Robust root-clustering of a matrix in intersections or unions of regions

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Abstract

The paper considers robust stability analysis for a matrix affected by unstructured complex uncertainty. A method is proposed to compute a bound on the amount of uncertainty ensuring robust root-clustering in a combination (intersection and/or union) of several possibly nonsymmetric half planes, discs, outsides of discs. In some cases to be detailed, this bound is not conservative. The conditions are expressed in terms of linear matrix inequalities (LMI) and derived through Lyapunov's second method. As a distinctive feature of the approach, the Lyapunov matrices proving robust root-clustering (one per subregion) are not necessarily positive definite, but have prescribed inertias depending on the number of roots in the corresponding subregions. As a special case, when root-clustering in a single half plane, disc or outside of disc is concerned, the whole clustering region reduces to only one convex subregion and the corresponding unique Lyapunov matrix has to be positive definite as usual.

1 Introduction

Robust stability has been raising much interest in the last three decades. Indeed, in a linear state-space context, it matters to attest whether an uncertain state matrix has its eigenvalues in the Open Left Half Plane (OLHP) for continuous-time analysis or in the Open Unit Disc (OUD) for discrete-time analysis. More precisely, assuming nominal stability, it can be useful to estimate the maximal size of the uncertainty domain for which stability is preserved.

The way to estimate this size obviously depends on the structure of the uncertainty. The structured (parametric) case should be distinguished from the unstructured (non parametric) one as pointed out in one of the first contributions due to Patel and Toda [1]. The present contribution is restricted to an unstructured uncertainty, namely the so-called norm-bounded uncertainty [2]. In this context, the maximal acceptable size of uncertainty was clearly defined, in continuous-time, as the stability radius [3]. Such a stability radius was shown to equal the reciprocal of the $\mathcal{H}_\infty$-norm of a strictly proper transfer in [4] and, thus, also appears to be the reciprocal of the maximal structured singular value μ [5] along frequency. The discrete-time counterpart is described in [6]. In these references, the computation of the stability radius could be carried out with iterative resolutions of Riccati equations [7, 8]. Another technique consists in computing μ while sweeping frequencies. With the emergence of convex optimization over Linear Matrix Inequalities (LMI), the stability radius can be computed owing to the LMI version of the Bounded Real Lemma [9].

It also matters to make a difference between the complex stability radius and the real one. The former concerns a complex uncertainty and can be computed with LMI software as just mentioned. The latter takes the realness of the uncertainty into account (what is more discerning to analyze practical plants in automatic control) and is a bit more difficult to obtain, involving a one-dimensional sweeping over the second largest singular value of a parametrized matrix [10, 11]. The present contribution is restricted to the complex case.

When further performances are required, such as transient ones, it might be shrewd to consider a more sophisticated region for the state matrix root-clustering, different from the OLHP or the OUD. Based on the notions of Ω-regions and Generalized Lyapunov Equations (GLE) due to Gutman and Jury [12], Yedavalli has proposed significant robustness bounds [13] (later improved by other authors) but the results are still conservative. The reader is also invited to see [14]. Moreover, the considered regions are usually connected, which might not be suitable for plants with separate dynamics or with specified robust damping ratio. One of the first attempts to consider unions of regions is provided in [15]. The concept of $\mathcal{D}_U$-stability (root-clustering in a some region $\mathcal{D}_U$ whose form encompasses many unions of possible disjoint and nonsymmetric subregions) enables more general results [16]. However, these results remain quite conservative.

This paper is an attempt to consider sophisticated clustering regions by extending the notion of complex stability radius to some combinations (unions and/or intersections) of half planes, discs, exteriors of discs. Besides, the conservatism of the previously proposed methods is reduced. In some typical cases to be further detailed, the exact value of the complex radius is reached.

The paper is organized as follows: section 2 states the problem, presenting the clustering regions and extending the concept of complex $\mathcal{D}$-stability radius to the case where $\mathcal{D}$ is some combination of regions. Section 3 introduces the notion of $\partial\mathcal{D}$-regularity of a nominal matrix, which is the non-membership of the matrix eigenvalues to a geometric curve $\partial\mathcal{D}$. Such a property can often be checked through the derivation of a Lyapunov matrix which is not necessarily positive definite but is just nonsingular with constant inertia. When $\mathcal{D}$-stability is concerned, the Lyapunov matrix is required to be strictly positive or negative definite. In section 4, the considered matrix is affected by an additive norm-bounded uncertainty. Based upon the notion of $\partial\mathcal{D}$-regularity, a method to reach the complex $\mathcal{D}$-stability radius is proposed. In some cases to be detailed, the exact value is obtained. Numerical examples are provided in section 5 before conclusion.

Notations

M' denotes the transpose conjugate of matrix M. Hence, s' is the conjugate of complex number s. M^H is the Hermitian matrix $M + M'$. The 2-norm of M induced by the Euclidean vector norm (maximal singular value) is denoted by $||M||_2$. $\mathbb{I}_n$ is the identity matrix of order n and $\mathbb{O}$ is a null matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Löwner *i.e.* >0 (resp. <0) means positive (resp. negative) definite. Symbol $\mathbf{i}$ denotes the imaginary unit and $\lambda(A)$ denotes the spectrum of square matrix A. $\mathfrak{I}$ and $\mathfrak{C}$ are the imaginary axis and the unit circle respectively. At last, the vector $\mathrm{In}(M) = [n_+ \; n_- \; n_0]$ is the inertia of a square matrix M if n_+, n_-, n_0 are the numbers of eigenvalues of M with positive, negative and zero real part, respectively.

2 Problem statement

First, the form of the uncertain matrix to be analyzed is given. Then, the clustering region is introduced. At last, the problem to be solved is stated.

2.1 The uncertain matrix

The considered matrix reads:

$$A_c \;=\; A + B\Delta C \;\in \mathbb{C}^{n\times n}. \tag{1}$$

In the above expression, Δ is constant, unknown and assumed to belong to $\mathcal{B}(\rho)$, the ball of all matrices $\Delta \in \mathbb{C}^{q\times r}$ satisfying $||\Delta||_2 \leq \rho$. Matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times q}$ and $C \in \mathbb{C}^{r\times n}$ are perfectly known. Such a description is referred to as norm-bounded uncertainty [2].

