \in \mathcal{S}.$$

The above definition is necessary to apply the so-called generalized S-procedure [24, Theorem 1] implicitly used in forthcoming Theorem 6. These definition and procedure are not the only borrowings from [24]. Other insights of this discerning paper are exploited in the sequel.

Lemma 3 *Let W and Z be complex matrices of the same size and $R \in \mathbb{C}^{2\times 2}$ be defined by:*

$$R = R' = \begin{bmatrix} r_{00} & r_{10} \\ r_{10}' & r_{11} \end{bmatrix}.$$

Then the following statements are equivalent.

(i)

$$\mathcal{F}_R(W, Z) = 0. \tag{10}$$

(ii) There exists a matrix E such that $Z = WE$ and

$$\mathcal{F}_R(\mathbf{I}, E) = 0. \tag{11}$$

Proof. (i)$\Rightarrow$(ii): first assume that $r_{11} = 0$ then from (10)

$$\begin{aligned} r_{00}WW' + r_{10}'WZ' + r_{10}ZW' &= W\left(\frac{1}{2}r_{00}W + r_{10}Z\right)' + \left(\frac{1}{2}r_{00}W + r_{10}Z\right)W' \\ &= WY' + YW' = 0 \end{aligned}$$

with $Y = \left(\frac{1}{2}r_{00}W + r_{10}Z\right)$. By virtue of [26, Lemma 3, statement (iii)], this implies the existence of U such that $UU' = \mathbf{I}$ and $Y = W(\mathbf{I} - U)(\mathbf{I} + U)^{-1} = W\bar{U}$. Hence

$$\bar{U} + \bar{U}' = (\mathbf{I} + U')^{-1}\left\{(\mathbf{I} + U')(\mathbf{I} - U) + (\mathbf{I} - U')(\mathbf{I} + U)\right\}(\mathbf{I} + U)^{-1}$$

$$= (\mathbf{I}+U')^{-1}\left\{\mathbf{I}+U'-U-U'U+\mathbf{I}-U'+U-U'U\right\}(\mathbf{I}+U)^{-1}=0.$$

From $Y = W\bar{U}$, it can be deduced that W and Z comply with

$$W\left(\bar{U}-\frac{1}{2}r_{00}\mathbf{I}\right)=r_{10}Z,$$

which can be written as $Z = WE$ with

$$E=\frac{1}{r_{10}}\left(\bar{U}-\frac{1}{2}r_{00}\mathbf{I}\right) \tag{12}$$

(note that $r_{10}\neq 0$ otherwise W and Z equal 0). Since $\bar{U}+\bar{U}'=0$,

$$r_{00}\mathbf{I}+r_{10}'E'+r_{10}E=0 \Leftrightarrow \mathcal{F}_R(\mathbf{I},E)=0.$$

Now assume that $r_{11}\neq 0$, then (10) becomes

$$\hat{r}_{00}WW'+\hat{r}_{10}'WZ'+\hat{r}_{10}ZW'+ZZ'=0$$

with $\hat{r}_{i0}=\dfrac{r_{i0}}{r_{11}}$ for $i=0,1$. The previous equation can be rewritten as

$$\left(Z+\hat{r}_{10}'W\right)\left(Z+\hat{r}_{10}'W\right)'=\left(\hat{r}_{10}\hat{r}_{10}'-\hat{r}_{00}\right)WW'$$

from which, by virtue of lemma [26, Lemma 3, statement (i)], one gets

$$Z+\hat{r}_{10}'W=\sqrt{(\hat{r}_{10}\hat{r}_{10}'-\hat{r}_{00})}WU$$

for some U such that $UU'=\mathbf{I}$. Finally it leads to $Z = WE$ with

$$E=\left(-\hat{r}_{10}'\mathbf{I}+\sqrt{(\hat{r}_{10}\hat{r}_{10}'-\hat{r}_{00})}U\right). \tag{13}$$

It is obvious that E satisfies

$$E+\hat{r}_{10}'\mathbf{I}=\sqrt{(\hat{r}_{10}\hat{r}_{10}'-\hat{r}_{00})}U$$

which yields

$$\begin{array}{c}\left(E+\hat{r}_{10}'\mathbf{I}\right)\left(E+\hat{r}_{10}'\mathbf{I}\right)'=\left(\hat{r}_{10}\hat{r}_{10}'-\hat{r}_{00}\right)\mathbf{I}\\ \Leftrightarrow \mathcal{F}_R(\mathbf{I},E)=0.\end{array}$$

To prove (ii)⇒(i), just left and right-multiply (11) by W and W' respectively to recover (10). □
An extension of [24, Lemma 2] is now presented.

