An Approach for Robust Matrix Root-Clustering Analysis in a Union of Regions

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Jérôme Bosche[a], jerome.bosche@u-picardie.fr
Olivier Bachelier[b], Olivier.Bachelier@univ-poitiers.fr
Driss Mehdi[b], Driss.Mehdi@univ-poitiers.fr

[a]Crea, Iut, Département Informatique, Avenue des facultés 80 000 Amiens, France
[b]Laii-Esip, Bâtiment de Mécanique, 40 avenue du recteur Pineau, 86022 Poitiers Cedex, France

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Abstract

This paper addresses the problem of robust matrix root-clustering analysis in a union of regions. The considered matrices are complex and subject to both polytopic and parameter-dependent norm-bounded uncertainties. The clustering regions are unions of convex and possibly disjoint and nonsymmetric subregions of the complex plane. The proposed clustering conditions are formulated in terms of Linear Matrix Inequalities, which enables an easy computation of Lyapunov matrices (possibly parameter-dependent) that ensure the clustering property. The results are an improvement of a pevious result form the two last authors. Come connections to classical results of the literature are also provided.

1 Introduction

Matrix root-clustering, also known as $\mathcal{D}$-stability, is known as a possible way to specify the shape of the transient response of a linear model. Indeed, checking that the whole spectrum of the state matrix lies in a region $\mathcal{D}$ of the complex plane can guarantee some performances such as settling time, damping ratio, ...

For this reason, the literature offers many tractable conditions for nominal matrix $\mathcal{D}$-stability [1, 2, 3]. Unfortunately, it becomes harder to test this property when the matrix is subject to an uncertainty and, most of the time, only sufficient conditions are derived.

For structured parametric uncertainty, the polytopic description had led to nice results [2]. For unstructured uncertainty, based upon the results of [4] and the notion of Ω-regions [1], reference [5] proposes a technique to compute robustness bounds. Those are bounds on the uncertainty domain that guarantee robust $\mathcal{D}$-stability. They can be seen as lower bounds of the non conservative bound called $\mathcal{D}$-stability radius.

Most of the clustering regions are connected and even often convex. Nevertheless, it might not always be discerning to specify a single clustering region containing the whole spectrum, especially for models with seperate dynamics (slow ones and fast ones: for instance, an aircraft model may contain both fast electric and slow mechanical dynamics) or when one wants to ensure that the damping ratio remains in a specified range. For such specifications, a union of possibly disjoint subregions is more suitable than a simple convex region. Literature suffers from a lack of results concerning non connected regions. Let us however quote [6, 7, 8] or the more recent work [9].

In this paper, it is aimed at proposing a quite general approach for the computation of robust $\mathcal{D}$-stability bounds in the presence of both polytopic and norm-bounded uncertainties. The originality and the generality of the approach consists in a wide class of clustering regions, possibly non connected, are proposed. The conditions are expressed in terms of $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$) and encompass some previous results from the literature. Those results are coherent with some other powerful ones in the lietrature [3] and can also be seen as an improvement compared with [8].

Section 2 is dedicated to the problem statement and to preliminaries. Section 3 defines the concept of $\mathcal{D}_U$-stability. $\mathcal{D}_U$-satbility of a matrix is the membership of its whole spectrum to a so-called $\mathcal{EEMI}$-region ($\mathcal{EEMI}$ for $\mathcal{E}$xtended $\mathcal{E}$llipsoidal $\mathcal{M}$atrix $\mathcal{I}$nequality). Such a formulation of a region, recalled in the sequel, encompasses many possible unions of subregions of the complex plane. Section 4 tackles the problem where only a norm-bounded uncertainty affects the matrix. It shows how some particular cases of the presented conditions can be seen as a reformulation of some results of [3]. Section 5 considers in addition a parametric (polytopic) uncertainty. An little illustration is provided in section 6 before concluding.

Notations : We denote by M', the transpose conjugate of M, by $\mathcal{H}(M)$ the Hermitian expression $M + M'$ and by M^+ the Moore-Penrose pseudo-inverse of M. The Kronecker product is denoted by $\otimes$. $||M||_2$ is the 2-norm of matrix M. $\mathbf{I}_n$ is the identity matrix of order n, $\mathbf{0}$ is a null matrix of suitable dimension and $\mathbf{1}_{u,v}$ is a matrix of dimension $u \times v$ whose all entries equaling 1. Matrix inequalities are considered in the sense of Löewner *i.e.* "$\prec 0$" ("$\preceq 0$") means negative (semi-)definite and "$\succ 0$" ("$\succeq 0$") positive (semi-)definite. Matrix $M^\perp$ is such that $M^\perp M = \mathbf{0}$ and $M^\perp M^{\perp\prime} \succ 0$. HPD stands for Hermitian Positive Definite. The set $\mathcal{A}/\mathcal{B}$ is the subset of $\mathcal{A}$ obtained by removing from $\mathcal{A}$ the elements of $\mathcal{B}$. Matrix $[f_k]_{k=1}^m$ denotes $[f_1 \dots f_m]$. At last, $\mathbf{i}$ denotes the imaginary unit and $\mathrm{Re}(z)$ is the real part of z.

2 Problem Statement and preliminaries

First, the uncertain matrix is presented. Then the precise purpose is formulated. At last some lemmas are recalled as mathematical preliminaries.