REMARK 1 *In this work, all the matrices are assumed to be complex. However, in practice, when transient performances of a linear state-space model are to be analyzed, state matrix A_c is real. In that case, Δ is then real and $\mathcal{B}(\rho)$ must be restricted to real matrices. Actually, this restriction is more difficult to take into account and this case is not investigated in the paper although some interesting results exist [10, 11].*

2.2 Clustering region $\mathcal{D}$

Consider the following geometric curves:

$$\left\{\begin{array}{l} \partial\mathcal{D}_k = \left\{ s \in \mathbb{C} | f_k(s) = r_{k_{00}} + (r_{k_{10}} s)^H + r_{k_{11}} s' s = 0 \right\} \\ \\ \{r_{k_{00}}, r_{k_{10}}, r_{k_{11}}\} \in \mathbb{R}\times\mathbb{C}\times\mathbb{R} \end{array}\right. \quad \forall k \in \{1, ..., m\}. \tag{2}$$

Relevant curves are lines or circles. Each curve $\partial\mathcal{D}_k$ enables to define an associated region:

$$\mathcal{D}_k \;=\; \{s \in \mathbb{C} \,|\, f_k(s) < 0\} \quad \forall k \in \{1, ..., m\}. \tag{3}$$

Clearly, $\mathcal{D}_k$ denotes either one side or the other side of the boundary $\partial\mathcal{D}_k$. It can then be a half plane, a disc or the outside of a disc. It is an open region (*i.e.* not including $\partial\mathcal{D}_k$) in order to encompass the concept of asymptotic stability for LTI systems. $\mathcal{D}_k$ can actually correspond to the scalar case of regions defined in [17] or to a special case of second order Ω-regions [12].

Also define the region $\mathcal{D}$ as a combination, *i.e.* any union and/or intersection of the various subregions $\mathcal{D}_k$. Such a formulation of $\mathcal{D}$ clearly enables a very large choice of clustering regions.

2.3 Problem Statement

This contribution aims at computing the complex $\mathcal{D}$-stability radius.

More precisely, assume that A is $\mathcal{D}$-stable *i.e* that the whole of its spectrum lies in $\mathcal{D}$. Define $r_{\mathcal{D}}$ as the largest value of ρ, the radius of $\mathcal{B}(\rho)$, such that A_c defined in (1) remains $\mathcal{D}$-stable for any $\Delta \in \mathcal{B}(\rho)$. Such a value is the so-called complex $\mathcal{D}$-stability radius. A lower bound $\rho^\star$ of $r_{\mathcal{D}}$, as tight as possible, is to be computed. For this purpose, the concept of $\partial\mathcal{D}$-regularity is introduced in the next section.

3 $\partial\mathcal{D}$-regularity

In this section, only nominal matrices are considered. The concepts of matrix $\partial\mathcal{D}$-regularity and matrix $\partial\mathcal{D}$-singularity are introduced. A necessary and sufficient condition for matrix $\partial\mathcal{D}$-regularity to be satisfied when $\partial\mathcal{D}$ is defined as in (2) is expressed in terms of LMI. After preliminary notions and assumptions in subsection 1, subsection 2 presents this condition through a first theorem. In subsection 3, the distribution of the matrix eigenvalues with respect to the boundary $\partial\mathcal{D}$ is connected to the inertia of the solution to the LMI, owing to a second theorem. Subsection 4 is devoted to a discussion about these theorems.

3.1 Preliminaries

Definition 1 *Let $\partial\mathcal{D}$ be any curve in the complex plane, then matrix $A \in \mathbb{C}^{n\times n}$ is called:*

- *$\partial\mathcal{D}$-singular when $\lambda(A) \cap \partial\mathcal{D} \neq \emptyset$;*
- *$\partial\mathcal{D}$-regular when $\lambda(A) \cap \partial\mathcal{D} = \emptyset$.*

Remark 2 *Assume that $\partial\mathcal{D}$ is a boundary separating two open regions $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ (then $\mathbb{C} = \mathcal{D} \cup \partial\mathcal{D} \cup \bar{\mathcal{D}}^C$). Matrix A is $\mathcal{D}$-stable if and only if it is $\partial\mathcal{D}$-regular and the whole of its spectrum lies in $\mathcal{D}$. Otherwise, it is $\mathcal{D}$-unstable.*

Now assume that $\partial\mathcal{D}$ here reduces to only one curve as defined in (2) *i.e.*:

$$\partial\mathcal{D} = \{s \in \mathbb{C} \,|\, f(s) \;=\; r_{00} + (r_{10}s)^H + r_{11}s's = 0\}. \tag{4}$$

In parallel with the work of Hill [18], we state two theorems in the next two parts. The result provided in [18] is based on Ostrowski and Schneider's theorem [19] and on Frobenius's theorem. Our contribution is more part of the framework relevant to Lyapunov's second method [20] and its extensions to root-clustering [12].

3.2 LMI condition for $\partial\mathcal{D}$-regularity

Theorem 2 *Let A and $\partial\mathcal{D}$ be respectively a complex square matrix of dimension n and a curve of the complex plane as defined in (4). Matrix A is $\partial\mathcal{D}$-regular if and only if there exists an Hermitian matrix $P \in \mathbb{C}^{n\times n}$ such that:*

$$F(A,P) \;=\; r_{00}P \;+\; (r_{10}PA)^H \;+\; r_{11}A'PA \;<\; 0. \tag{5}$$

Proof. Some arguments are inspired from [21, 22].

Sufficiency: first assume that there exists a suitable P such that (5) holds. Also assume that A has $\bar{n} \leq n$ distinct eigenvalues. It suffices to prove the non membership of these $\bar{n}$ eigenvalues to $\partial\mathcal{D}$. Let λ_j denote the j-th eigenvalue of A. There exists a non zero vector v_j such that $Av_j = \lambda_j v_j$, so it comes:

$$v_j' F(A,P) v_j \;<\; 0 \quad \forall j \in \{1, ..., \bar{n}\}$$

$$\Leftrightarrow\ r_{00}v_j'Pv_j\ +(r_{10}v_j'PAv_j)^H\ +r_{11}v_j'A'PAv_j\ <\ 0\quad \forall j\in\{1,...,\bar{n}\}$$
$$\Leftrightarrow\ (r_{00}\ +\ (r_{10}\lambda_j)^H\ +\ r_{11}\lambda_j'\lambda_j)(v_j'Pv_j)\ <\ 0\quad \forall j\in\{1,...,\bar{n}\}$$

which implies

$$f(\lambda_j)\ =\ r_{00}\ +\ (r_{10}\lambda_j)^H\ +\ r_{11}\lambda_j'\lambda_j\neq 0\quad \forall j\in\{1,...,\bar{n}\}$$

and hence

$$\lambda_j\notin\partial\mathcal{D}\quad \forall j\in\{1,...,\bar{n}\}\Leftrightarrow\lambda_i\notin\partial\mathcal{D}\quad \forall i\in\{1,...,n\}.$$

Matrix A is then $\partial\mathcal{D}$-regular.

