Computing a robust $\mathcal{D}$-stability bound using a parameter-dependent Lyapunov approach

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Version 1

Résumé

In this paper, the problem of robust matrix root-clustering against additive structured uncertainty is addressed. A bound on the size of the uncertainty domain preserving matrix $\mathcal{D}$-stability is derived from an $\mathcal{LMI}$ approach. A recently proposed sufficient condition for robust matrix $\mathcal{D}$-stability with respect to convex polytopic uncertainty is used. It is relevant to the framework dealing with parameter-dependent Lyapunov functions. Using this condition, the problem of computing the robustness bound is formulated as a generalized eigenvalue problem, what enables to maximize the bound value.

1 Introduction

We aim to analyse the robustness of transient performances of an LTI system. It is well known that they are strongly dependent on the location of the closed-loop state matrix eigenvalues in the complex plane. To ensure some particular performances on settling time or on damping ratio, those eigenvalues can be expected to lie in some specified region $\mathcal{D}$ of the complex plane. This property is referred to as $\mathcal{D}$-stability and can be seen as an extension of the concept of asymptotic stability. When $\mathcal{D}$ belongs to a given set of regions, given a known matrix A, the $\mathcal{D}$-stability of A can be easily testable from a computational point of view. For instance, when $\mathcal{D}$ is a transformable Ω-region [1], a necessary and sufficient condition (NSC) for the eigenvalues of A to lie in $\mathcal{D}$ is the existence of a single Hermitian positive definite matrix solution to a $\mathcal{G}$eneralized $\mathcal{L}$yapunov $\mathcal{E}$quation ($\mathcal{GLE}$) [2, 3, 4, 1, 5]. We refer to this work as the "Lyapunov approach". When $\mathcal{D}$ belongs to some set of convex regions called $\mathcal{LMI}$-regions ($\mathcal{LMI}$ for $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequality) as defined in [6], the NSC is the feasibility of an $\mathcal{LMI}$ system [6]. We refer to this framework as the "$\mathcal{LMI}$ approach". Thanks to the notion of $\mathcal{GLMI}$-regions ($\mathcal{G}$ for $\mathcal{G}$eneralized) introduced in [7], the $\mathcal{LMI}$ approach is extended to non-connected regions resulting from the union of several convex subregions [7, 8]. This can be of interest in the study of multi-time scale systems or to specify a robust damping ratio (see [8] for a better explanation).

Unfortunately, all those tests are not available anymore in the presence of an additive uncertainty matrix E affecting A. The $\mathcal{D}$-stability of the uncertain matrix $\mathbf{A} = A + E$ can not be tested by the previously mentionned conditions. A possible attempt to go round this obstacle is to try to estimate a lower bound on the maximal size of the tolerable uncertainty domain preserving the $\mathcal{D}$-stability of $\mathbf{A}$. Such bounds are called robust $\mathcal{D}$-stability bounds (or robustness bounds) and are usually deduced from a nominal $\mathcal{D}$-stability NSC. They constitute conservative conditions and it matters to improve existing results and to get closer to the ideal bounds (which could be called the $\mathcal{D}$-stability radii).

The first results relevant to Lyapunov approach are robust stability bounds proposed by Patel and Toda for both unstructured and structured uncertainties [9] and later improved in [10, 11, 12, 13, 14]. Concerning robust stability bounds and structured uncertainty, the reader should refer to the work proposed by Yedavalli [15], using the notion of bialternate product and improved in [16]. In the structured case, a robust stability bound corresponds in fact to a lower bound of k_m [17], the inverse of μ [18, 19]. Still dealing with Lyapunov approach, the first extensions towards robust $\mathcal{D}$-stability bounds is the case where $\mathcal{D}$ is a sector. It is tackled in [20, 21] and further investigated in [22]. But the first real generalization to any transformable Ω-region (including the disk *i.e.* Schur stability), and available for both unstructured and structured uncertainty, is due to Yedavalli [23]. It was later improved in [24, 25, 26, 27]. The $\mathcal{LMI}$ approach has also led to robust $\mathcal{D}$-stability bounds ($\mathcal{D}$ being an $\mathcal{LMI}$-region) and is very competitive against Lyapunov approach [28, 29]. It has the advantage, thanks to $\mathcal{GLMI}$-regions, to enable to consider the case of non-connected regions [30, 31, 8].