Lemma 4 *Let two complex vectors f and $g\neq 0$ be given. Also let $\partial\mathcal{D}$ be defined by (8) with $k=m=1$ (so the dependence of R, P, Φ_l and Q_l on k is omitted). The following statements are equivalent.*

(i) $\exists s\in\partial\mathcal{D}=\partial\Gamma_1\,|\,f=sg.$

(ii) $\mathcal{Z}=\begin{bmatrix} g\\ f\end{bmatrix}'\left(R\otimes P-\sum_{l=1}^{p}(\Phi_l\otimes Q_l)\right)\begin{bmatrix} g\\ f\end{bmatrix}\leq 0,\ \forall P=P'\ \&\ \forall Q_l=Q_l'>0,\ l=1,\ldots,p.$

Proof.

$$(\text{i}) \Rightarrow \mathcal{Z} = \mathcal{F}_R(1,s)g'Pg - \sum_{l=1}^{p}(\mathcal{F}_{\Phi_l}(1,s)g'Q_l g).$$

Since $s \in \partial\Gamma_1$, relations

$$-\sum_{l=1}^{p}(\mathcal{F}_{\Phi_l}(1,s)) \leq 0 \quad \text{and} \quad \mathcal{F}_R(1,s) = 0$$

are satisfied. Besides, since $Q_l > 0$, $\forall l \in \{1,...,p\}$, one gets $\mathcal{Z} \leq 0$ and then (ii) is true.
Reciprocally, (ii) $\Rightarrow$

$$\text{tr}\left(\left[\begin{array}{c} g \\ f \end{array}\right]\left[\begin{array}{cc} g' & f' \end{array}\right]\left(R \otimes P - \sum_{l=1}^{p}(\Phi_l \otimes Q_l)\right)\right) \leq 0$$

$$\Leftrightarrow \text{tr}\left(\left[\begin{array}{c} g \\ f \end{array}\right]\left[\begin{array}{cc} g' & f' \end{array}\right]\left[\begin{array}{c|c} r_{00}P - \sum_{l=1}^{p}(\phi_{l_{00}}Q_l) & r_{10}P - \sum_{l=1}^{p}(\phi_{l_{10}}Q_l) \\ \hline r'_{10}P - \sum_{l=1}^{p}(\phi'_{l_{10}}Q_l) & r_{11}P - \sum_{l=1}^{p}(\phi_{l_{11}}Q_l) \end{array}\right]\right) \leq 0$$

$$\Leftrightarrow \text{tr}(\mathcal{F}_R(g,f)P) - \text{tr}(\sum_{l=1}^{p}(\mathcal{F}_{\Phi_l}(g,f)Q_l)) \leq 0. \tag{14}$$

Clearly, since (14) is true for all $P = P'$ and $Q_l = Q'_l > 0$, it holds for any choice $Q_l = w_l I$, $w_l > 0$, $l = 1,...,p$, and $P = \alpha\mathbf{I}$ with $\alpha > 0$ possibly very large. Hence

$$\left\{\begin{array}{l} \mathcal{F}_R(g,f) = 0 \\ \\ \sum_{l=1}^{p}(w_l\mathcal{F}_{\Phi_l}(g,f)) \geq 0 \quad \forall w_l > 0,\ l = 1,...,p. \end{array}\right. \tag{15}$$

From the equality in (15) and Lemma 3, a scalar $s^\bullet \in \partial\mathcal{D}_1$ verfies $f = s^\bullet g$.
Now consider the inequality in (15). Since $f = s^\bullet g$, it comes:

$$\sum_{l=1}^{p}(w_l\mathcal{F}_{\Phi_l}(1,s^\bullet)) \geq 0 \ \ \forall w_l > 0, l = 1,...,p.$$

The above inequality holds for any set of $w_l > 0$ if and only if $\mathcal{F}_{\Phi_l}(1,s^\bullet) \geq 0$ for all l in $\{1,...,p\}$. Thus, $s^\bullet$ belongs to $\partial\Gamma_1$. $\square$

Now, still inspired from [24], the next lemma is proposed.

