# $\partial\mathcal{D}$-regularity for robust matrix root-clustering

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**Index Terms**

Matrix $\mathcal{D}$-stability, Matrix $\partial\mathcal{D}$-regularity, $\mathcal{LMI}$.

**Abstract**

*This paper addresses the problem of robust matrix root-clustering. The considered uncertainty is the so-called norm-bounded one. The clustering regions are unions of possibly disjoint and nonsymmetric disks and half planes. The concept of $\partial\mathcal{D}$-regularity is introduced. Owing to this notion, a sufficient (but not too pessimistic) $\mathcal{LMI}$ condition for robust matrix root-clustering is obtained.*

## 1 Introduction

The problem of robust matrix root-clustering in a region $\mathcal{D}$ of the complex plane (let us say matrix $\mathcal{D}$-stability) has been widely investigated in the last three decades. Indeed, checking the robust $\mathcal{D}$-stability of the state matrix proves to be very useful to analyse the transient response of the linear system. Moreover, such a property can also be taken into account in a design procedure. It enables, for instance, to reach some specified damping ratio and/or some settling time provided that $\mathcal{D}$ is discerningly chosen.
Since the well-known results on stability analysis of linear models, many extensions were provided to stability regions different from the left open half plane (*e.g.* stability analysis of discrete-time models where the suitable region is the unit disk). Among all the approaches for $\mathcal{D}$-stability analysis, a well known necessary and sufficient condition for a matrix to have all its eigenvalues in a transformable $\Omega$-region (this set enables a large choice of regions) is the existence of a single symmetric and positive definite matrix solution to a Generalized Lyapunov Equation ($\mathcal{GLE}$)[1]. Some interesting results are available for other descriptions of clustering regions, such $\mathcal{LMI}$-regions ($\mathcal{LMI}$ for Linear Matrix Inequalitiy) [2] or $\mathcal{EMI}$-regions ($\mathcal{EMI}$ for Ellipsoidal Matrix Inequality) [3]. For these formulations, the condition for matrix $\mathcal{D}$-stability is the existence of a symmetric and positive definite matrix solution to an $\mathcal{LMI}$. Most of the regions reading such descriptions are connected and even convex for $\mathcal{LMI}$ and $\mathcal{EMI}$-regions. However, it can be discerning, for precise root-clustering analysis, to specify a more sophisticated region that can result from the union of several simple subregions [4]. The concepts of $\mathcal{GLMI}$-regions or $\mathcal{EEMI}$-regions are used in [5, 4, 6] to handle such regions.

The $\mathcal{D}$-stability problem becomes harder when the matrix is uncertain. Necessity of the condition is hard, often impossible, to preserve. It matters to derive sufficient but not too pessimistic conditions. A way to tackle the problem is to try to estimate the greatest uncertainty domain (in the sense of a norm) such that some condition ensuring the nominal $\mathcal{D}$-stability keeps holding for the uncertain matrix. This is the idea that appeared in [7] where unstructured or parametric uncertainties, Hurwitz stability and Lyapunov equations are considered. These results were improved in [8], later extended to any transformable $\Omega$-region in [9]. In the meanwhile, another approach was dedicated to the norm-bounded uncertainty. In this case, the ideal non pessimistic bound on the uncertainty domain that preserves stability is called a stability radius [10]. This approach has led to very significant results. The stability radius was proved to be strongly related to the $\mathcal{H}_\infty$ and $\mu$-theories (see [11] anf the references therein) and owing to the $\mathcal{LMI}$ formulation of the Bounded Real Lemma, the complex stability radius can be easily computed as a $\mathcal{H}_\infty$-norm. However, although this result enables to consider Hurwitz or Schur stability, its extension to sophisticated regions is not obvious. Actually, root-clustering of $A \in \mathbb{C}^{n\times n}$ in a union of several subregions was first tackled in [12] with a union of $n$ necessarily disjoint disks. More general results are proposed in [4, 6]. This paper is an attempt to use the benefit of the very good results based upon stability radius and $\mathcal{H}_\infty$-theory to analyse robust $\mathcal{D}$-stability for sophisticated $\mathcal{D}$ *i.e.* a union of possibly nonsymmetric and nonconnected disks and half planes. To achieve this goal, we introduce the concept $\partial\mathcal{D}$-regularity and $\partial\mathcal{D}$-singularity of a matrix. Owing to this notion, a procedure to compute a robust $\mathcal{D}$-stability bound is proposed. This bound is a lower bound of the complex $\mathcal{D}$-stability radius and is expected to be very tight to this radius. Section 2 introduces some mathematical preliminaries. Section 3 states the problem. Section 4 introduces the notions of $\partial\mathcal{D}$-regularity or $\partial\mathcal{D}$-singularity. These notions are used in section 5 to derive the robust $\mathcal{D}$-stability bound. section 6 illustrates the technique before concluding.