2.1 The uncertain matrix

Consider a complex uncertain matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ defined by:

$$\mathbf{A} = A + JEL \tag{1}$$

in which $E \in \mathbb{C}^{q \times r}$ is unknown and belongs to $\mathcal{B}(\rho)$, the ball of $(q \times r)$ complex matrices checking $||E||_2 \leq \rho$. Define matrix M by:

$$M = \begin{bmatrix} A & J \\ L & \mathbf{0} \end{bmatrix} \tag{2}$$

Matrix $M \in \mathbb{C}^{(n+r)\times(n+q)}$ is assumed to belong to a polytope $\mathcal{M}$:

$$\begin{cases} \mathcal{M} = \left\{ M = M(\theta) | M = \sum_{i=1}^{N}(\theta_i M_i) = \sum_{i=1}^{N}\left(\theta_i \begin{bmatrix} A_i & J_i \\ L_i & \mathbf{0} \end{bmatrix} \right); \theta \in \Theta \right\} \\ \Theta = \left\{ \theta = \begin{bmatrix} \theta_1 & \dots & \theta_N \end{bmatrix}' \in \{\mathbb{R}^+\}^N | \sum_{i=1}^{N} \theta_i = 1 \right\} \end{cases} \tag{3}$$

where M_i, $i = 1, ..., N$ are the vertices of $\mathcal{M}$ and Θ is the set of coordinates. This formulation encompasses the case where M linearly depends on some parameter deflections around a nominal value. In that case, $\mathcal{M}$ is actually an "orthotope of matrices". If no polytopic dependence is assumed ($i.e.$ $M = M_1$) then the uncertainty is just norm-bounded [10].

2.2 The precise purpose

Given a matrix $\mathbf{A}$ defined in (1) and a clustering region $\mathcal{D}$ of the complex plane that might be the union of several possibly disjoint and non symmetric subregions (a possible formulation is to be given in the next section), it is aimed at determining, owing to $\mathcal{LMI}$ tools, a bound $\rho^\diamond$, as large as possible, such that for any radius $\rho \leq \rho^\diamond$, $\mathcal{D}$-stability of $\mathbf{A}$ is guaranteed, for any $\theta \in \Theta$ and for any $E \in \mathcal{B}(\rho)$. $\rho^\diamond$ is a robust $\mathcal{D}$-stability bound taking both unstructured and parametric uncertainty into account. It must be mentioned that the presented results are given for a matrix $\mathbf{A}$ that is complex and thus, so is E. The practical cases encountered in automatic control correspond to real matrices. However, it is difficult to impose the realness of E (*i. e.* the case where $E \in \mathcal{B}(\rho) \cap \mathbb{R}^{q\times r}$) and the results are stated for the general case of complex matrix.

2.3 Preliminaries

This section is dedicated to the presentation of two lemmas that will be useful for the various reasonings and proofs in the sequel.

Lemma 1 *[11] Let Z, E, Δ and F be complex matrices with appropriate dimensions. Assume that Z is Hermitian then*

$$Z + \mathcal{H}(E\Delta F) \prec 0 \quad \forall \Delta \,|\, \Delta'\Delta \preceq \mathbf{I} \tag{4}$$

if and only if there exists a scalar number $\epsilon > 0$ satisfying

$$Z + \epsilon EE' + \frac{1}{\epsilon}F'F \prec 0 \tag{5}$$

This lemma can be proved owing to the well known $\mathcal{S}$-procedure.

Lemma 2 *[12, theorem 2.3.12] Let $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{k\times n}$ and $Q = Q' \in \mathbb{C}^{n\times n}$ be given. Then the following statements are equivalent:*

(i) There exists a matrix Y satisfying

$$Q + \mathcal{H}(BYC) \prec 0 \tag{6}$$

(ii) The following two conditions hold

$$\begin{cases} B^{\perp}QB^{\perp'} \prec 0 \quad \text{or} \quad BB' \succ 0 \\ C'^{\perp}QC'^{\perp'} \prec 0 \quad \text{or} \quad C'C \succ 0 \end{cases} \tag{7}$$

3 Matrix $\mathcal{D}_U$-stability

In this section, the notion of $\mathcal{EEMI}$-regions ($\mathcal{EEMI}$ for $\mathcal{E}$xtended $\mathcal{E}$llipsoidal $\mathcal{M}$atrix $\mathcal{I}$nequality) introduced in [8] is recalled and modified. Then, the concept of $\mathcal{D}_U$-stability is reminded to the reader and some conditions for $\mathcal{D}_U$-stability to hold are given. These conditions are a basis for the derivation of the results of this paper.

Definition 3 *: Let $\mathcal{R}$ be a set of m Hermitian matrices R_k defined by:*

$$R_k = R_k' = \begin{bmatrix} R_{k_{00}} & R_{k10} \\ R_{k_{10}}' & R_{k11} \end{bmatrix} \in \mathbb{C}^{2d \times 2d},\ R_{k_{11}} \succeq 0,\ R_{k_{11}} \in \mathbb{C}^{d \times d} \forall k \in \{1, ..., m\} \tag{8}$$

The set of points $\mathcal{D}_U$ defined by:

$$\begin{aligned} \mathcal{D}_U \;=\; \Bigg\{ & z \in \mathbb{C} \mid \exists w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \in \{\mathbb{R}^{+*}\}^m \mid \\ & f_{\mathcal{D}_U}(w,z) = \sum_{k=1}^{m} \left(w_k \begin{bmatrix} \mathbf{I}_{d_k} & \tilde{z}\mathbf{I}_{d_k} \end{bmatrix} R_k \begin{bmatrix} \mathbf{I}_{d_k} \\ z\mathbf{I}_{d_k} \end{bmatrix} \right) \prec 0 \Bigg\} \end{aligned} \tag{9}$$

is called an open $\mathcal{EEMI}$-region of degree d.

$\mathcal{D}_U$ is the generic name for an $\mathcal{EEMI}$-region. When $m = 1$, this description reduces to $\mathcal{EMI}$-formulation ($\mathcal{EMI}$ for $\mathcal{E}$llipsoidal $\mathcal{M}$atrix $\mathcal{I}$nequality) proposed in [13]. In this case, the region is convex and is rather denoted by $\mathcal{D}_R$ (since there is only one matrix $R_k = R$). $\mathcal{D}_R$ can be a half plane, a disc, the interior of an ellipse, a classical or hyperbolic sector, a vertical or horizontal strip... possibly non symmetric with respect to the real axis since matrix R here is assumed to be complex. In practice, assume that the clustering region to be described is the union of m possibly non symmetric and disjoint subregions, each of them reading an $\mathcal{EMI}$-formulation deduced from an original scalar inequality, then this union always reads some $\mathcal{EEMI}$-formualtion.