Lemma 5 *Let some matrix R defined in (2), some matrices Φ_l defined in (6) (with $k = 1$) and a matrix $F \in \mathbb{C}^{2n\times q}$ be given. Also let a subset of Hermitian matrices $\mathcal{S}$ be defined by*

$$\mathcal{S} := \left\{F'\left(R \otimes P - \sum_{l}^{p}(\Phi_l \otimes Q_l)\right)F \ :\ P = P', Q_l = Q'_l > 0, l = 1,...,p\right\}. \tag{16}$$

Then $\mathcal{S}$ is lossless.

Proof. Refering to definition 2, properties (a) and (b) are obvious. For property (c), consider a matrix $H = H' \geq 0$ such that $\mathrm{tr}(HS) \leq 0$, $\forall S \in \mathcal{S}$ and $H = GG'$ where $G \in \mathbb{C}^{q\times r}$ is of rank r. One gets :

$$\mathrm{tr}\left(GG'F'\left(R\otimes P - \sum_{l}^{p}(\Phi_l \otimes Q_l)\right)F\right) \leq 0 \;\forall P = P', Q_l = Q_l' > 0$$

$$\Leftrightarrow \mathrm{tr}\left(FGG'F'\left(R\otimes P - \sum_{l=1}^{p}(\Phi_l \otimes Q_l)\right)\right) \leq 0 \;\forall P = P', Q_l = Q_l' > 0. \tag{17}$$

Define matrices W and Z by

$$\begin{bmatrix} W \\ Z \end{bmatrix} = FG. \tag{18}$$

Thus, inequality (17) can be written

$$\mathrm{tr}\left(\begin{bmatrix} W \\ Z \end{bmatrix}\begin{bmatrix} W \\ Z \end{bmatrix}'\left(R\otimes P - \sum_{l=1}^{p}(\Phi_l \otimes Q_l)\right)\right) =$$

$$\mathrm{tr}\left(\begin{bmatrix} WW' & WZ' \\ ZW & ZZ' \end{bmatrix}'\left(R\otimes P - \sum_{l=1}^{p}(\Phi_l \otimes Q_l)\right)\right) =$$

$$\mathrm{tr}\left(\begin{bmatrix} WW' & WZ' \\ ZW' & ZZ' \end{bmatrix}(R\otimes P)\right) - \mathrm{tr}\left(\sum_{l=1}^{p}\left(\begin{bmatrix} WW' & WZ' \\ ZW' & ZZ' \end{bmatrix}(\Phi_l \otimes Q_l)\right)\right) \leq 0 \,\forall P = P', Q_l = Q_l' > 0.$$

As in the proof of lemma 4, it can be deduced that

$$\begin{cases} \mathcal{F}_R(W,Z) = 0 \\ \displaystyle\sum_{l=1}^{p}(w_l \mathcal{F}_{\Phi_l}(W,Z)) \geq 0, \quad \forall w_l > 0,\; l = 1,...,p. \end{cases} \tag{19}$$