**Notations :**
$M'$ denotes the transpose conjugate of matrix $M$. Hence, $s'$ is the conjugate of complex number $s$. The 2-norm of $M$ induced by the Euclidean vector norm (maximum singular value) is denoted by $||M||_2$. $\mathbb{I}_n$ is the identity matrix of order $n$ and $\mathbb{O}$ is a null matrix of appropriate dimensions. Matrix inequalities are considered in the sense of Loëwner *i.e.* $>0$ (resp. $<0$) means positive (resp. negative) definite. $M^\perp$ is the left nullspace of $M$ *i.e.* $M^\perp$ is such that $M^\perp M = \mathbb{O}$ and $M^\perp M^{\perp'} > 0$. $(\bullet)'$ is used to avoid repeated writings of entries in Hermitian matrices. $\mathcal{C}(M,T)$ denotes the congruence operation $T'MT$. $\mathcal{H}(M)$ stands for $M+M'$. At last, $\mathbf{i}$ denotes the imaginary unit and $\sigma(A)$ is the spectrum of $A$.

## 2 Preliminaries

This section introduces three lemmas that will be useful in the sequel.

**Lemma 1** *[13] Let $Z$, $E$, $\Delta$ and $F$ be complex matrices with appropriate dimensions. Assume that $Z$ is Hermitian then*

$$Z + \mathcal{H}(E\Delta F) < 0 \quad \forall\Delta|\Delta'\Delta < \mathbb{I} \tag{1}$$

*if and only if there exists a scalar $\epsilon > 0$ such that*

$$Z + \epsilon EE' + \frac{1}{\epsilon}F'F < 0 \tag{2}$$

**Lemma 2** *[14] (elimination procedure) Let $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{k\times n}$ and $Q = Q' \in \mathbb{C}^{n\times n}$ be given. Then the following statements are equivalent:*

*(i) There exists a matrix $X$ satisfying*

$$Q + \mathcal{H}(BXC) < 0 \tag{3}$$

*(ii) The following two conditions hold*

$$\begin{cases} B^{\perp}QB^{\perp'} < 0 \quad \text{or} \quad BB' > 0 \\ C'^{\perp}QC'^{\perp'} < 0 \quad \text{or} \quad C'C > 0 \end{cases} \tag{4}$$

**Lemma 3** *(Schur's lemma) Let $R$, $Q$ and $S$ be complex matrices with appropriate dimensions. Assume that $Q$ and $R$ are Hermitian, then:*

$$\begin{bmatrix} Q & S' \\ S & R \end{bmatrix} < 0 \quad \Leftrightarrow \quad \begin{cases} R < 0 \\ R - S'QS < 0 \end{cases} \tag{5}$$

# 3 Problem statement

In this section the problem is stated. A first part is dedicated to the presentation of the uncertain matrix whose $\mathcal{D}$-stability is to be tested. Then, the description of the clustering regions $\mathcal{D}$ is given. At last, the purpose is clearly formulated.

## 3.1 The uncertain matrix

The considered matrix reads:

$$\mathbf{A} = A + B\Delta C \in \mathbb{C}^{n\times n} \tag{6}$$

$A \in \mathbb{C}^{n\times n}$, the nominal term in this expression, is precisely known. Matrices $B \in \mathbb{C}^{n\times q}$ and $C \in \mathbb{C}^{r\times n}$ are precisely known and introduced to specify the structure of the uncertainty. Matrix $\Delta$ is unknown and is assumed to belong to $\mathcal{B}(\rho)$, the ball of $(q \times r)$ complex matrices checking $||\Delta||_2 \leq \rho$. When $\rho$ is assumed to equal 1, this uncertainty becomes the well known norm-bounded one.