All those bounds suffer from some conservatism that is due, for most of them, to two reasons :
- First, those bounds are strongly dependent on the notion of "quadratic stability"[32] *i.e.* they are established considering a strong restriction : the robust $\mathcal{D}$-stability is attested by a single solution to either the $\mathcal{GLE}$ or the $\mathcal{LMI}$ for the whole uncertainty domain ;
- Second, the expressions of those bounds have been derived from the use of triangular inequality on the norms.

In this paper, we restrict our study to the affine structured parametric uncertainty and we propose to compute a robust $\mathcal{D}$-stability bound in a very different manner. To achieve our goal, we take benefit of a condition recently introduced in [33] and relying on the use of parameter-dependent Lyapunov functions [34, 35, 36, 37, 38, 39, 33, 40]. This framework helps us to reduce the first source of conservatism. We also formulate our problem as a generalized eigenvalue one. Hence, no expression or robustness bound is required. The use of triangular inequality is avoided what leads to the suppression of the second source of conservatism.

The paper is organized as follows. After this introduction, the second section is dedicated to the problem statement with the presentation of the uncertainty structure as well as our precise purpose, the characterization of the considered regions and an helpful background due to [33]. In the third section, the main result is given : a robust $\mathcal{D}$-stability bound is computed by solving a generalized eigenvalue problem. A step towards robust synthesis is also made. Numerical examples are proposed in the fourth section before concluding in the fifth one.

Notations : Throughout the paper, we denote by M^*, the transpose conjugate of M (only transpose if M is real), by M^{-*}, the matrix $(M^*)^{-1} = (M^{-1})^*$ when M is square non singular matrix, by $\otimes$, the Kronecker product and by $\bullet$, the Hadamard product [41]. $\mathbf{I}_n$ is the identity matrix of order n and $\mathbf{0}_{u,v}$ (resp. $\mathbf{0}_u$) is a null matrix of dimension $u \times v$ (resp. $u \times u$). $\mathbf{1}_{u,v}$ is a matrix of dimension $u \times v$ all the entries of which equal 1. In the matrix inequalities, "< 0" ("≤ 0") stands for negative (semi negative) definiteness and "> 0" ("≥ 0") stands for positive (semi positive) definiteness. At last, $\mathbf{i}$ denotes the imaginary unit.

2 Problem statement

In this section, we introduce the structure of the uncertainty that is taken into account, with two possible formulations. Then, we state the precise problem we want to tackle and we characterize the set of regions we will consider in the sequel. At last, we recall an helpful condition due to [33] that is a basis for this contribution.

2.1 The uncertainty structure

In this paragraph, we present a structure for the uncertainty we want to deal with. The first formulation is a simplifed one for the structured parametric uncertainty [42, 15]. We then express it as a particular case of the convex polytopic uncertainty.

The uncertain matrix is denoted by :

$$\mathbf{A} = (A + E) \in \mathbb{R}^{n\times n} \tag{1}$$

A is the nominal matrix (precisely known) and E is an additive uncertainty structured as follows :

$$E = \sum_{i=1}^{p}(\delta_i E_i) \tag{2}$$

where $\delta = [\delta_1 \cdots \delta_i \cdots \delta_p]^* \in \mathbb{R}^p$, is an unknown parameter vector and where E_i, $i \in \{1,...,p\}$ are known matrices specifying which matrix entries are affected by the parameter variation. In many contributions, δ is assumed to belong to an hyperrectangular set defined by :