From Lemma 3 and from equality in (19), there exists some matrix E such that

$$Z = WE \tag{20}$$

and

$$\mathcal{F}_R(\mathbf{I}, E) = 0. \tag{21}$$

Such a matrix E is given by either (12) and (13). Since U satisfies $UU' = \mathbf{I}$, it is easy to check that it has no eigenvalue with geometric multiplicity greater than 1. Then it can be written

$$U = V\Lambda_U V^{-1},$$

where V is a modal matrix and Λ_U is diagonal. Matrix $\bar{U}$ involved in (12) complies with

$$\bar{U} = V(\mathbf{I} - \Lambda_U)(\mathbf{I} + \Lambda_U)^{-1}V^{-1}.$$

so it has a similar diagonal form. Expressions (12) and (13) enable to check that E has also a similar diagonal form with V as a modal matrix. Thus let r vectors v_i, $i = 1,...,r$, be the columns of V, one can write the definition of the right eingenstructure of E as follows:

$$Ev_i = \lambda_i v_i,\ \lambda_i \in \mathbb{C}, \quad \forall i \in \{1, ..., r\}.$$

The eigenvectors v_i can be chosen as the orthonormal basis of a subspace of dimension r so that $\sum_{i=1}^{r}(v_i v_i') = \mathbf{I}$. Then, inequality (21) yields

$$v_i' \mathcal{F}_R(I, E) v_i = \mathcal{F}_R(1, \lambda_i) v_i' v_i = 0 \quad \forall i \in \{1, ..., r\}. \tag{22}$$

The above equality proves that each λ_i belongs to $\partial\mathcal{D}_1$ characterized by R.

Besides, from inequality given in (19), relation (20) enables to write:

$$v_i' \left(\sum_{l=1}^{p} (w_l \mathcal{F}_{\Phi_l}(W, WE)) \right) v_i =$$

$$\left(\sum_{l=1}^{p} (w_l \mathcal{F}_{\Phi_l}(1, \lambda_i)) \right) (v_i' W W' v_i) \geq 0 \quad \forall i \in \{1, ..., r\},\ \forall w_l > 0,\ l = 1, ...p$$

$$\Rightarrow \sum_{l=1}^{p} (w_l \mathcal{F}_{\Phi_l}(1, \lambda_i)) \geq 0 \quad \forall i \in \{1, ..., r\},\ \forall w_l > 0,\ l = 1, ...p$$

$$\Rightarrow \mathcal{F}_{\Phi_l}(1, \lambda_i) \geq 0 \quad \forall l = 1, ...p. \tag{23}$$

The above inequality proves that each $\lambda_i \in \partial\mathcal{D}_1$ belongs to $\bar{\mathcal{E}}_1$ and therefore to $\partial\Gamma_1$.
The remaining part of the proof is analogous to the one in [24, Lemma 3]. The sum and the Kronecker product in the expressions of $\mathcal{F}_R$ and $\mathcal{F}_{\Phi_l}$ are no real obstacles and bring only slight modifications. □

3.2 Extended KYP lemma

Now that the set $\mathcal{S}$ is proved to be lossless, an appropriate application of the generalized S-procedure [24, Theorem 1] enables to state the next theorem which might be considered as the main result of the paper.

Theorem 6 *let $\partial\mathcal{D} = \partial\Gamma_1$ be defined by (8) ($k = m = 1$ so the dependence of R, P, Φ_l and Q_l on k is still omitted). Also let matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$ and $\Theta = \Theta' \in \mathbb{C}^{(n+m)\times(n+m)}$ be given. Assume that A is $\partial\mathcal{D}$-regular. Then the following statements are equivalent.*

(i)

$$\begin{bmatrix} (s\mathbf{I} - A)^{-1}B \\ \mathbf{I} \end{bmatrix}' \Theta \begin{bmatrix} (s\mathbf{I} - A)^{-1}B \\ \mathbf{I} \end{bmatrix} < 0 \quad \forall s \in \partial\mathcal{D}. \tag{24}$$

(ii) There exist $n \times n$ complex Hermitian matrices P and $Q_l = Q_l > 0$, $l = 1, ..., p$, such that

$$\begin{bmatrix} \mathbf{I} & 0 \\ A & B \end{bmatrix}' \left(R \otimes P + \sum_{l=1}^{p} (\Phi_l \otimes Q_l) \right) \begin{bmatrix} \mathbf{I} & 0 \\ A & B \end{bmatrix} + \Theta < 0. \tag{25}$$