## 3.2 The clustering region $\mathcal{D}$

The clustering region $\mathcal{D}$ is defined by:

$$\mathcal{D} = \bigcup_{k=1}^{m} \mathcal{D}_k \tag{7}$$

$$\mathcal{D}_k = \{ \ s \in \mathbb{C} \,|\, r_{k_{00}} + \mathcal{H}(r_{k_{10}}s) + r_{k_{11}}s's < 0 \ \} \tag{8}$$

with $\{r_{k_{00}}; r_{k_{11}}\} \in \mathbb{R} \times \mathbb{R}^+$ and $r_{k_{10}} \in \mathbb{C}\ \forall k$.
$\mathcal{D}$ is a union of $m$ subregions $\mathcal{D}_k$. Various $\mathcal{D}_k$ are specified owing to parameters $r_{k_{00}}$, $r_{k_{10}}$ and $r_{k_{11}}$. Such a description is a particular case of second order $\Omega$-regions [1]. Besides, since $r_{k_{11}} \geq 0$, $\mathcal{D}_k$ is also a first order $\mathcal{EMI}$ complex regions [3]. If $r_{k_{10}} \notin \mathbb{R}$, $\mathcal{D}_k$ is not symmetric with respect to the real axis. In practice, two kinds of regions might be of interest : half planes (not necessarily vertical) and disks (not necessarily centred on the real axis). The whole union $\mathcal{D}$ might be non symmetric as well but actual problems with $\mathbf{A}$ real (*i.e.* with a self conjugate spectrum) would lead to a choice of a symmetric $\mathcal{D}$ with possibly non symmetric various $\mathcal{D}_k$. Note that $\mathcal{D}$ is open and thus, does not include its boundary $\partial\mathcal{D}$.

### 3.3 Precise purpose

Assuming that $A$ is $\mathcal{D}$-stable, it is aimed at computing $r_{\mathbb{C}}^{[\mathcal{D}]}$, the complex $\mathcal{D}$-stability radius associated with the triple of matrices $(A, B, C)$. This criterion is the highest radius of the ball $\mathcal{B}(\rho)$ such that $\mathcal{D}$-stability of $\mathbf{A}$ is preserved. This value is not easily computable so a lower bound $\rho^\star$ will be derived. $\rho^\star$ is expected to be as close as possible to $r_{\mathbb{C}}^{[\mathcal{D}]}$. The $\mathcal{D}$-stability radius can be defined by:

$$r_{\mathbb{C}}^{[\mathcal{D}]} = \inf\{||\Delta||_2 \,|\, \Delta \in \mathbb{C}^{q\times r} \text{ and } \mathbf{A} \text{ is } \mathcal{D}\text{-unstable}\} \tag{9}$$

$$\Leftrightarrow r_{\mathbb{C}}^{[\mathcal{D}]} = \inf\{||\Delta||_2 | \Delta \in \mathbb{C}^{q\times r} \text{ and } \sigma(\mathbf{A}) \cap \partial\mathcal{D} \neq \emptyset\} \tag{10}$$

Considering that $\partial\mathcal{D} \subset (\cup\partial\mathcal{D}_k)$ yields to:

$$\begin{aligned}\rho^\bullet = \inf\{||\Delta||_2 | \Delta \in \mathbb{C}^{q\times r}\\ \text{and } \sigma(\mathbf{A}) \cap (\cup\partial\mathcal{D}_k) \neq \emptyset\} \leq r_{\mathbb{C}}^{[\mathcal{D}]}\end{aligned} \tag{11}$$

The inequality comes from the fact that if the various $\mathcal{D}_k$ have an intersection, $\mathbf{A}$ can have an eigenvalue located on some $\mathcal{D}_k$ while preserving matrix $\mathcal{D}$-stability, provided that this eigenvalue belongs to the intersection. If the various subregions are disjoint, $r_{\mathbb{C}}^{[\mathcal{D}]} = \rho^\bullet$. Equation given by (11) can be written:

$$\rho^\bullet = \min_{k=1,\ldots,m} \{\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}\} \tag{12}$$

where each $\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}$ is defined by:

$$\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]} = \inf\{||\Delta||_2 \,|\, \Delta \in \mathbb{C}^{q\times r} \text{ and } \sigma(\mathbf{A}) \cap \partial\mathcal{D}_k \neq \emptyset\} \tag{13}$$

A way to solve the problem is to determine $\rho^\star$ as a lower bound of $\rho^\bullet$. It then matters to approximate each $\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}$. To achieve such an approximation, the notions of matrix $\partial\mathcal{D}_k$-regularity and $\partial\mathcal{D}_k$-singularity are discussed in the next section.

## 4 Matrix $\partial\mathcal{D}$-regularity and $\partial\mathcal{D}$-singularity

In this section, the concept of matrix $\partial\mathcal{D}$-regularity and $\partial\mathcal{D}$-singularity is introduced.