$$\Delta = \{\delta = \begin{bmatrix}\delta_1\\ \vdots \\ \delta_p\end{bmatrix} \in \mathbb{R}^p \,|\, (\underline{\psi}_i\tilde{\delta} < \delta_i < \bar{\psi}_i\tilde{\delta}\,,\, \forall i \in \{1,...,p\})\} \tag{3}$$

where $\underline{\psi}_i$ and $\bar{\psi}_i$ are positive real specified numbers that define the form of Δ in the $\mathbb{R}^p$-space and $\tilde{\delta}$ can be seen as the "size" of this hyperrectangle. The nominal value is obtained with $\delta = \mathbf{0}_{p,1}$, *i.e.* :

$$\mathbf{A}(\mathbf{0}_{p,1}) = A \tag{4}$$

Note that Δ is a convex hull that can be defined by its vertices :

$$\Delta = \mathrm{Co}\{\delta_{[1]},...,\delta_{[N]}\} \tag{5}$$

where the vertices are

$$\delta_{[j]} = (-(\Phi_j \bullet \underline{\Psi}) + ((\mathbf{1}_{p,1} - \Phi_j) \bullet \overline{\Psi}))\tilde{\delta} \qquad \forall j \in \{1,...,N = 2^p\}, \tag{6}$$

and where

$$\Phi_j = \begin{bmatrix}\phi_{j_1}\\ \vdots \\ \phi_{j_p}\end{bmatrix} \in \mathbb{R}^p \tag{7}$$

is such that each entry ϕ_{j_i} of Φ_j takes the value 0 or 1 and where $\underline{\Psi}$ ans $\overline{\Psi}$ are column vectors defined by :

$$\underline{\Psi} = \begin{bmatrix} \underline{\psi}_1 \\ \vdots \\ \underline{\psi}_p \end{bmatrix} \in \{\mathbb{R}^+\}^p \quad \& \quad \overline{\Psi} = \begin{bmatrix} \overline{\psi}_1 \\ \vdots \\ \overline{\psi}_p \end{bmatrix} \in \{\mathbb{R}^+\}^p \tag{8}$$

Using these notations, $\mathbf{A}(\delta)$ is an element of $\mathcal{A}$ where $\mathcal{A}$ is a convex polytope defined by :

$$\mathcal{A} = \{\mathbf{A}(\xi) \,|\, \mathbf{A}(\xi) = \sum_{j=1}^{N}(\xi_j A_j); \xi \in \Xi\} \tag{9}$$

with :

$$\Xi = \{\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_N \end{bmatrix} \in \mathbb{R}^N \,|\, (\xi_j \geq 0, \forall j \in \{1, ..., N\}\,; \sum_{j=1}^{N} \xi_j = 1)\} \tag{10}$$

Note that :

$$A_j = A + \tilde{\delta}\mathcal{E}_j = \mathbf{A}(\delta_{[j]}) \quad \forall j \in \{1, ..., N\} \tag{11}$$

where

$$\mathcal{E}_j = \sum_{i=1}^{p}((-\phi_{j_i}\underline{\psi}_i + (1-\phi_{j_i})\overline{\psi}_i)E_i) \quad \forall j \in \{1, ..., N\} \tag{12}$$

This last form (9) will be useful for the derivation of our main result.

2.2 Robust matrix root-clustering

In the paper, we are interested in the $\mathcal{D}$-stability of $\mathbf{A}$. We assume that A is already $\mathcal{D}$-stable and we are looking for a condition on E that preserves the $\mathcal{D}$-stability of $\mathbf{A}$. As explained in introduction, this condition is a robust $\mathcal{D}$-stability bound *i.e.* a bound on the uncertainty domain that must not be exceeded. In the previous paragraph, we have defined this domain Δ and highlighted the scalar number $\tilde{\delta}$ as its size.