Proof. The proof is similar to the one of the finite frequency KYP lemma proposed in [24]. However, an outline is proposed for clarity.
Statement (i) is equivalent to $\zeta'\Theta\zeta < 0 \quad \forall \zeta \in \mathcal{G}$, where

$$\mathcal{G} = \left\{ \zeta = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^{n+m} \,|\, y \neq 0, sx = Ax + By \text{ for some } s \in \partial\mathcal{D} \right\}.$$

Note that with $F = \begin{bmatrix} \mathbf{I} & 0 \\ A & B \end{bmatrix}$ and by virtue of lemma 4, the set $\mathcal{G}$ complies with

$$\mathcal{G} = \left\{ \zeta \in \mathbb{C}^{n+m} \,|\, \zeta \neq 0,\ \ \zeta' S \zeta \leq 0,\ \ \forall S \in \mathcal{S} \right\},$$

where $\mathcal{S}$ is defined in (16). Since it is lossless from lemma 5, the generalized S-procedure [24] can be applied to prove that there exists $S \in \mathcal{S}$ such that $\Theta < S$ wich is equivalent to (ii). □

REMARK **1** *The assumption of $\partial\mathcal{D}$-regularity of A is necessary to ensure that $(sI - A)^{-1}$ exists. However, by virtue of the results of [23] related to the inertia of P, this assumption can be omitted when applying the theorem for the implication $(ii) \Rightarrow (i)$.*

For special applications of this theorem, the following cases can be considered :

(a) $p = 1,\ R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \Phi_1 = 0;$

(b) $p = 1,\ R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ \Phi_1 = 0;$

(c) $p = 1,\ R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \Phi_1 = \begin{bmatrix} -1 & 0 \\ 0 & \omega_0^2 \end{bmatrix}.$

Case (a) and (b) correspond to the continuous-time and discrete-time versions of the well-known KYP lemma [25, 26]. Case (c) corresponds to the so-called finite frequency KYP lemma proposed in [24]. It is not surprising since, as already mentioned, a part of this paper is based on ideas borrowed from [24]. Note that a recent paper [27] generalize the results in [24]. The reader is strongly advised to refer to this article. In comparison with [24, 27], the present contribution is mostly the involvement of the sum in (25) *i.e.* the possibility to consider several sets $\bar{\mathcal{E}}_{k,l}$ (*i.e.* $p > 1$).

3.3 Application to the computation of $r_{\mathcal{D}}$

Theorem 6 is now exploited to solve the stated problem.

Theorem 7 *Let A_c, $\mathcal{D}$, $\partial\mathcal{D}$ be an uncertain matrix as defined in (1), a clustering region and its boundary that reads definition (9) respectively. Assume that A is $\partial\mathcal{D}$-regular. Then A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exist m Hermitian matrices P_k and positive definite Hermitian matrices $Q_{k,l}$, $k = 1, ..., m$ and $l = 1, ..., p_k$, such that*

$$\begin{bmatrix} \mathbf{I} & 0 \\ A & B \end{bmatrix}' \left(R_k \otimes P_k + \sum_{l=1}^{p_k} (\Phi_{k,l} \otimes Q_{k,l}) \right) \begin{bmatrix} \mathbf{I} & 0 \\ A & B \end{bmatrix} + \begin{bmatrix} C'C & 0 \\ 0 & -\gamma \mathbf{I} \end{bmatrix} < 0 \ \forall k \in \{1, ..., m\}, \tag{26}$$

where $\gamma = \rho^{-2}$.
Moreover, if A is assumed to be $\mathcal{D}$-stable then A_c remains $\mathcal{D}$-stable over $\mathcal{B}(\rho)$.

Proof. Inequality (26) corresponds to statement (ii) in Theorem 6 with some special choice of Θ and with respect to all sub-boundaries $\partial\Gamma_k$, $k = 1, ..., m$. Equivalent statement (i) holds *i.e.*

$$\Leftrightarrow ||C(s\mathbf{I} - A)^{-1}B||_2 < \sqrt{\gamma} = \rho^{-1} \quad \forall s \in \partial\Gamma_k.$$

At this stage, arguments similar to those in [17, 23] conclude the proof. □
Any LMI solver can reach $\rho^\star$, the maximum value of $\rho = (\sqrt{\gamma})^{-1}$ under LMI constraint (26) and then

$$\rho^\star = \varrho_{\partial\mathcal{D}} = \min_{k=1,...,m} \varrho_{\partial\Gamma_k}, \tag{27}$$

Furthermore, if A is $\mathcal{D}$-stable, then one gets $\rho^\star = r_{\mathcal{D}}$.