Now assume that the union of m_1 first degree $\mathcal{EMI}$-subregions and m_2 second degree (with available descriptions deduced from a scalar inequality) $\mathcal{EMI}$-subregions is to be considered. then the union reads definition 3 with $m = m_1 + m_2$ and $d = m + 1$. If only m first degree $\mathcal{EMI}$-subregions are considered (half planes and discs), then $d = 1$. This last case will induce special reasonings in the sequel.

Remark 4 *The set of $\mathcal{EEMI}$-regions is the same as the one of $\mathcal{LMI}$-regions [2]. Only the formulation is slightly different. The interest in this change in the formulation will appear in Section 5 where parameter uncertainties are considered.*

A Necessary and sufficient condition for the $\mathcal{D}_U$-stability of a complex matrix is now provided:

Theorem 5 *: Let $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region as introduced in definition 3. A matrix $A \in \mathbb{C}^{n \times n}$ is $\mathcal{D}_U$-stable if and only if there exists a set $\mathcal{P}$ of m HPD matrices $P_k \in \mathbb{C}^{n \times n}$, $k = 1, ..., m$, such that:*

$$\mathbb{U}(A, \mathcal{P}) \;=\; \left[\begin{array}{c|c} \mathbf{I}_{dn} & \mathbf{I}_d \otimes A' \end{array}\right] \mathfrak{U}(\mathcal{P}) \left[\begin{array}{c} \mathbf{I}_{dn} \\ \hline \mathbf{I}_d \otimes A \end{array}\right] \prec 0 \tag{10}$$

with:

$$\mathfrak{U}(\mathcal{P}) \;=\; \sum_{k=1}^{m} (R_k \otimes P_k) \tag{11}$$

The complete proof is proposed in [8] following original arguments borrowed from [14, 1, 2]. However, definition of an $\mathcal{EEMI}$-region proposed in [8] is now replaced by definition 3.

Note that this condition consists in finding m Lyapunov matrices P_k (as much as subregions in practice). Besides, the result of [13] is a special case of theorem 5 for which $m = 1$ and $\mathcal{R}$ reduces to only one matrix $R \in \mathbb{R}^{d \times d}$ (*i.e.* for one single symmetric $\mathcal{EMI}$-region).

Since inequality (11) provides a NSC for nominal matrix $\mathcal{D}_U$-stability, it enables to define both the concepts of robust and quadratic $\mathcal{D}_U$-stability of $\mathbf{A}$.

Definition 6 *Let an uncertain matrix* $\mathbf{A}$ *be defined as in (1) and* $\mathcal{D}_U$ *be an* $\mathcal{EEMI}$*-region as defined in (9). Matrix* $\mathbf{A}$ *is robustly (or quadratically)* $\mathcal{D}_U$*-stable if and only if for any* $\theta \in \Theta$ *and for any* $E \in \mathcal{B}(\rho)$*, there exists a set* $\mathcal{P}(\theta, E)$ *of* m *HPD matrices* $P_k(\theta, E) \in \mathbb{C}^{n \times n}$*,* $k = 1, ..., m$ *(resp. a set* $\mathcal{P}$ *of* m *HPD matrices* P_k*), such that:*

$$\mathbb{U}(\mathbf{A}, \mathcal{P}(\theta, E)) \prec 0 \quad \textit{(resp.} \quad \mathbb{U}(\mathbf{A}, \mathcal{P}) \prec 0\textit{)} \tag{12}$$

Clearly, robust $\mathcal{D}_U$-stability is the property to be tested whereas quadratic stability is restrictive since it corresponds to the existence of a single set of Lyapunov matrices, independent on θ and E, ensuring property over the whole polytope $\mathcal{M}$. It is a natural extension of the well known quadratic stability [15] which is only sufficient for robust $\mathcal{D}_U$-stability.

In the sequel, a robust $\mathcal{D}_U$-stability bound based upon quadratic $\mathcal{D}_U$-stability will be denoted by $\rho^\star$ whereas a bound allowing Lyapunov matrices to depend on θ will be denoted by $\rho^\diamond$.

4 Complex quadratic $\mathcal{D}_U$-stability analysis

In this section, no polytopic uncertainty is considered so M is a <u>known</u> and <u>constant</u> matrix. A lower bound $\rho^\star$ of the maximum value of ρ, the radius of ball $\mathcal{B}(\rho)$, such that $\mathbf{A}$ is quadratically $\mathcal{D}_U$-stable, is computed. Some connections to previous results, especially relevant to $\mathcal{H}_\infty$ theory, are also highlighted.

4.1 Complex quadratic $\mathcal{D}_U$-stability bound

Theorem 7 *Let* $\mathbf{A} \in \mathbb{C}^{n \times n}$ *be a matrix as defined in (1) with* M *constant and* $\mathcal{D}_U$ *be an* $\mathcal{EEMI}$*-region as formulated in definition 3. Matrix* $\mathbf{A}$ *is quadratically* $\mathcal{D}_U$*-stable with respect to* $\mathcal{B}(\rho)$ *if* **(and only if when** $d = 1$**)** *there exists a set* $\mathcal{P}$ *of* m *HPD matrices* P_k*,* $k = 1, ..., m$ *such that:*

$$\mathbb{Q}_U(M, \mathcal{P}) = \left[\begin{array}{c|c|c|c} \mathbf{I}_{dn} & \mathbf{I}_d \otimes A' & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_d \otimes J' & \mathbf{I}_{dq} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{dr} \end{array}\right] Q(\mathcal{P}, \gamma) \left[\begin{array}{c|c|c} \mathbf{I}_{dn} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{I}_d \otimes A & \mathbf{I}_d \otimes J & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_{dq} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_{dr} \end{array}\right] \prec 0 \tag{13}$$

with:

$$Q(\mathcal{P},\gamma) = \left[\begin{array}{cc|c|c} \multicolumn{2}{c|}{\mathcal{U}(\mathcal{P})} & \mathbf{0} & \begin{array}{c}\mathbf{I}_d\otimes L' \\ \hline \mathbf{0}\end{array} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & -\mathbf{I}_{dq} & \mathbf{0} \\ \hline \mathbf{I}_d\otimes L & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_{dr} \end{array}\right] \tag{14}$$

where $\gamma = \rho^{-2}$ and $\mathcal{U}(\mathcal{P})$ defined in (11).