Necessity: now assume that A is $\partial\mathcal{D}$-regular. Then:

$$f(\lambda_i)\ \neq\ 0\quad \forall i\in\{1,...,n\}. \tag{6}$$

It means that $f(\lambda_i)$ is either strictly positive or strictly negative. Define matrix Λ as:

$$\Lambda\ =\ \operatorname*{diag}_{i=1,\dots,n}\{\lambda_i\}\ =\left[\begin{array}{cc}\Lambda_1 & \mathbb{O}\\ \mathbb{O} & \Lambda_2\end{array}\right]$$

where $\Lambda_1\in\mathbb{C}^{n_-}$ contains the various λ_i for which $f(\lambda_i)>0$ and $\Lambda_2\in\mathbb{C}^{n_+}$ contains those for which $f(\lambda_i)<0$. With

$$\mathcal{I}=\left[\begin{array}{cc}-\mathbb{I}_{n_-} & \mathbb{O}\\ \mathbb{O} & \mathbb{I}_{n_+}\end{array}\right], \tag{7}$$

it holds:

$$F(\Lambda,\mathcal{I})\ =\ r_{00}\left[\begin{array}{cc}-\mathbb{I}_{n_-} & \mathbb{O}\\ \mathbb{O} & \mathbb{I}_{n_+}\end{array}\right]+\left(r_{10}\left[\begin{array}{cc}-\Lambda_1 & \mathbb{O}\\ \mathbb{O} & \Lambda_2\end{array}\right]\right)^H+r_{11}\left[\begin{array}{cc}-\Lambda_1'\Lambda_1 & \mathbb{O}\\ \mathbb{O} & \Lambda_2'\Lambda_2\end{array}\right]<\ 0. \tag{8}$$

Let $\mathcal{J}$ be the Jordan canonical form of A. There exists a sequence of full rank matrices T_l such that [22]:

$$\lim_{l\to\infty}(T_l\mathcal{J}T_l^{-1})\ =\ \Lambda \tag{9}$$

(for example, if $\mathcal{J}$ is a single Jordan block $\left[\begin{array}{cc}\lambda & 1\\ 0 & \lambda\end{array}\right]$, T_l equals $\left[\begin{array}{cc}\frac{1}{l} & 0\\ 0 & 1\end{array}\right]$). Since F is continuous with respect to its first argument, from (8) it comes:

$$\lim_{l\to\infty}F(T_l\mathcal{J}T_l^{-1},\mathcal{I})\ <\ 0. \tag{10}$$

Let $\mathcal{T}$ denote T_l for a sufficiently large $l<\infty$ (*i.e.* such that $\mathcal{T}^{-1}$ exists) so that

$$F(\mathcal{T}\mathcal{J}\mathcal{T}^{-1},\mathcal{I})<0.$$

Applying congruence on the previous inequality leads to:

$$\mathcal{T}'F(\mathcal{T}\mathcal{J}\mathcal{T}^{-1},\mathcal{I})\mathcal{T}=F(\mathcal{J},\mathcal{T}'\mathcal{I}\mathcal{T})<0.$$

Now assume that V denotes a matrix such that $\mathcal{J}=VAV^{-1}$. Left and right multiplying the previous inequality respectively by V' and V leads to (5) with:

$$P\ =\ V'\mathcal{T}'\mathcal{I}\mathcal{T}V. \tag{11}$$

Note that if there exists a matrix V such that $\Lambda=VAV^{-1}$ (or equivalently $\Lambda=\mathcal{J}$), then the above reasoning stands with $T_l=\mathbb{I}_n\,\forall l$. This completes the proof. $\square$

3.3 Root-distribution and the inertia of P

Theorem 3 *Let A and $\partial\mathcal{D}$ be respectively a complex square matrix of dimension n and a curve of the complex plane as defined in (4). Matrix A is $\partial\mathcal{D}$-regular with n_+ eigenvalues in $\mathcal{D}$ and n_- eigenvalues outside $\mathcal{D}$ if and only if any Hermitian solution P to LMI (5) has inertia* $\mathrm{In}(P) = [n_+,\ n_-,\ 0]$.

Proof. Some possible expression of P is given by (11). Since inertia is invariant under congruence, $\mathrm{In}(P) = \mathrm{In}(\mathcal{I}) = [n_+,\ n_-,\ 0]$ where $\mathcal{I}$ is defined in (7). The question is to know if it is possible that there exists some other solution to LMI (5) with a different inertia. If it is possible, by continuity of the convex set of solutions, there exists a singular X such that:

$$F(A, X) < 0.$$

Let $\mathcal{J}$ be a Jordan canonical form similar to A and V be the associated similarity matrix. By congruence, it comes:

$$F(\mathcal{J}, Z) < 0,$$

where $Z = (V')^{-1}XV^{-1}$ is also a singular matrix. Taking the sequence of full rank matrices T_l introduced in (9) into account enables to deduce:

$$F(\lim_{l\to\infty}(T_l^{-1}\Lambda T_l), Z) < 0,$$

which by continuity of F with respect to its first argument leads to

$$\lim_{l\to\infty} F(T_l^{-1}\Lambda T_l, Z) < 0.$$

Once again, let $\mathcal{T}$ denote T_l for a sufficiently large value of $l < \infty$ so that

$$F(\mathcal{T}^{-1}\Lambda\mathcal{T}, Z) < 0.$$

Applying congruence on the previous inequality leads to

$$F(\Lambda, Y) < 0, \tag{12}$$

where $Y = \begin{bmatrix} Y_1 & Y_3 \\ Y_3' & Y_2 \end{bmatrix} = (\mathcal{T}')^{-1}Z\mathcal{T}^{-1}$ is singular. At this stage a distinction is made between two cases:

- $\underline{r_{11} = 0}$: in this case, $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ are half planes. By Schur factorization, there exists a unitary matrix U such that $Y = U'SU$ where S is a diagonal matrix whose diagonal entries are the eigenvalues of Y. With no loss of generality, Y being singular, the first diagonal entry is assumed to be zero. By congruence, one gets

$$F(U\Lambda U', S) < 0.$$

The previous inequality never holds since the first diagonal entry of the left member is zero. Hence, in this case, Y or X can not be singular and the inertia of the solution is invariant.

- $\underline{r_{11} \neq 0}$: in this case, first note that $\partial\mathcal{D}$ can be defined another way:

$$\partial\mathcal{D} = \{s \in \mathbb{C} \,|\, f(s) = \tilde{r}_{00} + r_{11}(s' + \frac{r_{10}}{r_{11}})(s + \frac{r_{10}'}{r_{11}}) = 0\} \tag{13}$$

with

$$\tilde{r}_{00} = r_{00} - \frac{r_{10}r'_{10}}{r_{11}}.$$

Note that $\frac{\tilde{r}_{00}}{r_{11}}$ is strictly negative otherwise $\partial\mathcal{D}$ reduces to $\emptyset$. In terms of matrix inequality, it leads to rewriting (12) as follows:

$$\tilde{F}(\tilde{\Lambda}, Y) = \tilde{r}_{00}Y + r_{11}\tilde{\Lambda}'Y\tilde{\Lambda} < 0$$

where $\tilde{\Lambda} = \Lambda + \frac{r'_{10}}{r_{11}}\mathbb{I}_n$ is $\partial\tilde{\mathcal{D}}$-regular, $\partial\tilde{\mathcal{D}}$ being the circle centered around the new origin and of radius $\sqrt{-\tilde{r}_{00}/r_{11}}$. Two diagonal blocks can be extracted from the previous inequality:

$$\left\{ \begin{array}{l} \tilde{F}(\tilde{\Lambda}_1, Y_1) < 0 \\ \tilde{F}(\tilde{\Lambda}_2, Y_2) < 0. \end{array} \right.$$