**Definition 4** *Let $\mathcal{D}$ be a region of the complex plane and $\partial\mathcal{D}$ denote its boundary. $A$ is said:*
- *$\partial\mathcal{D}$-regular when $\sigma(A) \cap \partial\mathcal{D} = \emptyset$;*
- *$\partial\mathcal{D}$-singular when $\sigma(A) \cap \partial\mathcal{D} \neq \emptyset$.*

Now assume that $\mathcal{D}$ is a simple convex region as previously presented *i.e.*

$$\mathcal{D} = \{s \in \mathbb{C} \,|\, f(s) = r_{00} + \mathcal{H}(r_{10}s) + r_{11}s's < 0\} \tag{14}$$

A nessary and sufficient condition for a nominal matrix $A$ to be $\partial\mathcal{D}$-regular is now proposed.

**Theorem 5** *Let $\mathcal{D}$ be a region as defined in (14). Let $\partial\mathcal{D}$ denote its boundary. Matrix $A \in \mathbb{C}^{n\times n}$ is $\partial\mathcal{D}$-regular if and only if there exists an Hermitian full rank matrix $P \in \mathbb{C}^{n\times n}$* **(that does not need to be positive definite)** *such that:*

$$F(A, P) = r_{00}P + \mathcal{H}(r_{10}PA) + r_{11}A'PA < 0 \tag{15}$$

**Proof.** Some arguments are inspired from [2]
Sufficiency: first assume that there exists a suitable $P$ such that (15) holds. Let $\lambda_i$ denote the $i^{\text{th}}$ eigenvalue of $A$. There exists a non zero vector $v_i$ such that $Av_i = \lambda_i v_i$ so it comes:

$$v_i' F(A,P) v_i <\Leftrightarrow r_{00} v_i' P v_i +$$

$$\mathcal{H}(r_{10} v_i' P A v_i) + r_{11} v_i' A' P A v_i < 0 \Leftrightarrow$$

$$(r_{00} + \mathcal{H}(r_{10}\lambda_i) + r_{11}\lambda_i'\lambda_i)(v_i' P v_i) < 0$$

$P$ is non singular $\Rightarrow$ $v_i' P v_i \neq 0$. Hence:

$$f(\lambda_i) \; = \; r_{00} \; + \; \mathcal{H}(r_{10}\lambda_i) \; + \; r_{11}\lambda_i'\lambda_i \neq 0$$

This expression shows that $\lambda_i \notin \partial\mathcal{D}$. This reasoning is available for all $i$ and $A$ is then $\partial\mathcal{D}$-regular.

Necessity: now assume that $A$ is $\partial\mathcal{D}$-regular$\Leftrightarrow$

$$f(\lambda_i) \neq 0 \quad \forall i \in \{1, ..., n\} \tag{16}$$

It means that $f(\lambda_i)$ is either strictly positive or strictly negative. Define matrix $\Lambda$ by:

$$\Lambda \;=\; \underset{i=1,\ldots,n}{\mathrm{diag}}\{\lambda_i\} \;=\; \begin{bmatrix} \Lambda_1 & \mathbb{O} \\ \mathbb{O} & \Lambda_2 \end{bmatrix}$$

where $\Lambda_1$ contains the various $\lambda_i$ for which $f(\lambda_i) > 0$ et $\Lambda_2$ contains those for which $f(\lambda_i) < 0$. It is clear that (16) is equivalent to

$$\underset{i=1,\ldots,n}{\mathrm{diag}} \; \{f(\lambda_i)\} \;\neq\; \mathbb{O}$$

that can be written:

$$M = \begin{bmatrix} -r_{00}\mathbb{I} & \mathbb{O} \\ \mathbb{O} & r_{00}\mathbb{I} \end{bmatrix} + \begin{bmatrix} -r_{10}'\Lambda_1' & \mathbb{O} \\ \mathbb{O} & r_{10}'\Lambda_2' \end{bmatrix} +$$

$$\begin{bmatrix} -r_{10}\Lambda_1 & \mathbb{O} \\ \mathbb{O} & r_{10}\Lambda_2 \end{bmatrix} + \begin{bmatrix} -r_{11}\Lambda_1'\Lambda_1 & \mathbb{O} \\ \mathbb{O} & r_{11}\Lambda_2'\Lambda_2 \end{bmatrix} < 0 \tag{17}$$

Let $J$ be the Jordan canonical form of $A$. There exists a sequence of full rank matrices $T_l$ such that:

$$\lim_{l\to\infty}(T_l J) \;=\; \lim_{l\to\infty}(\Lambda T_l)$$

For example, if $J$ is a single Jordan block $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $T_l$ equals $\begin{bmatrix} \frac{1}{l} & 0 \\ 0 & 1 \end{bmatrix}$.
Considering the structure of $\Lambda$, let $T_l$ be expressed as follows:

$$T_l \;=\; \begin{bmatrix} T_{l_1} & T_{l_2} \\ T_{l_3} & T_{l_4} \end{bmatrix}$$