4 Examples

4.1 First example

This simple example highlights the improvement in comparison with [23]. Consider (1) with

$$A = \begin{bmatrix} -1.1388 & 1.8993 \\ -2.5882 & -2.8612 \end{bmatrix}, \quad B = C = \mathbf{I}.$$

The nominal spectrum is $\lambda(A) = \{-2.5; -1.5\}$. Region $\mathcal{D}$ is the union of two intersecting open discs $\mathcal{D}_1$ and $\mathcal{D}_2$ centred around the eigenvalues of A and both of radius 1. Thus, A is $\mathcal{D}$-stable but is both $\partial\mathcal{D}_1$-singular and $\partial\mathcal{D}_2$-singular. That is why LMI condition (5) is found **infeasible**. The reason is that the parts of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ that belong to $\mathcal{D}$ cannot be excluded before the $\partial\mathcal{D}$-regularity test is achieved. In the present work, one can consider, firstly, $\partial\mathcal{D}_1$ intersecting the outside of $\mathcal{D}_2$ and, secondly, $\partial\mathcal{D}_2$ intersecting the outside of $\mathcal{D}_1$. Then one can apply Theorem 7. It leads to the actual value of the complex $\mathcal{D}$-stability radius:

$$\rho^\star = \varrho_{\partial\mathcal{D}} = r_{\mathcal{D}} = 0.39212.$$

500 instances of uncertainties Δ checking $||\Delta||_2 \leq r_{\mathcal{D}}$ are generated using standard MATLAB function `rand`. The associated uncertain root-locus is plotted on Figure 2.

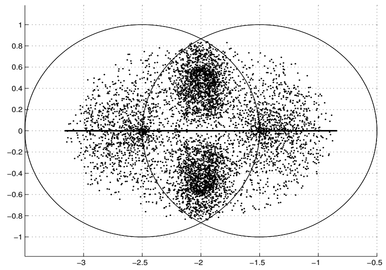

Figure 2: Root-locus perturbed with acceptable uncertainties

4.2 Second example

Before this example is detailed, it is important to notice that Theorem 6 can be used for various purposes different from pole placement analysis. For instance, assume that A is Hurwitz-stable and that matrix Θ involved in Theorem 6 is

$$\Theta = \begin{bmatrix} C'C & D'C \\ D'C & D'D - \gamma\mathbf{I} \end{bmatrix}.$$

Then, Theorem 6 becomes an extension of the well-known LMI version of the bounded real lemma [28] dedicated to the computation of the $\mathcal{H}_\infty$-norm of a linear model. Indeed, this $\mathcal{H}_\infty$-norm is nothing else than $\sqrt{\gamma}$. In this case, since A is Hurwitz-stable, P is positive definite.

To illustrate the possibilities offered by Theorem 6, a simple SISO model is considered:

$$A = \begin{bmatrix} -1.5000 & -1.0100\times10^4 & -1.50050\times10^4 & -1.0000\times10^6 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 10^6 \end{bmatrix}.$$

This realization actually corresponds to the transfer function

$$G(s) = \frac{1}{(1 + 0.01s + 0.01s^2)(1 + 5 \times 10^{-5}s + 10^{-4}s^2)},$$

which shows that there are two peaks of magnitude in the frequency response, one at $\omega_1 = 10rad/s$ and the other one at $\omega_2 = 100rad/s$. Theorem 6 will be used to analyze the magnitude of $G(s)$ in low frequencies, and around the peaks. The principle consists in considering the imaginary axis as the only curve $\partial\mathcal{D}_1$. To select frequencies, one can make a discerning choice of subregions $\bar{\mathcal{E}}_k$. Several tests are achieved.