Proof:

Sufficiency: first apply lemma 2 to inequality (13), then there exists some matrix $G_U \in \mathbb{C}^{d(2n+q+r)\times dn}$ such that:

$$\mathbb{G}_U(M,\mathcal{P},G_U) = Q(\mathcal{P},\gamma) + \mathcal{H}\left(G_U \left[\begin{array}{c|c|c|c} \mathbf{I}_d\otimes A & -\mathbf{I}_{mdn} & \mathbf{I}_d\otimes J & \mathbf{0}\end{array}\right] \right) \prec 0 \tag{15}$$

Define matrix $S(M,E)$ by:

$$S(M,E) = \left[\begin{array}{c|c|c|c} \mathbf{I}_{dn} & \mathbf{I}_d\otimes\mathbf{A}' & \mathbf{I}_d\otimes(L'E') & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{dr}\end{array}\right]$$

Left and right mutliply (15) by $S(M,E)$ and $S'(M,E)$ respectively yields

$$\left[\begin{array}{c|c} \mathbb{U}(\mathbf{A},\mathcal{P}) - \mathbf{I}_d\otimes(L'E'EL) & \mathbf{I}_d\otimes L' \\ \hline \mathbf{I}_d\otimes L & -\gamma\mathbf{I}_{dr}\end{array}\right] \prec 0$$

By virtue of Schur's lemma, it is equivalent to:

$$\mathbb{U}(\mathbf{A},\mathcal{P}) \prec \mathbf{I}_d\otimes(L'E'EL) - \rho^2\mathbf{I}_d\otimes(L'L)$$

Since E belongs to $\mathcal{B}(\rho)$, it comes $E'E \preceq \rho^2\mathbf{I}_n$ what proves sufficiency.

Necessity part (when $d=1$) : the reader is reminded of the fact that $d = 1$ means that $\mathcal{D}_U$ is a union of discs anf half planes. Assume $\mathbf{A}$ is quadratically $\mathcal{D}_U$-stable. Then, there exists a set $\tilde{\mathcal{P}}$ of m HPD matrices $\tilde{P}_k$ such that $\mathbb{U}(\mathbf{A},\tilde{\mathcal{P}}) \prec 0$. Consider the scalar numbers $v_k = \sqrt{R_{k_{11}}}$, $\forall k\in\{1,\ldots,m\}$. By virtue of Schur's lemma, this last inequality is equivalent to:

$$Z(\mathbf{A},\tilde{\mathcal{P}}) = \begin{bmatrix} \sum_{k=1}^{m}(R_{k_{00}}\tilde{P}_k + \mathcal{H}(R_{k_{10}}\tilde{P}_k\mathbf{A})) & [v_k\mathbf{A}'\tilde{P}_k]_{k=1}^m \\ ([v_k\mathbf{A}'\tilde{P}_k]_{k=1}^m)' & -\operatorname*{diag}_{k=1,\ldots,m}\{\tilde{P}_k\}\end{bmatrix} \prec 0$$

Defining $\tilde{E} = \frac{E}{\rho}$ and separating the uncertain part of the previous inequality from the nominal one, it comes:

$$Z(A,\tilde{\mathcal{P}}) = \mathcal{H}\left(\left[\begin{array}{c} \sum_{k=1}^{m}(R_{k_{10}}\tilde{P}_kJ) \\ \hline ([v_kJ'\tilde{P}_k]_{k=1}^m)'\end{array}\right] \tilde{E} \left[\begin{array}{c|c} \rho L & \mathbf{0}\end{array}\right] \right) \prec 0$$

At this stage, by virtue of lemma 1, there exists $\epsilon > 0$ such that:

$$Z(A,\tilde{\mathcal{P}}) + \epsilon \left[\begin{array}{c} \sum_{k=1}^{m}(R_{k_{10}}\tilde{P}_kJ) \\ \hline ([v_kJ'\tilde{P}_k]_{k=1}^m)'\end{array}\right] \left[\begin{array}{c|c} \sum_{k=1}^{m}(R_{k_{10}}J'\tilde{P}_k) & [v_kJ'\tilde{P}_k]_{k=1}^m\end{array}\right] +$$

$$\frac{1}{\epsilon}\left[\begin{array}{c}\rho L'\\ \hline \mathbf{0}\end{array}\right]\left[\begin{array}{c|c}\rho L & \mathbf{0}\end{array}\right] \prec 0$$

Mutliplying this inequality by ϵ and applying Schur's lemma leads to:

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{Z(A,\mathcal{P})} & \sum_{k=1}^{m}(R_{k_{10}}P_kJ) & L' \\ & & ([v_kJ'P_k]_{k=1}^m)' & \mathbf{0} \\ \hline \sum_{k=1}^{m}(R'_{k_{10}}J'P_k) & [v_kJ'P_k]_{k=1}^m & -\mathbf{I}_q & \mathbf{0} \\ \hline L & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_r \end{array}\right] \prec 0 \qquad (16)$$

in which $P_k = \epsilon\tilde{P}_k$, $\forall k \in \{1,...,m\}$. Now define matrix $T(A,J)$ by:

$$T(A,J) = \left[\begin{array}{c|c|c} \mathbf{I}_n & \mathbf{0} & \mathbf{0} \\ \hline v_k(\mathbf{1}_{m,1}\otimes A) & v_k(\mathbf{1}_{m,1}\otimes J) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_q & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I}_r \end{array}\right]$$

Left and right multiply (16) respectively by $T'(A,J)$ and $T(A,J)$ yields inequality (13). $\square$

It must be noticed that condition (13) is an $\mathcal{LMI}$ system with respect to $\mathcal{P}$ whose feasibility can be checked owing to various $\mathcal{LMI}$ softwares. Besides, those softwares allow to consider γ as a decision variable to be minimized while preserving feasibility. Denote by $\gamma^\star$, this minimum value. Thus, $\rho^\star = (\gamma^\star)^{-1/2}$ is the greatest value of ρ such that quadratic $\mathcal{D}_U$-stability can be attested by our condition. Since necessity is proved for $d = 1$ (*i.e.* for a union of discs and half planes), this robust $\mathcal{D}_U$-stability bound, in this case, is also the complex quadratic $\mathcal{D}_U$-stability radius. However, when $d > 1$, it can be proved that $\rho^\star$ is not always the quadratic $\mathcal{D}_U$-stability radius.