Clearly, $\tilde{\Lambda}_1 = \Lambda_1 + \frac{r'_{10}}{r_{11}}\mathbb{I}_{n_-}$ and $\tilde{\Lambda}_2 = \Lambda_2 + \frac{r'_{10}}{r_{11}}\mathbb{I}_{n_+}$ are such that $\tilde{\Lambda}_2$ is $\tilde{\mathcal{D}}$-stable and $\tilde{\Lambda}_1$ has its whole spectrum outside $\tilde{\mathcal{D}}$. It is clear, from the beginning of the proof, that there exists a Hermitian matrix $\tilde{Y}_1 < 0$ such that $\tilde{F}(\tilde{\Lambda}_1, \tilde{Y}_1) < 0$. It will be now shown that any other solution Y_1 remains negative definite by continuity *i.e.* never becomes singular. First note that since $\tilde{\Lambda}_1$ has its spectrum outside of the disc $\tilde{\mathcal{D}}$, it can be inverted and it comes :

$$r_{11}Y_1 + \tilde{r}_{00}(\tilde{\Lambda}_1^{-1})'Y_1\tilde{\Lambda}_1^{-1} < 0.$$

Assume that Y_1 is negative semi-definite. Achieve Schur factorization of Y_1 such that $Y_1 = U'_1S_1U_1$. The diagonal entries of S_1 are negative except the first one that is zero. By congruence, it comes

$$S_1 + \frac{\tilde{r}_{00}}{r_{11}}U_1(\tilde{\Lambda}_1^{-1})'U'_1S_1U_1\tilde{\Lambda}_1^{-1}U'_1 < 0.$$

Since $\frac{\tilde{r}_{00}}{r_{11}} < 0$, the first diagonal entry of the left handside member turns to be positive or zero thus inequality never holds. Y_1 is then necessarily strictly negative definite.
In the same way, there also exists some Hermitian matrix $\tilde{Y}_2 > 0$ such that $\tilde{F}(\tilde{\Lambda}_2, \tilde{Y}_2) < 0$. Is it possible that, by continuity, some other solution Y_2 becomes singular *i.e.* positive semi-definite? Assume that such a Y_2 exists and by Schur factorization $Y_2 = U'_2S_2U_2$ (S_2 is diagonal with the first diagonal entry zero and the other ones positive) and by congruence, it comes:

$$\tilde{F}(U_2\tilde{\Lambda}_2U'_2, S_2) < 0.$$

Since $\tilde{\Lambda}_2$ and S_2 are diagonal matrices and since the first diagonal entry of S_2 is zero, then the first diagonal entry of the left handside member is positive or zero which leads to a contradiction. Thus, Y_2 is necessarily strictly positive definite.
Now coming back to Y, using Schur complement, one gets:

$$\mathrm{In}(Y) = \mathrm{In}\left(\left[\begin{array}{c|c} Y_1 & \mathbb{O} \\ \hline \mathbb{O} & Y_2 - Y'_3Y_1^{-1}Y_3 \end{array} \right] \right).$$

Clearly since $Y_2 > 0$ and $Y_1 < 0$, none of the diagonal blocks of the above matrix can be singular. Hence, $\mathrm{In}(Y) = \mathrm{In}(X) = \mathrm{In}(P)$ and the proof is completed. $\square$

As mentioned in subsection 3.1, our result is an alternative to results by Hill [18]. However, our result is summarized under the form of one single theorem valid both for any line and any circle. It does not require any preliminary mathematical lemma except Sylvester's well known theorem ($\mathrm{In}(H) = \mathrm{In}(MHM')$ for any nonsingular matrix M). Although, it is formulated as in Lyapunov's framework, the proofs basically require simple algebraic manipulations which, the authors hope, will help extension to other curves. In that sense, it is closer to the result of [21]. Nevertheless, some differences are pointed out in the discussion provided in the next subsection.

Corollary 4 *Let A and $\partial\mathcal{D}$ be respectively a complex square matrix of dimension n and a curve of the complex plane as defined in (4). Also let $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ be the regions defined by $f(s) < 0$ and $f(s) > 0$ respectively. Matrix A is $\mathcal{D}$-stable (respectively $\bar{\mathcal{D}}^C$-stable) if and only if there exists an Hermitian positive (respectively negative) definite matrix $P \in \mathbb{C}^{n\times n}$ such that (5) holds.*

Proof. It is direct from theorem 3 with considering inertia $\text{In}(P) = [n\ \ 0\ \ 0]$ (respectively $\text{In}(P) = [0\ \ n\ \ 0]$). $\square$

As a special cases, $\mathcal{D}$ can be the OLHP or the OUD. In those cases, Lyapunov's theorem [20] and Stein's theorem [23] are respectively recovered.

Remark **3** *Since any solution to (5) is necessarily nonsingular then there is no need to specify it as a constraint and (5) is a simple LMI in P easy to test with various existing LMI software. Note that the non strict LMI cannot be considered here because it would allow $\partial\mathcal{D}$-singularity.*

3.4 Discussion

With appropriate changes, the above proof could be adapted to any second order Ω-transformable region. In that sense, this could be seen as a special case of [21]. However, although [21, theorem 1] seems suitable to prove the first statement in theorem 2, we do not agree with the proof of [21, theorem 2] related to the inertia of P. Indeed after having proven that some solution P to an LMI exists and has expression [21, eq. (6)], it is claimed that for any choice of the negative definite right member of the associated equality, the solution to this equality keeps the same expression. We do not agree with that point. Perhaps the same doubt led Jury to achieve a special proof for the nonsingularity of the solution of a GLE [24, theorem 3.16]. The notion of Ω-transformability was required to prove this nonsingularity (see also [12, theorem 12]). It could at first sight directly be derived from [21, theorem 2] but we also think that this special proof was necessary.

Going on with Ω-transformability, it is interesting to see that although transformability seemed to be required to prove the nonsingularity of the solution to a GLE [12], we here show that, owing to the notion of $\partial\mathcal{D}$-regularity (rather than just $\mathcal{D}$-stability), $\mathcal{D}$ and $\bar{\mathcal{D}}^C$ are considered altogether (the reader is here reminded of the fact that $\mathcal{D}$-stability and $\bar{\mathcal{D}}^C$-stability are only special cases of $\partial\mathcal{D}$-regularity as stated in corollary 4). As a consequence, the outside of a disc is a non Ω-transformable region for which it is impossible to find a singular solution to a corresponding GLE. Otherwise, it would be in contradiction with remark 3. In other words, any solution to a GLE attesting matrix root-clustering is necessarily nonsingular (existence and uniqueness of the solution to a GLE is another problem : see [12]). The outside of the disc is then a region for which Ω-transformability is not required to guarantee the nonsingularity of the solution to a corresponding GLE. It is what was illustrated by an example proposed in [25].

Apart from our doubt on the proof of [21, theorem 2], we would like to add that this contribution seems to have been overlooked. Actually, [21, theorem 1] is nothing but an LMI test for matrix root-clustering in an Ω-region. In 1971, such a test was not tractable from a computational point of view (at about the same time, Willems was just beginning to warn the control community about the great interest in handling LMI [26]). For this reason, it mattered to “convert” this LMI test into a GLE [12]. Now that LMI have become classical tools, although some significant contributions enabled to test matrix root-clustering via LMI conditions [22], many authors should remember the pioneer work of [21].