(a) The static gain is obtained by considering only one region $\bar{\mathcal{E}}_1$ which is the disc centred around the origin with a radius 0.0001. Then, Theorem 6 corresponds to the finite frequency KYP lemma [24] and here yields

$$\gamma \simeq 1.0000 \Rightarrow T = 20\log(\sqrt{\gamma}) = 3.4375 \times 10^{-6} dB.$$

(b) The magnitude of the first peak is found by considering $\bar{\mathcal{E}}_1$ as the disc centred around $10\mathbf{i}$ with a radius 1. The result is

$$\gamma = 1.0228 \times 10^2 \Rightarrow T = 20\log(\sqrt{\gamma}) = 20.098dB.$$

The same value is reached when no region $\bar{\mathcal{E}}_k$ is considered since in that case Theorem 6 is reduced to the classical bounded real lemma and the $\mathcal{H}_\infty$-norm is $\sqrt{\gamma} = 10.11$.

(c) The magnitude of the second peak is analyzed with the disc $\bar{\mathcal{E}}_1$ centred around $100\mathbf{i}$ with a radius 10. It leads to

$$\gamma = 4.0881 \Rightarrow T = 20\log(\sqrt{\gamma}) = 6.1077dB.$$

(d) Both peaks can be analyzed together. One can consider $\bar{\mathcal{E}}_1$ as the intersection of two subregions $\bar{\mathcal{E}}_{1,1}$ and $\bar{\mathcal{E}}_{1,2}$. Here, $\bar{\mathcal{E}}_{1,1}$ is chosen as the disc centred around $55\mathbf{i}$ with a radius 47 while $\bar{\mathcal{E}}_{1,2}$ is chosen as the outside of the disc centred around $55\mathbf{i}$ with a radius 43. Of course the highest magnitude due to the first peak is recovered, *i.e.* $20.098dB$.

To check the above results, the Bode diagram is given by Figure 3

5 Conclusions

In this paper, a technique is proposed to compute the complex $\mathcal{D}$-stability radius when $\mathcal{D}$ is a combination (intersection and/or union) of open half planes, discs, outsides of discs. This point was already addressed in [23] but an overlap of various subregions could lead to a conservative value. In the present contribution, the inconvenient is circumvented. This is made possible by the use of some extension of the KYP lemma. Such a lemma was deduced by considering insightful results of [24], particularly its generalized S-procedure. This procedure appears to be a catalyst for many future works. The reading of [24] and the more recent paper [27] is highly recommended.

As possible further investigations, the case where A, B and C are affected by polytopic uncertainty could be tackled using the matrix elimination lemma [29, Theorem 2.3.12]. Both the application of this approach in a design context and its possible exploitation for model reduction are a challenge.

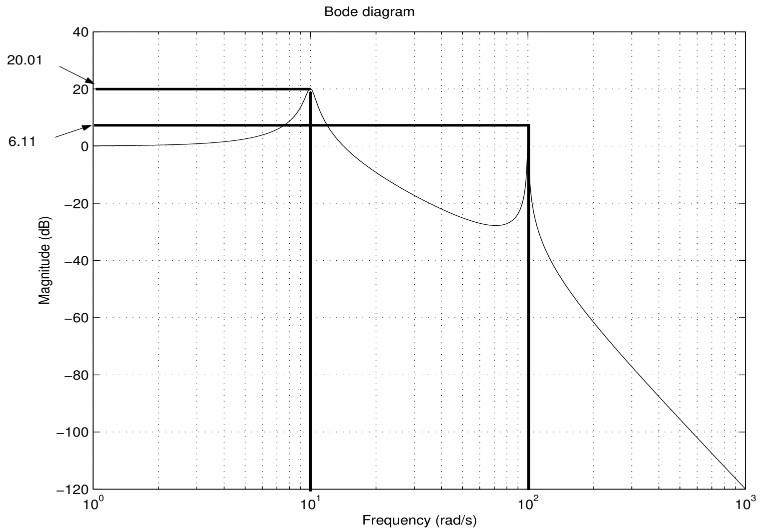

Figure 3: Bode diagram of the model

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