When $\mathbf{A}$ is complex or when some R_k is complex ($R_k \in \mathbb{C}^{2d\times 2d}$ corresponds to the case where the assocaited subregion $\mathcal{D}_k$ is not symmetric with respect to the real axis), then $\mathcal{LMI}$ condition (13) involves complex matrices. such $\mathcal{LMI}$ can be solved by building equivalent augmented real $\mathcal{LMI}$ (see [16]).

Remark 8 *For the special case where $m = 1$ (one single region), theorem 7 falls out as reformulation of a result in [3] dedicated to $\mathcal{LMI}$-regions. If, furthermore, $m = d = 1$ (in practice, root-clustering in a disc or a half plane), $\rho^\star$ can be proved to be the complex stability radius. Indeed, considering Hurwitz or Schur stability as particular case, our condition becomes equivalent to the $\mathcal{LMI}$ formulation of the Bounded Real Lemma applied to computation of the $\mathcal{H}_\infty$-norm of the system described by the triple (A, J, L). Besides, the inverse of such norm is known to be the complex (or complex quadratic, or real quadratic) stability radius (see [17, 18, 19, 3] and the references therein). Other discs and half planes can be handled in the same way after appropriate translation/rotation and/or homothety.*

To sum up this section, taking the above remark into account, it can be assessed that $\rho^\star$ is:
- a quadratic (so robust)-$\mathcal{D}_U$-stability bound for any $\mathcal{EEMI}$-region $\mathcal{D}_U$;
- the complex (or complex quadratic or real quadratic) $\mathcal{D}_R$-stability radius for any $\mathcal{EMI}$-region $\mathcal{D}_R$ of degree $d = 1$ (*i.e.* any half plane or disc);
- the quadratic $\mathcal{D}_U$-stability radius for any $\mathcal{EEMI}$-region $\mathcal{D}_U$ such that $d = 1$ (*i.e.* any union of discs and half planes).

5 Parameter-dependent $\mathcal{D}_U$-stability analysis

In this section, it is assumed that M is somewhere in the polytope $\mathcal{M}$. The aim is to compute $\rho^\diamond$, a robust $\mathcal{D}_U$-stability bound. It will first be shown that condition (13) can still be handled to derive a quadratic $\mathcal{D}_U$-stability bound $\rho^\star$. Nevertheless, attesting only quadratic $\mathcal{D}_U$-stability over the whole polytope is likely to be conservative. Thus it matters to take parameter deflections into account. This is why another proposed $\mathcal{LMI}$ condition leading to $\rho^\diamond$ will implicitly involve Parameter-Dependent Lyapunov Matrices (PDLM). The first paragraph is dedicated to the computation of $\rho^\star$; the second one tackles the computation of $\rho^\diamond$; the third one emphasizes the interest of $\rho^\diamond$ in both closed-loop robustness and fragility analysis.

5.1 Quadratic $\mathcal{D}_U$-stability bound $\rho^\star$

Theorem 9 *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a matrix as defined in (1) and $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region as formulated in definition 3. Matrix $\mathbf{A}$ is quadratically $\mathcal{D}_U$-stable with respect to $\mathcal{B}(\rho)$ and $\mathcal{M}$ if there exists one single set $\mathcal{P}$ made up by m HPD matrices P_k, $k = 1, ..., m$ such that:*

$$\mathbb{Q}_U(M_i, \mathcal{P}) \prec 0 \quad \forall i \in \{1, ..., N\} \tag{17}$$

where $\gamma = \rho^{-2}$.

Proof:
Start from inequality (17). There exists $\alpha > 0$ such that, $\forall\, i \in \{1, ..., N\}$:

$$\mathbb{Q}_U(M_i,\mathcal{P}) + \alpha \sum_{k=1}^{m} \left(\left[\begin{array}{c} \mathbf{I}_d \otimes A_i' \\ \hline \mathbf{I}_d \otimes J_i' \\ \hline \mathbf{0} \end{array}\right] (\mathbf{I}_d \otimes P_k) \left[\begin{array}{c|c|c} \mathbf{I}_d \otimes A_i & \mathbf{I}_d \otimes J_i & \mathbf{0} \end{array}\right] \right) \prec 0 \Leftrightarrow$$

$$\left[\begin{array}{c|c|c} \Psi_i & \sum_{k=1}^{m}(R_{k_{10}} \otimes (P_k J_i)) & \mathbf{I}_d \otimes L_i' \\ \hline \sum_{k=1}^{m}(R_{k_{10}}' \otimes (J_i' P_k)) & -\mathbf{I}_{dq} & \mathbf{0} \\ \hline \mathbf{I}_d \otimes L_i & \mathbf{0} & -\gamma \mathbf{I}_{dr} \end{array}\right] +$$

$$\left[\begin{array}{c} \mathbf{I}_d \otimes A_i' \\ \hline \mathbf{I}_d \otimes J_i' \\ \hline \mathbf{0} \end{array}\right] (\sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes P_k))(\sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes P_k))^{-1}(\sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes P_k)) \left[\begin{array}{c} \mathbf{I}_d \otimes A_i' \\ \hline \mathbf{I}_d \otimes J_i' \\ \hline \mathbf{0} \end{array}\right]' \prec 0$$

with

$$\Psi_i = \sum_{k=1}^{m}(R_{k_{00}} \otimes P_k + \mathcal{H}(R_{k_{10}} \otimes (P_k A_i)))$$

and

$$\hat{R}_{k_{11}} = R_{k_{11}} + \alpha \mathbf{I}_d > 0,\ \forall k \in \{1, ..., m\}.$$