Furthermore, if solving an LMI is not an obstacle, results in [21] can be used to consider problem 85 formulated by Wang in the electronic book proposed by Blondel and Megretski [27]. In problem 85, the analysis of root-clustering in Ω-transformable regions of order greater than 2 is concerned and expressed in terms of GLE. Of course, it is an interesting challenge from a mathematical point of view. At first sight, as far as root-clustering analysis is concerned, the LMI approach, as used in this paper or in [21], could be exploited to solve the problem and the GLE approach could seem useless. However it is not really true because the problem is not purely theoretical. When problems involving very high dimensions are to be solved, the LMI approach generates heavy computations and can lead to numerical failure while some P actually exists. This is why the GLE approach preserves all its interest. To solve a GLE requires to choose a right handside member in order to get a simple linear system. Thus it matters to be sure that this member can be arbitrarily chosen and will always lead to a unique nonsingular solution P. Such a possible arbitrary choice is probably strongly related to Ω-transformability and that is exactly the point presented in problem 85. To sum up this part of the discussion, for low dimensional problems, the LMI approach might enable to free of the transformability assumption and of GLE but for high dimensional problems, those notions are still fundamental.

Chapter 17 of [28] describes the guardian map as an alternative tool for checking $\mathcal{D}$-stability or $\partial\mathcal{D}$-regularity. The guardian map is a mapping ν such that $\nu(A) \neq 0$ if $A \in \mathcal{A}$ and $\nu(A) = 0$ if $A \notin \mathcal{A}$. For example the mapping $\nu(A) = \det A$ guards the sets of non-singular matrices, and $\nu(A) = \det H(p)$ guards the sets of Hurwitz stable matrices, where $H(p)$ is the Hurwitz matrix of characteristic polynomial $p(s) = \det\,(sI - A)$. As shown in [28], checking properties like robust regularity or stability then amounts to computing eigenvalues of matrices related with guardian maps. Our approach can be viewed as an alternative to guardian map techniques, in the sense that the robustness certificate for regularity or stability is the existence of a Lyapunov matrix with given inertia solving a given LMI.

4 Complex $\mathcal{D}$-stability radius

In this section, the uncertain case is studied. First, a necessary and sufficient condition for the uncertain A_c defined in (1) to be $\partial\mathcal{D}$-regular when $\partial\mathcal{D}$ complies to (4) is given. This condition is then used to compute the $\mathcal{D}$-stability radius when $\mathcal{D}$ is some combination of regions as defined in subsection 2.2.

Theorem 5 *Let A_c and $\partial\mathcal{D}$ be respectively an uncertain matrix as defined in (1) and a geometric curve as defined in (4). Assume that nominal matrix A is $\partial\mathcal{D}$-regular. Matrix A_c is robustly $\partial\mathcal{D}$-regular against the ball $\mathcal{B}(\rho)$ if and only if there exists some Hermitian matrix P with inertia* $\mathrm{In}(P) = [n_+,\ n_-,\ 0]$ *such that*

$$\mathcal{Q}(P,\gamma) = \left[\begin{array}{cc} r_{00P} + (r_{10}PA)^H + r_{11}A'PA + C'C & r_{10}PB + r_{11}A'PB \\ r'_{10}B'P + r_{11}B'PA & r_{11}B'PB - \gamma\mathbb{I}_q \end{array}\right] < 0. \qquad (14)$$

with $\gamma = \rho^{-2}$. In this event, A_c keeps n_+ eigenvalues inside $\mathcal{D}$ and n_- outside $\mathcal{D}$.

Proof. First assume that $r_{11} = 0$. Then $\partial\mathcal{D}$ is a line. By appropriate rotation and shifting, it can be transformed into the imaginary axis $\partial\hat{\mathcal{D}} = \Im$. Apply the same mapping on matrices A and then the necessary and sufficient condition for the new matrix $\hat{A}_c = \hat{A} + B\Delta C$ to be $\partial\hat{\mathcal{D}}$-regular can be deduced from lemma 7 recalled in appendix. Indeed, consider lemma 7. Condition (14) corresponds to (28) with A substituted by $\hat{A}$ and

$$M = \left[\begin{array}{cc} C'C & \mathbb{O} \\ \mathbb{O} & -\gamma\mathbb{I}_q \end{array}\right], \qquad (15)$$

which, is equivalent to (27). It comes

$$(C(\mathbf{i}\omega\mathbb{I} - \hat{A})^{-1}B)'(C(\mathbf{i}\omega\mathbb{I} - \hat{A})^{-1}B) < \gamma\mathbb{I}_q \quad \forall\omega \in \mathbb{R} \qquad (16)$$

In the same way, if $r_{11} \neq 0$ then $\partial\mathcal{D}$ is a circle. With appropriate scaling and shifting, $\partial\mathcal{D}$ can be transformed into the unit circle $\partial\hat{\mathcal{D}} = \mathfrak{C}$. Apply the same mapping on matrix A in order to get the new matrix $\hat{A}$. Uncertain matrix $\hat{A}_c = \hat{A} + B\Delta C$ has to be $\mathfrak{C}$-regular, what can be tested owing to lemma 8 given in appendix. Condition (14) is equivalent to (30) with $A = \hat{A}$ and M given by (15), which is equivalent to (29). It comes

$$(C(e^{\mathbf{i}\omega}\mathbb{I} - \hat{A})^{-1}B)'(C(e^{\mathbf{i}\omega}\mathbb{I} - \hat{A})^{-1}B) < \gamma\mathbb{I}_q \quad \forall\omega \in \mathbb{R}. \tag{17}$$

Both cases then lead to the equivalent condition:

$$\Leftrightarrow ||C(s\mathbb{I}_n - \hat{A})^{-1}B||_2 < \sqrt{\gamma} = \rho^{-1} \quad \forall s \in \partial\hat{\mathcal{D}}.$$

Using classical arguments on singular values yields

$$\left[\inf\left\{||\Delta||_2 | \Delta \in \mathbb{C}^{q\times r} \text{ and } \det(\mathbb{I}_q - \Delta C(s\mathbb{I}_n - \hat{A})^{-1}B) = 0\right\}\right]^{-1} < \rho^{-1} \quad \forall s \in \partial\hat{\mathcal{D}},$$

which is the definition of the structured singular value. Using properties of det(.), the rewriting of the above expression leads to:

$$\left[\inf\left\{||\Delta||_2 | \Delta \in \mathbb{C}^{q\times r} \text{ and } \det(s\mathbb{I}_n - (\hat{A} + B\Delta C)) = 0\right\}\right]^{-1} < \rho^{-1} \quad \forall s \in \partial\hat{\mathcal{D}} \Leftrightarrow$$

$$\eta^{-1}(s) = \left[\inf\left\{||\Delta||_2 | \Delta \in \mathbb{C}^{q\times r} \text{ and } s \in \lambda(\hat{A} + B\Delta C)\right\}\right]^{-1} < \rho^{-1}\ \forall s \in \partial\hat{\mathcal{D}} \tag{18}$$

$$\Leftrightarrow \eta(s) > \rho \quad \forall s \in \partial\hat{\mathcal{D}}.$$