It then yields:

$$\lim_{l\to\infty} \begin{bmatrix} T_{l_1}J_1 & T_{l_2}J_2 \\ T_{l_3}J_1 & T_{l_4}J_2 \end{bmatrix} = \lim_{l\to\infty} \begin{bmatrix} \Lambda_1 T_{l_1} & \Lambda_1 T_{l_2} \\ \Lambda_2 T_{l_3} & \Lambda_2 T_{l_4} \end{bmatrix} \tag{18}$$

Since $T_l$ is non singular for all $l$, it comes:

$$\lim_{l\to\infty}(\mathcal{C}(M,T_l)) < 0 \tag{19}$$

Taking (18) into account, (19), after some tedious but not difficult calculation, can be written:

$$\lim_{l\to\infty}(r_{00}\mathcal{T}_l + \mathcal{H}(r_{10}\mathcal{T}_l J) + r_{11}J'\mathcal{T}_l J) < 0 \quad \text{with}$$

$$\mathcal{T}_l = \left[\begin{array}{c|c} T'_{l_3}T_{l_3} - T'_{l_1}T_{l_1} & T'_{l_3}T_{l_4} - T'_{l_1}T_{l_2} \\ \hline T'_{l_4}T_{l_3} - T'_{l_2}T_{l_1} & T'_{l_4}T_{l_4} - T'_{l_2}T_{l_2} \end{array}\right]$$

Let denote $\mathcal{T}$, an instance of $\mathcal{T}_l$ for a sufficiently large $l$, it comes:

$$r_{00}\mathcal{T} + \mathcal{H}(r_{10}\mathcal{T}J) + r_{11}J'\mathcal{T}J < 0$$

Now assume $V$ satisfies $J = V\Lambda V^{-1}$. Left and right multiply the above inequality respectively by $V'$ and $V$ leads to (15) with $P = \mathcal{C}(\mathcal{T}, V)$. Note that if there exists a matrix $V$ such that $\Lambda = V\Lambda V^{-1}$ (or equivalently $\Lambda = J$), then the above reasoning stands with $T_l = \mathbb{I}_n\, \forall l$.$\square$

It matters to notice that condition (15) consists in an $\mathcal{LMI}$ system whose feasibility can be tested owing to $\mathcal{LMI}$ softwares.

# 5 Robust $\mathcal{D}$-stability analysis

In this section, the $\mathcal{D}$-stability of $\mathbf{A}$ introduced in (6) with $\mathcal{D}$ defined in (7) is to be studied. More precsiely, $A$, the nominal matrix, is assumed to be $\mathcal{D}$-stable and it is aimed to compute a lower bound $\rho^\star$ of the $\mathcal{D}$-stability radius $r_{\mathbb{C}}^{[\mathcal{D}]}$ associated with the triple of matrices $(A, B, C)$.
Considering paragaph 3.3, $r_{\mathbb{C}}^{[\mathcal{D}]}$ is actually the greatest radius $\rho$ of the ball $\mathcal{B}(\rho)$ such that no element $\Delta$ of $\mathcal{B}(\rho)$ makes matrix $\mathbf{A}$ be $\partial\mathcal{D}$-singular. Moreover, it is equivalent to estimate the greatest value of $\rho$ such that $\mathbf{A}$ remains $\partial\mathcal{D}_k$-regular for all $k \in \{1, ..., m\}$. A solution consists in analysing $\partial\mathcal{D}_k$-regularity for all $k$, *i.e.* to find a lower bound $\rho_k^\star$ for each "$\partial\mathcal{D}_k$-regularity radius" $\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}$ and then to select the minimum value among those bounds.
We first introduce the natural concept of quadratic $\partial\mathcal{D}_k$-regularity and then we propose a corresponding necessary and sufficient condition. Second, we deduce $\rho^\star$ from this result.