Applying Schur's lemma yields:

$$\left[\begin{array}{c|c|c|c} \Psi_i & \sum_{k=1}^{m}(R_{k_{10}} \otimes (P_k J_i)) & \mathbf{I}_d \otimes L_i' & \sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes (A_i' P_k)) \\ \hline \sum_{k=1}^{m}(R_{k_{10}}' \otimes (J_i' P_k)) & -\mathbf{I}_{dq} & \mathbf{0} & \sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes (J_i' P_k)) \\ \hline \mathbf{I}_d \otimes L_i & \mathbf{0} & -\gamma \mathbf{I}_{dr} & \mathbf{0} \\ \hline \sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes (A_i' P_k)) & \sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes (J_i' P_k)) & \mathbf{0} & -\sum_{k=1}^{m}(\hat{R}_{k_{11}} \otimes P_k) \end{array}\right] \prec 0$$

what, noting that $-\hat{R}_{k_{11}} = R_{k_{11}} - 2\hat{R}_{k_{11}} + \alpha\mathbf{I}_d$, can be written:

$$\left[\begin{array}{c|c|c|c} \Phi_i & \sum_{k=1}^{m}(F_k(\mathbf{I}_d \otimes J_i)) & \mathbf{I}_d \otimes L_i' & \beta_i \\ \hline \sum_{k=1}^{m}((\mathbf{I}_d \otimes J_i')F_k') & -\mathbf{I}_{dq} & \mathbf{0} & \sum_{k=1}^{m}(\mathbf{I}_d \otimes (J_i'G_k')) \\ \hline \mathbf{I}_d \otimes L_i & \mathbf{0} & -\gamma\mathbf{I}_{dr} & \mathbf{0} \\ \hline \beta_i' & \sum_{k=1}^{m}(G_k(\mathbf{I}_d \otimes J_i)) & \mathbf{0} & \Omega_i \end{array}\right] +$$

$$\left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \sum_{k=1}^{m}(\alpha\mathbf{I}_d \otimes P_k) \end{array}\right] \prec 0$$

where

$$\left\{\begin{array}{lcl} \Phi_i & = & \sum_{k=1}^{m}(R_{k_{00}} \otimes P_k + \mathcal{H}(F_k(\mathbf{I}_d \otimes A_i))) \\ \beta_i & = & \sum_{k=1}^{m}((\mathbf{I}_d \otimes A_i')G_k' + R_{k_{10}} \otimes P_k - F_k) \\ \Omega_i & = & \sum_{k=1}^{m}((R_{k_{11}} \otimes P_k) - G_k - G_k') \end{array}\right.$$

with $F_k = R_{k_{10}} \otimes P_k$ and $G_k = \hat{R}_{k_{11}} \otimes P_k$, $\forall k \in \{1, ..., m\}$. By permutation on the rows and columns of the previous matrix inequality and taking into account that the second term above is positive semi-definite, it comes:

$$\left[\begin{array}{c|c|c|c} \Phi_i & \beta_i & \sum_{k=1}^{m}(F_k(\mathbf{I}_d \otimes J_i)) & \mathbf{I}_d \otimes L_i' \\ \hline \beta_i' & \Omega_i & \sum_{k=1}^{m}((\mathbf{I}_d \otimes J_i')G_k') & \mathbf{0} \\ \hline \sum_{k=1}^{m}((\mathbf{I}_d \otimes J_i')F_k') & \sum_{k=1}^{m}((\mathbf{I}_d \otimes J_i')G_k') & -\mathbf{I}_{dq} & \mathbf{0} \\ \hline \mathbf{I}_d \otimes L_i & \mathbf{0} & \mathbf{0} & -\gamma\mathbf{I}_{dr} \end{array}\right] \prec 0$$

The previous inequality can be written

$$\mathbb{G}_U(M_i, \mathcal{P}, G_U) \prec 0 \tag{18}$$

with:

$$G_U' = \sum_{k=1}^{m} \left[\begin{array}{c|c|c} F_k & G_k & \mathbf{0} \end{array}\right]' = \left[\begin{array}{c|c|c} F & G & \mathbf{0} \end{array}\right]' \tag{19}$$

For any $\theta \in \Theta$, since $\theta_i \geq 0$, it comes

$$\sum_{j=1}^{N}(\theta_i \mathbb{G}_U(M_i, \mathcal{P}, G_U)) = \mathbb{G}_U(M(\theta), \mathcal{P}, G_U) \prec 0$$

Left and rightmultiply the previous inequality by $S'(M(\theta), E)$ and $S(M(\theta), E)$ respectively and follow the same reasoning as in the "sufficiency part" of the proof of theorem 7. G_U is eliminated and it leads to $\mathbb{U}(\mathbf{A}, \mathcal{P}) \prec 0$.□

As in the previous section, γ can be considered as an $\mathcal{LMI}$ variable and be minimized in order to derive the bound $\rho^\star$ which is however not proved, this time, to be the quadratic $\mathcal{D}_U$-stability radius, even when $d = 1$.

5.2 Parameter-dependent robust $\mathcal{D}_U$-stability bound $\rho^\diamond$

In this section, the main result is stated with the help of all the previoulsy presented results. Now the conservatism induced by the quadratic $\mathcal{D}_U$-stability is reduced by the use of PDLM. For this purpose, inequality (15) is prefered to (13).

Theorem 10 *Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a matrix as defined in (1) and $\mathcal{D}_U$ be an $\mathcal{EEMI}$-region as formulated in definition 3. $\mathbf{A}$ is robustly $\mathcal{D}_U$-stable with respect to $\mathcal{B}(\rho)$ and $\mathcal{M}$ if there exist N sets $\mathcal{P}_i$, each one made up by m HPD matrices P_{k_i}, $k = 1, ..., m$, as well as $G_U \in \mathbb{C}^{d(2n+q+r)\times dn}$ such that:*

$$\mathbb{G}_U(M_i, \mathcal{P}_i, G_U) \prec 0 \quad \forall i \in \{1, ..., N\} \tag{20}$$

where $\gamma = \rho^{-2}$.