The above inequality shows that any uncertain matrix Δ inducing the $\partial\hat{\mathcal{D}}$-singularity of matrix $\hat{A}_c$ is such that $||\Delta||_2 > \rho$. As a consequence, $\hat{A}_c$ is $\partial\hat{\mathcal{D}}$-regular over $\mathcal{B}(\rho)$. Because each eigenvalue of A is implicitly subject to the same mapping as A, the root-distribution of A with respect to $\partial\mathcal{D}$ is the same as the root-distribution of $\hat{A}$ with respect to $\partial\hat{\mathcal{D}}$. From the block $(1,1)$, in (14), it can be seen that P guarantees the $\partial\mathcal{D}$-regularity of A. In virtue of theorem 3, the root-distribution of A with respect to $\partial\mathcal{D}$ is given by the inertia of P. Since A_c never becomes $\partial\mathcal{D}$-singular over $\mathcal{B}(\rho)$, the inertia of P gives the root-distribution of A_c over $\mathcal{B}(\rho)$. $\Box$

The value of ρ obtained when minimizing γ under LMI constraints (14) is the largest acceptable value of ρ. Thus, it is what can be called the complex $\partial\mathcal{D}$-regularity radius. It will be denoted by $\varrho_{\partial\mathcal{D}}$ in the sequel.

Now come back to region $\mathcal{D}$ defined as a combination of several regions $\mathcal{D}_k$ (see subsection 2.2). Referring to previous works on stability radii [3, 10], the complex $\mathcal{D}$-stability radius can also be defined as follows:

$$r_{\mathcal{D}} = \inf\{||\Delta||_2 \,|\, \Delta \in \mathbb{C}^{q\times r} : A + B\Delta C \text{ is } \mathcal{D}\text{-unstable}\} \tag{19}$$

The complex $\partial\mathcal{D}$-regularity of a complex matrix A (not necessarily $\mathcal{D}$-stable) is here defined by:

$$\varrho_{\partial\mathcal{D}} = \inf\{||\Delta||_2 \,|\, \Delta \in \mathbb{C}^{q\times r} : \lambda(A + B\Delta C) \cap \partial\mathcal{D} \neq \emptyset\} \tag{20}$$

where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$. The formulation of $r_{\mathcal{D}}$ implicitly assumes that A is $\mathcal{D}$-stable. That is the basic difference with the formulation of $\varrho_{\partial\mathcal{D}}$ for which no assumption of nominal $\mathcal{D}$-stability is made. For this reason, it comes :

$$A \text{ is } \mathcal{D}\text{-stable} \Rightarrow \varrho_{\partial\mathcal{D}} = r_{\mathcal{D}}. \tag{21}$$

Also define the set $\partial\mathcal{U}$ by:

$$\partial\mathcal{U} = \bigcup_{k=1}^{m} \partial\mathcal{D}_k. \tag{22}$$

It is clear that $\partial\mathcal{D} \subset \partial\mathcal{U}$. Hence, A_c has no eigenvalue on $\partial\mathcal{D}$ if it has no eigenvalue on $\partial\mathcal{U}$ so it comes:

$$\varrho_{\partial\mathcal{U}} = \min_{k\in\{1,\dots,m\}} \varrho_{\partial\mathcal{D}_k} \leq \varrho_{\partial\mathcal{D}}. \tag{23}$$

Moreover, if $\partial\mathcal{U} = \partial\mathcal{D}$, the above inequality becomes an equality. Besides, if A is assumed to be $\mathcal{D}$-stable and if $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$, it can be assessed that:

$$\varrho_{\partial\mathcal{U}} = r_{\mathcal{D}}. \tag{24}$$

To make the previous reasoning clear, it is illustrated by figures 1 and 2 where region $\mathcal{D}$ is hatched.

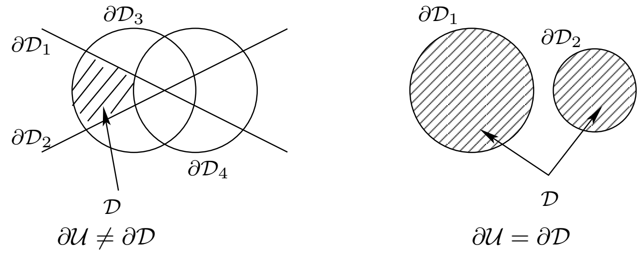

Figure 1: $\varrho_{\partial\mathcal{D}} = \varrho_{\partial\mathcal{U}}$?

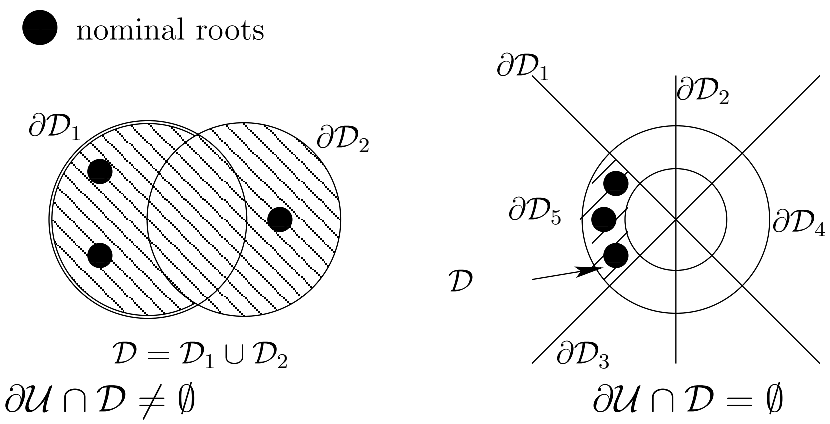

Figure 2: $\varrho_{\partial\mathcal{U}} = r_{\mathcal{D}}$?

Figure 1 shows the two cases where $\partial\mathcal{D}$ is either different from $\partial\mathcal{U}$ (then $\varrho_{\partial\mathcal{D}} \neq \varrho_{\partial\mathcal{U}}$) or equal to $\partial\mathcal{U}$ (then $\varrho_{\partial\mathcal{D}} = \varrho_{\partial\mathcal{U}}$). Figure 2 highlights the fact that if A is $\mathcal{D}$-stable, and if $\partial\mathcal{U} \cap \mathcal{D}$ is empty, then $\varrho_{\partial\mathcal{U}}$ equals $r_{\mathcal{D}}$.