## 5.1 Quadratic $\mathcal{D}_k$-regularity of a matrix

Let an uncertain matrix $\mathbf{A}$ be defined as in (6). Also let $\mathcal{D}_k$ and $\partial\mathcal{D}_k$ be respectively a region and its boundary as defined above. $\mathbf{A}$ is robustly $\partial\mathcal{D}_k$-regular against $\mathcal{B}(\rho)$ if and only if :

$$\begin{array}{c} \forall\Delta \in \mathcal{B}(\rho),\ \exists P(\Delta) = P'(\Delta)\,| \\ F(\mathbf{A}, P(\Delta)) < 0 \ \&\ 0 \notin \sigma(P(\Delta)). \end{array} \tag{20}$$

It is said quadratically $\partial\mathcal{D}_k$-regular if and only if there exists one single constant matrix $P = P'$ such that:

$$\exists P = P'\,|\,0 \notin \sigma(P)\ \&\ \forall\ \Delta \in \mathcal{B}(\rho), F(\mathbf{A}, P) < 0 \tag{21}$$

Such a definition is analogous to the definition of quadratic stability. It is *a priori* restrictive compared with robust $\partial\mathcal{D}_k$-regularity since the same matrix $P$ enables to check the property over the whole of the uncertainty domain.

As mentionned in the beginning of this section, it matters to compute $\rho_k^\star$, a lower bound for each $\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}$.

**Theorem 6** *Let* $\mathbf{A}$, $\mathcal{D}_k$ *and* $\partial\mathcal{D}_k$ *be respectively an uncertain matrix as defined in (6), a region of the complex plane as defined in (8) and its boundary. Assume that nominal matrix* $A$ *is* $\partial\mathcal{D}_k$*-regular. Matrix* $\mathbf{A}$ *is quadratically* $\partial\mathcal{D}_k$*-regular if and only if there exists some non singular Hermitian matrix* $P_k$ *such that:*

$$\begin{bmatrix} F_k(A,P_k) & r_{k_{10}}P_kB + r_{k_{11}}A'P_kB & C' \\ (\bullet)' & r_{k_{11}}B'P_kB - \mathbb{I}_q & \mathbb{O} \\ (\bullet)' & (\bullet)' & -\gamma_k\mathbb{I}_r \end{bmatrix} < 0 \tag{22}$$

*with* $\gamma_k = \rho_k^{-2}$.

**Proof:**
<u>Sufficiency :</u> first note that (22) can be written:

$$\mathcal{C}\left(\underbrace{\begin{bmatrix} r_{k_{00}}P_k & r_{k_{10}}P_k & \mathbb{O} & C' \\ r'_{k_{10}}P_k & r_{k_{11}}P_k & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & -\mathbb{I}_q & \mathbb{O} \\ C & \mathbb{O} & \mathbb{O} & -\gamma_k\mathbb{I}_r \end{bmatrix}}_{Q_k}, \begin{bmatrix} \mathbb{I}_n & \mathbb{O} & \mathbb{O} \\ A & B & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_q & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_r \end{bmatrix}\right) < 0$$

From lemma 2, $\exists\, G_k$ such that:

$$\mathcal{G} = Q_k + \mathcal{H}(G_k \begin{bmatrix} A & -\mathbb{I}_n & B & \mathbb{O} \end{bmatrix}) \tag{23}$$

Hence, it comes

$$\mathcal{C}\left(\mathcal{G}, \begin{bmatrix} \mathbb{I}_n & \mathbb{O} \\ \mathbf{A} & \mathbb{O} \\ \Delta C & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_r \end{bmatrix}\right) =$$

$$\begin{bmatrix} F(\mathbf{A},P_k) - C'\Delta'\Delta C & C' \\ C & -\gamma_k\mathbb{I}_r \end{bmatrix} < 0$$

By virtue of lemma 3, this last inequality is equivalent to:

$$F(\mathbf{A},P_k) < C'\Delta'\Delta C - \rho_k^2 C'C$$

Since $||\Delta||_2 \leq \rho_k$, $\mathbf{A}$ is quadratically $\partial\mathcal{D}_k$-regular with respect to $\mathcal{B}(\rho_k)$ by virtue of theorem 5.

<u>Necessity:</u> now assume that $\mathbf{A}$ is quadratically $\mathcal{D}_k$-regular over $\mathcal{B}(\rho_k)$. Then, $\exists \tilde{P}_k \,|\, 0 \notin \sigma(\tilde{P}_k)\,\&\, \forall\Delta \in \mathcal{B}(\rho_k)$:

$$F(\mathbf{A},\tilde{P}_k) < 0 \qquad \underset{\text{lemma 3}}{\Leftrightarrow}$$

$$\begin{bmatrix} r_{k_{00}}\tilde{P}_k + \mathcal{H}(r_{k_{10}}\tilde{P}_k\mathbf{A}) & (\bullet)' \\ \sqrt{r_{k_{11}}}\tilde{P}_k^{1/2}\mathbf{A} & -\mathbb{I}_n \end{bmatrix} < 0$$