Proof: for any vector of coordinates $\theta \in \Theta$, consider:

$$\mathbb{G}_U(M(\theta), \mathcal{P}(\theta), G_U) = \sum_{j=1}^{N} (\theta_i \mathbb{G}_U(M_i, \mathcal{P}_i, G_U)) \prec 0$$

where the set $\mathcal{P}(\theta)$ is made up by m HPD matrices $P_k(\theta) = \sum_{i=1}^{N}(\theta_i P_{k_i})$. From this step, follow the same reasoning as in the "sufficiency part" of the proof of theorem 7. □

As in the previous conditions, $\rho = \gamma^{-1/2}$ can be easily maximized to $\rho^\diamond$, which is a robust $\mathcal{D}_U$-stability bound. Note that the Lyapunov matrices are parameter-dependent. For each instance $\theta \in \Theta$, matrices $P_k(\theta)$ is a convex combination of the Lyapunov matrices P_{k_i} associated with each vertex. Each $P_k(\theta)$ therefore describes a polytope of matrices when θ describes Θ. Such a dependence is enabled by the addition of G_U. Rather than keeping $\mathcal{P}$ constant, it is better to keep G_U constant and to allow the set $\mathcal{P}(\theta)$ to vary in a polytopic way. This is an idea that was originally proposed in [20]. It is here exploited to solve our root-clustering problem.

The $\mathcal{EMI}$ and $\mathcal{EEMI}$ formulations reveal very useful since they read expressions such as $B^\perp Q B^{\perp'} \prec 0$ anf therefore, lemma 2 can be very easily applied to introduce G_U. Hence, such a formulations of regions are, to the authors'mind, as far as robust analysis is concerned, more discerning than $\mathcal{LMI}$-regions to derive PDLM.

Remark 11 *It can be proved that matrix G_U can always be taken of the form (19) with no loss of generality. The proof is not provided here for conciseness. Such an assertion can be used to reduce the time required for the computation of $\rho^\diamond$.*

Remark 12 *To be convinced that condition based upon PDLM is always at least less conservative than the one based upon quadratic $\mathcal{D}_U$-stability, just have a look to the beginning of the proof of theorem 9 up to equations (18) and (19) to note that, starting from (17), but with $\mathcal{P}_i = \mathcal{P}\, \forall i$, would lead to (20) with G_U given by (19). As a consequence, it comes $\rho^\diamond \geq \rho^\star$.*

5.3 Application to closed-loop robustness and fragility analysis

In this paragraph, a system is associated with the triple of matrices $(A_0; B; C)$. These matrices are actually subject to a polytopic uncertainty as the one defined in (2)-(3). A static output feedback control law K is applied on this system. Because it is impossible in pratice to precisely implement this control law, it is reasonnable to assume that the feedback matrix actually reads $K + \Delta_K$ where Δ_K is an additive uncertain term. Thus, the closed-loop uncertain state matrix is given by:

$$\mathbf{A} = A_0 + BKC + B\Delta_K C \tag{21}$$

Now assume that either B or C is precisely known and that it matters to attest robust $\mathcal{D}_U$-stability of $\mathbf{A}$ for a specified $\mathcal{EEMI}$-region $\mathcal{D}_U$. This problem is the same as the one handled above. It suffices to compute $\rho^\diamond$ by solving (20) with $A = A_0 + BKC$, $J = B$, $L = C$ and $E = \Delta_K$. If this condition holds, then $\rho^\diamond$ can be seen as the maximal acceptable deviation (in terms of 2-norm) while implementing K. It is then a fragility criterion.

6 Numerical illustration

The model is inspired from an example in [21]. The computations are performed with MATLAB 6.1 and its LMITOOLBOX on a PC Pentium 1,7 MHz with 128Mo of RAM.
A satellite is modelled by two masses connected by a spring with torque Γ and viscous damping δ. State and input matrices are:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -10\Gamma & -10\delta & 10\Gamma & 10\delta \\ 0 & 0 & 0 & 1 \\ \Gamma & \delta & -\Gamma & -\delta \end{bmatrix} \; ; \; B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

The torque is assumed to read $\Gamma = \Gamma_0 + \Delta\Gamma$ where $\Gamma_0 = 0.35$ and where the uncertain term $\Delta\Gamma$ checks $|\Delta\Gamma| \leq 0.03$. Viscous damping δ checks $\delta = \phi\psi$. Function ϕ equals $\phi_0 + \Delta\phi$ where $\phi_0 = 0.15$ and $\Delta\phi$ is unknown. Function ψ is defined by $\psi = \sqrt{\Gamma/10}$. Thus, it comes:

$$\psi = \sqrt{\frac{\Gamma_0 + \Delta\Gamma}{10}} = \sqrt{\frac{\Gamma_0}{10}}\sqrt{1 + \frac{\Delta\Gamma}{\Gamma_0}} \simeq \sqrt{\frac{\Gamma_0}{10}}\left(1 + \frac{\Delta\Gamma}{2\Gamma_0}\right) = \psi_0 + \frac{1}{2\sqrt{10\Gamma_0}}\Delta\Gamma$$