From the previous reasoning, the next theorem is deduced:

Theorem 6 *Let A_c, $\mathcal{D}$, $\partial\mathcal{D}$ be respectively an uncertain matrix as defined in (1), a clustering region as defined in subsection 2.2 and its boundary. Then A_c is robustly $\partial\mathcal{D}$-regular against $\mathcal{B}(\rho)$ if there exist m Hermitian matrices P_k, $k = 1, \dots, m$, such that:*

$$\mathcal{Q}_k(P_k, \gamma) < 0 \quad \forall k \in \{1, \dots, m\}, \tag{25}$$

where

$$\mathcal{Q}_k(P_k,\gamma) = \begin{bmatrix} r_{k_{00}}P_k + (r_{k_{10}}P_kA)^H + r_{k_{11}}A'PA + C'C & r_{k_{10}}PB + r_{k_{11}}A'PB \\ r'_{k_{10}}B'P + r_{k_{11}}B'PA & r_{k_{11}}B'PB - \gamma \mathbb{I}_q \end{bmatrix}$$

and $\gamma = \rho^{-2}$.
In this event, A_c *keeps the same number of eigenvalues inside* $\mathcal{D}$ *for any* $\Delta \in \mathcal{B}(\rho)$.
Moreover, if the boundary of $\mathcal{D}$ *can be identified to the union of the boundaries of all subregions (*i.e. $\partial\mathcal{D} = \partial\mathcal{U}$*), then condition (25) is also necessary.*

Proof. LMI system (25) is found feasible if and only if A_c is $\partial\mathcal{D}_k$-regular for any k in $\{1, ..., m\}$, in virtue of theorem 5. Hence, it is equivalent to $\partial\mathcal{U}$-regularity. Another consequence of theorem 5 is that A_c has the same root-distribution as A with respect to $\partial\mathcal{D}_k$ (and *a fortiori* with respect to $\partial\mathcal{D}$). This proves the first and second statements in theorem 6. Besides, if $\partial\mathcal{U} = \partial\mathcal{D}$ as in the right part of figure 1, then $\partial\mathcal{U}$-regularity is equivalent to $\partial\mathcal{D}$-regularity, what proves the third statement. □

In the light of this theorem, the following statements, which can be seen as corollaries, can be formulated:

- γ can be minimized under LMI constraints (25) down to $\gamma^\star$ and then $\rho^\star = (\gamma^\star)^{-1/2}$ actually equals $\varrho_{\partial\mathcal{U}}$. It is the largest robust $\partial\mathcal{D}$-regularity bound provided by this approach. If A is $\mathcal{D}$-stable, then $\rho^\star$ is a robust $\mathcal{D}$-stability bound.
- If $\partial\mathcal{U} = \partial\mathcal{D}$, then $\rho^\star$ equals $\varrho_{\partial\mathcal{D}}$, the complex $\partial\mathcal{D}$-regularity radius.
- If A is $\mathcal{D}$-stable and if $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$, then $\rho^\star$ equals both $\varrho_{\partial\mathcal{D}}$ and $r_{\mathcal{D}}$, the complex $\mathcal{D}$-stability radius.

5 Numerical illustration

In this section, we propose a simple illustration of the presented technique in order to highlight its relevance. Computations are performed on a PC Pentium 1.7MHz with MATLAB LMI CONTROL TOOLBOX.

5.1 First example

This model is borrowed from [29]. The lateral dynamic of an aircraft is modeled by state and input matrices:

$$A_0 = \begin{bmatrix} -0.3400 & 0.0517 & 0.0010 & -0.9970 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ -2.6900 & 0.0000 & -1.1500 & 0.7380 & 0.0000 \\ 5.9100 & 0.0000 & 0.1380 & -0.5060 & 0.0000 \\ -0.3400 & 0.0517 & 0.0010 & 0.0031 & 0.0000 \end{bmatrix};$$

$$B_0 = \begin{bmatrix} 0.0755 & 0.0000 & 0.0246 \\ 0.0000 & 0.0000 & 0.0000 \\ 4.4800 & 5.2200 & -0.7420 \\ -5.0300 & 0.0998 & 0.9848 \\ 0.0755 & 0.0000 & 0.0246 \end{bmatrix}.$$

A static state feedback control law associated with matrix

$$K = \begin{bmatrix} -3.9063 & -0.2869 & 0.0006 & -1.5109 & -1.8135 \\ 0.8077 & -2.4178 & -0.9356 & -0.2877 & -0.0296 \\ -21.5282 & -1.2212 & -0.0424 & -10.1518 & -11.1581 \end{bmatrix}$$

is applied to the previous model so that:

$$\lambda(A = A_0 + B_0 K) = \{-0.5; -2 \pm 2\mathbf{i}; -3 \pm 2\mathbf{i}\}. \quad (26)$$

Matrix A is assumed to be subject to an additive uncertainty but no particular structure of this uncertainty is considered so B and C are both assumed to equal $\mathbb{I}_5$ in (1) or, equivalently, $A_c = A + \Delta$. To analyze the robustness of the pole location in the presence of the uncertainty, $\mathcal{D}$ is chosen as the union of 5 discs $\mathcal{D}_k$, $k = 1, ..., 5$, centered around the nominal eigenvalues given by (26) and all of radius 0.5. In [15], a robust $\mathcal{D}$-stability bound is computed: $\rho^\star = 0.0729$. This value is improved owing to our approach. Indeed, since $\mathcal{D}_2$ and $\mathcal{D}_3$ (respectively $\mathcal{D}_4$ and $\mathcal{D}_5$) are symmetric to each other with respect to the real axis, it is only required to compute three values, namely $\varrho_{\partial\mathcal{D}_1}$, $\varrho_{\partial\mathcal{D}_2} = \varrho_{\partial\mathcal{D}_3}$, $\varrho_{\partial\mathcal{D}_4} = \varrho_{\partial\mathcal{D}_5}$. After `2.58s` of computation time, it comes:

$$\rho^\star \;=\; \min_{k \in \{1,2,4\}} \varrho_{\partial\mathcal{D}_k} \;=\; \min\{0.3012; 0.1433; 0.1262\} \;=\; 0.1262 \;=\; r_{\partial\mathcal{D}}.$$

The bound of [15] is then increased by 73%. If many spectra are plotted for various values of complex Δ as in the first example, figure 3 is obtained (to proceed, it suffices to generate many random complex matrices using standard Matlab functions such as `rand` and to scale its entries so that its 2-norm is a random value in the range $[0; \rho^\star]$). It might seem that the bound is a bit conservative but this example is not trivial and since Δ is of dimension (5×5), it is hard to generate a random instance of Δ that corresponds to the worst case. However, the pole migration is not very far from $\partial\mathcal{D}$, especially from $\partial\mathcal{D}_4$ and $\partial\mathcal{D}_5$, and since A is $\mathcal{D}$-stable and since the five open subregions are disjoint (which implies that $\partial\mathcal{U} \cap \mathcal{D} = \emptyset$), $\rho^\star$ actually equals $r_{\mathcal{D}}$. Hence, in this case, the exact value of the complex $\mathcal{D}$-stability radius is obtained, without conservatism.

To compute $\rho^\star$ leads to derive five Lyapunov matrices P_k, $i = 1, ..., 5$. Each matrix P_k is such that $\mathrm{In}(P_k) = [1,\ 4,\ 0]$ since, for any $\Delta \in \mathcal{B}(\rho^\star)$, A_c has one root inside $\mathcal{D}_k$ and four roots outside $\mathcal{D}_k$.