First separating the uncertain term of the above inequality from the nominal one leads to:

$$\underbrace{\left[\begin{array}{cc} r_{k00}\tilde{P}_k + \mathcal{H}(r_{k10}\tilde{P}_k A) & (\bullet)' \\ \sqrt{r_{k11}}\tilde{P}_k^{1/2}A & -\mathbb{I}_n \end{array}\right]}_{Y_k} +$$

$$\mathcal{H}\left( \left[\begin{array}{c} r_{k10}\tilde{P}_k B \\ \sqrt{r_{k11}}\tilde{P}_k^{1/2}B \end{array}\right] \frac{\Delta}{\rho_k} \left[\begin{array}{cc} \rho_k C & \mathbb{O} \end{array}\right] \right) < 0$$

Lemma 1 $\Rightarrow \exists \epsilon > 0 \,|$

$$Y_k + \epsilon \left[\begin{array}{c} r_{k10}\tilde{P}_k B \\ \sqrt{r_{k11}}\tilde{P}_k^{1/2}B \end{array}\right] \left[\begin{array}{cc} r'_{k10}B'\tilde{P}_k & \sqrt{r_{k11}}B'\tilde{P}_k^{1/2} \end{array}\right] +$$

$$\frac{1}{\epsilon}\left[\begin{array}{c} \rho_k C' \\ \mathbb{O} \end{array}\right] \left[\begin{array}{cc} \rho_k C & \mathbb{O} \end{array}\right] < 0$$

Mutliply by $\epsilon$ and apply lemma 3 once more so that it comes:

$$Z_k = \left[\begin{array}{c|cc} \epsilon Y_k & \begin{array}{c} r_{k10}B'P_k \\ \sqrt{\epsilon r_{k11}}P_k^{1/2}B \end{array} & \begin{array}{c} C' \\ \mathbb{O} \end{array} \\ \hline (\bullet)' & \begin{array}{c} -\mathbb{I}_q \\ \mathbb{O} \end{array} & \begin{array}{c} \mathbb{O} \\ -\gamma_k \mathbb{I}_r \end{array} \end{array}\right] < 0 \text{ with } P_k = \epsilon\tilde{P}_k \;\Rightarrow$$

$$\mathcal{C}\left( Z_k, \left[\begin{array}{ccc} \mathbb{I}_n & \mathbb{O} & \mathbb{O} \\ \sqrt{\frac{r_{k11}}{\epsilon}}P_k^{1/2}A & \sqrt{\frac{r_{k11}}{\epsilon}}P_k^{1/2}B & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_q & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I}_r \end{array}\right] \right) < 0$$

which is equivalent to (22). $\square$

Inequality (22) is an $\mathcal{LMI}$ whose feasibility can be tested with minimizing $\gamma_k$ to $\gamma_k^\star$. Thus, $\rho_k^\star = {\gamma_k^\star}^{-1/2}$ is the quadratic $\partial\mathcal{D}_k$-regularity radius.

The reader may recognise in (22) an expression that is very close to the "$\mathcal{LMI}$ version" of the well known Bounded Real Lemma. Actually, applied to the open left half plane or to the unit disk, the $\mathcal{LMI}$ condition (22) can be used to compute $\mathcal{H}_\infty$-norm (equaling $\sqrt{\gamma_k^\star}$) of a plant described the triple $(A, B, C)$ provided that $P_k$ is assumed to be positive definite. Furthermore, it was proved that computing this $\mathcal{H}_\infty$-norm is equivalent to compute the complex (and even real in fact) stability radius (see [11] and references therein). For this reason, it can be conjectured that, in the present problem, $\rho_k^\star$ is the complex $\partial\mathcal{D}_k$-regularity radius $\tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}$. It would however require to adapt and generalize to any region the reasoning provided in [11].

## 5.2 Robust $\mathcal{D}$-stability bound

Coming back to the beginning of the section, a simple method to compute $\rho^\star$ is proposed. Assume that $\rho_k^\star$ was computed for all the subregions $\mathcal{D}_k$, $k = 1, ..., m$, owing to condition (22). Thus, any matrix $\Delta$ such that $||\Delta||_2 > \rho_k^\star$ might make an eigenvalue of $\mathbf{A}$ migrate on $\partial\mathcal{D}_k$ and therefore make $\mathbf{A}$ become $\partial\mathcal{D}_k$-singular. Under this value, because the eigenvalues of $\mathbf{A}$ are continuous functions of $\Delta$, the $\partial\mathcal{D}_k$-regularity of $\mathbf{A}$ is ensured. Hence, noting that $A$ is $\mathcal{D}$-stable (*a fortiori* $\partial\mathcal{D}$-regular and $\partial\mathcal{D}_k$-regular), it becomes clear that $\rho^\star$ must be chosen as follows:

$$\rho^\star = \min_{k\in\{1,\dots,m\}}\{\rho_k^\star\} \tag{24}$$

If we admit that $\rho_k^\star = \tilde{r}_{\mathbb{C}}^{[\mathcal{D}_k]}$ (or equivalently that quadratic $\partial\mathcal{D}_k$-regularity is not pessimistic compared with complex robust $\partial\mathcal{D}_k$-regularity) then $\rho^\star = \rho^\bullet$. If moreover, the various subregions are disjoint, $\rho^\star$ might be considered as the complex $\mathcal{D}$-stability radius $r_{\mathbb{C}}^{[\mathcal{D}]}$.

**Remark 7** *It must be noticed that with such a technique, the outside of the disk can also be handled since $\partial\mathcal{D}$-regularity enables to consider eigenvalues outside of a convex region.*

## 6 Numerical illustration

The model is borrowed from [12]. The lateral dynamic of an aircraft is modeled by state and input matrices:

$$A_0 = \begin{bmatrix} -0.34 & 0.0517 & 0.001 & -0.9970 & 0 \\ 0.00 & 0.0000 & 1.000 & 0.0000 & 0 \\ -2.69 & 0.0000 & -1.150 & 0.7380 & 0 \\ 5.91 & 0.0000 & 0.138 & -0.5060 & 0 \\ -0.34 & 0.0517 & 0.001 & 0.0031 & 0 \end{bmatrix};$$

$$B_0 = \begin{bmatrix} 0.0755 & 0.0000 & 0.0246 \\ 0.0000 & 0.0000 & 0.0000 \\ 4.4800 & 5.2200 & -0.7420 \\ -5.0300 & 0.0998 & 0.9848 \\ 0.0755 & 0.0000 & 0.0246 \end{bmatrix}$$

A static state feedback control law associated with matrix:

$$K = \begin{bmatrix} -3.9063 & -0.2869 & 0.0006 & -1.5109 & -1.8135 \\ 0.8077 & -2.4178 & -0.9356 & -0.2877 & -0.0296 \\ -21.5282 & -1.2212 & -0.0424 & -10.1518 & -11.1581 \end{bmatrix}$$

is applied so that the spectrum of closed-loop state matrix $A = A_0 + B_0K$ is:

$$\{-0.5; -2 \pm 2\mathbf{i}; -3 \pm 2\mathbf{i}\} \tag{25}$$

Matrix $A$ is assumed to be subject to an additive uncertainty but no particular structure of this uncertainty is considered so $B$ and $C$ are both assumed to equal $\mathbb{I}_5$. To analyse the robustness of the pole location, $\mathcal{D}$ is chosen as the union of 5 disks $\mathcal{D}_k$, $k = 1, ..., 5$, respectively centred on the nominal eigenvalues given by (25) and all of radius 0.5. In [12], a robust $\mathcal{D}$-stability bound is computed: $\eta = 0.0729$. This value is improved owing to our approach. Indeed, since $\mathcal{D}_2$ and $\mathcal{D}_3$ (respectively $\mathcal{D}_4$ and $\mathcal{D}_5$) are symmetric to each other with respect to the real axis, it is only required to compute three values, namely $\rho_1^\star$, $\rho_2^\star = \rho_3^\star$, $\rho_4^\star = \rho_5^\star$. After `2.58s` of computation time on a PC Pentium 1.7MHz, it comes:

$$\rho^\star = \min\{\rho_1^\star; \rho_2^\star; \rho_4^\star\} = \min\{0.3012; 0.1433; 0.1262\} = 0.1262$$

The result of [12] is then increased of 73%.

## 7 Conclusion

A very simple technique to compute a robust $\mathcal{D}$-stability bound associated with the norm-bounded uncertainty has been proposed. The originality of this work is that the clustering region $\mathcal{D}$ can be a union of possibly disjoint and non symmetric disks and half planes. Outsides of disks can also be handled. The robustness bound is expected to be very tight to the complex $\mathcal{D}$-stability radius (sometimes equal). Owing to the $\partial\mathcal{D}$-regularity concept, the $\mathcal{LMI}$ computation seems easier than with other conditions proposed in the literature. It would be interesting to know if such a result can be extended to a larger class of regions with more general descriptions. Moreover, in practice, the real $\mathcal{D}$-stability radius is more significant than the complex one. For this purpose, the work of [11] could be a starting point. Furthermore, this result can be adapted to the case where $A$, $B$, $C$ read a polytopic formulation deduced from parametric variations. We shall propose results for this problem in a near future.

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