A static state feedback matrix $K = [-17.499 \; -53.326 \; -78.501 \; -16.691]$ assigns the nominal spectrum $\{-2 \pm 2\mathbf{i}; -7; -6\}$. We choose $\mathcal{D}_U = \mathcal{D}_{R_1} \cup \mathcal{D}_{R_2} \cup \mathcal{D}_{R_3}$ where $\mathcal{D}_{R_1}$ and $\mathcal{D}_{R_2}$ are discs of radius 1 and of respective centers $-2 \pm 2\mathbf{i}$ and where $\mathcal{D}_{R_3}$ is the half plane defined by $\mathrm{Re}(z) < -4$. Of course, $\mathcal{D}_U$ contains the nominal spectrum. The uncertainty on $\Delta\Gamma$ generates a polytopic uncertainty while the one on $\Delta\phi$ generates a norm-bounded uncertainty. Thus, the closed-loop state matrix reads formulation (1). That is why theorems 10 and 9 are used to compute the bounds

$$\rho^\diamond = 2.63 \times 10^{-2}, \quad \text{and} \quad \rho^\star = 1.62 \times 10^{-2}$$

respectively obtained in `297.1s` and `18.1s`. This example emphasizes the relevance of the PDL approach versus the quadratic one as far as conservatism is concerned. On figure 1, the uncertain matrix roots are plotted for 2000 random values of $\Delta\phi$ such that $|\Delta\phi| \leq \rho^\diamond$. The left side corresponds to the practical case where $\Delta\phi$ is real. The right side corresponds to an unrealistic case where $\Delta\phi$ is complex. Such a figure highlights the tightness of $\rho^\diamond$ and enables to appreciate the conservatism induced by the approach. It also seems to confirm that quadratic $\mathcal{D}_U$-stability is not equivalent to complex robust $\mathcal{D}_U$-stability for all $\mathcal{EEMI}$ regions of degree $d = 1$.

Figure 1: Pole migration with $|\Delta\phi| \leq \rho^{\diamond}$: $\Delta\phi \in \mathbb{R}$ (left) and $\Delta\phi \in \mathbb{C}$ (right)

7 Conclusion

The problem of robust matrix root-clustering was tackled. The uncertainty is <u>both</u> parametric and norm-bounded and the clustering regions denoted by $\mathcal{D}_U$ are referred to as $\mathcal{EEMI}$-regions. This set includes numerous unions of possibly non symmetric and disjoint $\mathcal{EMI}$-subregions. This is an original point of the paper. Some $\mathcal{LMI}$ techniques to compute robust $\mathcal{D}_U$-stability bounds were presented. They implicitly involve parameter-dependent Lyapunov matrices what helps the bound to be tighter to the complex $\mathcal{D}_U$-stability radius. Besides, the proposed condition are general enough to encompass some strong results of the literature relevant to robust stability. Some drawback of this technique is that it is available for a complex matrix. In practice, $\mathbf{A}$ is real and so is uncertainty E. It is hard to take the realness of E into account but a further investigation could be the research of a good lower bound of the real $\mathcal{D}_U$-stability radius, using the arguments of [19] for instance. At last, another challenge is the discerning use of these robust transient performance criteria in a design context.

References

[1] S. Gutman and E. I. Jury, "A general theory for matrix root-clustering in subregions of the complex plane," *IEEE Transactions on Automatic Control*, vol. 26, no. 4, pp. 853–863, August 1981.

[2] M. Chilali and P. Gahinet, "H_∞ Design with Pole Placement Constraints : An LMI Approach," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 358–367, March 1996.

[3] M. Chilali, P. Gahinet, and P. Apkarian, "Robust pole placement in LMI regions," *IEEE Transactions on Automatic Control*, vol. 44, no. 12, pp. 2257–2269, December 1999.

[4] R. V. Patel and M. Toda, "Quantitative measures of robustness for multivariable systems," in *Joint Automatic Control Conference*, TP-8A, San Francisco, CA, 1980.

[5] R. K. Yedavalli, "Robust root clustering for linear uncertain systems using generalized Lyapunov theory," *Automatica*, vol. 29, no. 1, pp. 237–240, January 1993.

[6] C. R. Ashokkumar and R. K. Yedavalli, "Eigenstructure perturbation analysis in disjointed domains for linear uncertain systems," *International Journal of Control*, vol. 67, no. 6, pp. 887–899, 1997.

[7] O. Bachelier and B. Pradin, "Bounds for uncertain matrix root-clustering in a union of subregions," *International Journal of Robust and Nonlinear Control*, vol. 9, no. 6, pp. 333–359, May 1999.

[8] O. Bachelier and D. Mehdi, "Robust D_U-stability analysis," *International Journal of Robust and Nonlinear Control*, vol. 13, no. 6, pp. 533–558, May 2003.

[9] S. Wang, "Robust pole clustering in a good ride quality region of aircraft for matrices with structured uncertainties," *Automatica*, vol. 39, no. 3, pp. 525–532, March 2003.

[10] I. R. Petersen, "A stabilization algorithm for a class of uncertain linear systems," *Systems & Control Letters*, vol. 8, pp. 351–357, 1987.

[11] L. Xie, "Output feedback H_∞ control of systems with parameter uncertainties," *International Journal of Control*, vol. 63, no. 4, pp. 741–750, 1996.

[12] R. E. Skelton, T. Iwasaki, and K. Grigoriadis, *A unified approach to linear control design*. Taylor and Francis series in Systems and Control, 1997.

[13] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust D-stability condition for real convex polytopic uncertainty," *Systems & Control Letters*, vol. 40, no. 1, pp. 21–30, May 2000.

[14] J. L. Howland, "Matrix equations and the separation of matrix eigenvalues," *Journal of Mathematical Analysis and Applications*, vol. 33, pp. 683–691, 1971.

[15] B. R. Barmish, "Necessary and sufficient conditions for quadratic stabilizability of an uncertain system," *Journal of Optimization Theory and Applications*, vol. 46, no. 4, pp. 399–408, August 1985.

[16] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *LMI Control Toolbox*. The MathWorks, Inc., 1995.

[17] P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems : Quadratic stabilizability and H_∞ control theory," *IEEE Transactions on Automatic Control*, vol. 35, pp. 356–361, 1990.

[18] A. Packard and J. Doyle, "Quadratic stability with real and complex perturbations," *IEEE Transactions on Automatic Control*, vol. 35, no. 2, pp. 198–201, February 1990.

[19] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. Doyle, "A formula for computation of the real stability radius," *Automatica*, vol. 31, no. 6, pp. 879–890, 1995.

[20] J. C. Geromel, M. C. de Oliveira, and L. Hsu, "LMI characterization of structural and robust stability," *Linear Algebra and its Applications*, vol. 285, pp. 69–80, 1998.

[21] G. J. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*. Massachusetts ; Addison Wesley, 1986.