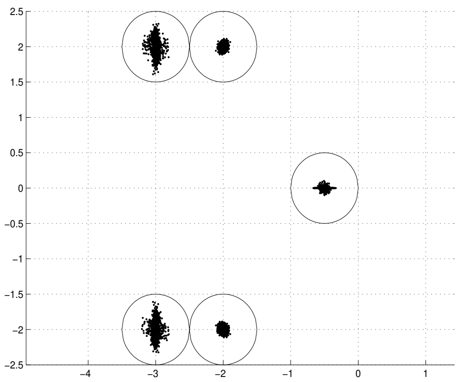

Figure 3: Pole migration of closed-loop aircraft model with norm-bounded complex uncertainty

5.2 Second example

This example is simpler but involves a clustering region that is of interest for the analysis of some plant models. It is inspired from [30] and is relevant to the short-period approximation of the F-16 dynamics (see [30] for further details). The open-loop model is

$$A_0 = \begin{bmatrix} -1.0188 & 0.0905 \\ 4.0639 & -0.7701 \end{bmatrix} \quad ; \quad B_0 = \begin{bmatrix} -0.0021 \\ -0.1692 \end{bmatrix}.$$

A static state feedback control law associated with a matrix K is applied such that the spectrum of the closed-loop state matrix

$$A = A_0 + B_0 K = \begin{bmatrix} -1.1319 & 0.8790 \\ -4.9643 & -2.8683 \end{bmatrix}$$

is $\sigma(A) = \{-2 \pm 1.9\mathbf{i}\}$. We aim at analyzing the non fragility induced by K. More precisely, assuming that K can be affected by an additive uncertainty Δ, then the uncertain closed-loop state matrix A_c is given by expression (1) with $B = B_0$ and $C = \mathbb{I}_2$. The transient performances are supposed to be preserved when the spectrum of A_c remains in the clustering region $\mathcal{D}$ defined by the intersection of the OLHP, the ring centered around the origin and of interior and exterior radii 2.47 and 3.05 and the symmetric horizontal strip with a half width equaling 2.15. Such a region is the so-called Good Ride Quality Region of an aircraft [30]. The present approach can be used to compute the $\mathcal{D}$-stability radius in `1.52s`:

$$\rho^\star = \varrho_{\partial\mathcal{D}} = r_{\mathcal{D}} = 2.39.$$

Plotting the spectrum of A_c for many random instances of Δ highlights the relevance of our approach to compute this non fragility criterion (see figure 4).

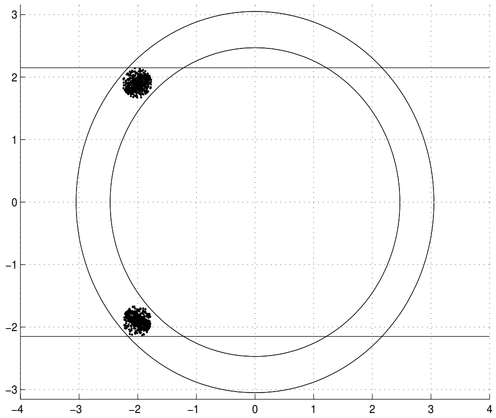

Figure 4: Pole migration of closed-loop aircraft model with norm-bounded complex uncertainty

6 Conclusion

In this paper, the concept of matrix $\partial\mathcal{D}$-regularity, *i.e.* absence of eigenvalues along a curve $\partial\mathcal{D}$ of the complex plane, was used to compute, through a Lyapunov approach, a robust $\mathcal{D}$-stability bound ,*i.e.* a bound on an uncertainty affecting a matrix ensuring that the eigenvalues of the uncertain matrix remain inside a subregion $\mathcal{D}$ of the complex plane.

An original point in the paper is the wide class of allowed clustering regions since $\mathcal{D}$ can be a combination (*i.e.* the union and/or intersection) of several possibly non symmetric half planes, discs and outsides of discs. Such an originality in the choice of the region is made possible by using Lyapunov matrices which are not necessarily positive definite, but preserve their inertia over the uncertainty domain. When root-clustering in a single disc, a half-plane or the outside of a disc is to be analyzed, the Lyapunov matrix has to be strictly positive or negative definite. The computed bound proves to be not very conservative. When the boundaries of various subregions do not intersect $\mathcal{D}$, this bound turns to be the exact complex $\mathcal{D}$-stability radius, that is the largest bound on a complex uncertainty preserving $\mathcal{D}$-stability, thus enabling efficient robust matrix root-location analysis. The bound is deduced from the solution of a very simple LMI problem.

As an extension of our work, a parametric structured uncertainty could therefore be very easily taken into account, especially through a polytopic description. Another challenge is the use of such an approach in a design context. More precisely the question is to know if the inertia of various Lyapunov matrices P_k can be discerningly exploited to distribute the poles in various subregions while determining a control law.

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Appendix

Two lemmas are here provided to the reader. They are exploited in the proof of theorem 5.

Lemma 7 *[31]. Given $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and $M = M' \in \mathbb{C}^{(n+m)\times(n+m)}$, with $\det(\mathbf{i}\omega \mathbb{I} - A) \neq 0, \forall \omega \in \mathbb{R}$ (i.e. A is $\mathfrak{I}$-regular), then the following two statements are equivalent:*

(i)

$$\left[\begin{array}{c} (\mathbf{i}\omega\mathbb{I} - A)^{-1}B \\ \mathbb{I} \end{array}\right]' M \left[\begin{array}{c} (\mathbf{i}\omega\mathbb{I} - A)^{-1}B \\ \mathbb{I} \end{array}\right] < 0 \quad \omega \in \mathbb{R} \tag{27}$$

(ii) There exists a matrix $P = P' \in \mathbb{C}^{n\times n}$ such that

$$M + \left[\begin{array}{cc} A'P + PA & PB \\ B'P & \mathbb{O} \end{array}\right] < 0. \tag{28}$$

The above lemma is known as the Kalman-Yakubovich-Popov lemma [26]. It is here expressed with complex matrices.

Lemma 8 *[31]. Given $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$ and $M = M' \in \mathbb{C}^{(n+m)\times(n+m)}$, with $\det(e^{\mathbf{i}\omega} \mathbb{I} - A) \neq 0, \forall \omega \in \mathbb{R}$ (i.e. A is $\mathfrak{C}$-regular), then the following two statements are equivalent:*

(i)

$$\left[\begin{array}{c} (e^{\mathbf{i}\omega}\mathbb{I} - A)^{-1}B \\ \mathbb{I} \end{array}\right]' M \left[\begin{array}{c} (e^{\mathbf{i}\omega}\mathbb{I} - A)^{-1}B \\ \mathbb{I} \end{array}\right] < 0 \quad \omega \in \mathbb{R}\cup\infty. \tag{29}$$

(ii) There exists a matrix $P = P' \in \mathbb{C}^{n\times n}$ such that

$$M + \left[\begin{array}{cc} A'PA - P & PB \\ B'P & B'PB \end{array}\right] < 0. \tag{30}$$

The above lemma is the discrete-time counterpart of the Kalman-Yakubovich-Popov lemma.