Our problem is to compute the maximum value of $\tilde{\delta}$ preserving the $\mathcal{D}$-stability of $\mathbf{A}$. $\tilde{\delta}^\star$ is therefore defined as the maximal value of $\tilde{\delta}$ for which $\mathbf{A}$ is $\mathcal{D}$-stable. In general, $\tilde{\delta}^\star$ is very hard to compute, even impossible, so it may be required to compute a lower bound $\bar{\delta} \leq \tilde{\delta}^\star$. $\bar{\delta}$ is a so-called robust $\mathcal{D}$-stability bound. Our goal is to find the tighter lower bound $\bar{\delta}^\star \leq \tilde{\delta}^\star$. This problem has already been addressed in [23, 24, 25, 22, 26, 31, 30, 8] using a quadratic stability framework. In this paper, parameter-dependent Lyapunov functions based conditions are used to reduce the conservatism of the computed robust $\mathcal{D}$-stability bound $\bar{\delta}^\star$.

2.2.1 Characterization of $\mathcal{D}$-regions and $\mathcal{D}$-stability

This paragraph is dedicated to the characterization of the regions we consider in this paper. Those regions were formulated in [33]. We recall that formulation by first defining the matrix R such that :

$$R_k = \begin{bmatrix} R_{00} & R_{10} \\ R_{10}^* & R_{11} \end{bmatrix} \quad \text{with :} \quad \begin{cases} R_{00} = R_{00}^* \in \mathbb{R}^{d\times d} \\ R_{11} = R_{11}^* \geq 0 \in \mathbb{R}^{d\times d} \end{cases} \tag{13}$$

From this matrix, the region $\mathcal{D}$ is defined as the subset of the complex plane described by :

$$\mathcal{D} = \{z \in \mathbb{C} \,|\, f_{\mathcal{D}}(z) = R_{00} + R_{10}z + R_{10}^* z^* + R_{11} z z^* < 0\} \tag{14}$$

Remarque 1 *: the assumption $R_{11} \geq 0$ is made for technical reasons. Actually, it ensures that $\mathcal{D}$ is a convex region. Then, this formulation is just slightly different from the more classical formulation of $\mathcal{LMI}$-regions [6].*

Remarque 2 *: the matrix R is assumed to be real because most of the practical pole placement problems require a convex symmetric region with respect to the real axis. However, it is possible to consider a complex matrix R with no strong modification in the forthcoming reasonning. This assumption was made in [43] where unions of possibly disjoint subregions are considered. We will come back to this extension further.*

Remarque 3 *: For many practical symmetric regions, the previous formulation can be easily deduced from the more classical polynomial one or from the $\mathcal{LMI}$ one [6]. Hence, it includes, for example, vertical shifted half planes, classical and hyperbolic sectors, vertical and horizontal strips, disks and ellipses ...*

An $\mathcal{LMI}$ characterization for $\mathcal{D}$-stability is then :

Théorème 4 *[33] : Let the region $\mathcal{D}$ be defined by (14) with (13). A matrix $A \in \mathbb{R}^{n \times n}$ is $\mathcal{D}$-stable* **if and only if** *there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that the following $\mathcal{LMI}$ holds :*

$$\mathcal{M}_{\mathcal{D}}(A, P) = R_{00} \otimes P + R_{10} \otimes (PA) + R_{10}^* \otimes (A^* P) + R_{11} \otimes (A^* P A) < 0 \tag{15}$$

We notice that the expression of $\mathcal{M}_{\mathcal{D}}(A, P)$ is easily deduced from $f_{\mathcal{D}}(z)$ by the substitution :

$$\{1; z; z^*; zz^*\} \leftrightarrow \{P; PA; A^*P; A^*PA\} \tag{16}$$

2.2.2 Robust $\mathcal{D}$-stability analysis

The polytope $\mathcal{A}$ is said to be robustly $\mathcal{D}$-stable, $\mathcal{D}$ being defined as in (14) with (13), if and only if $\mathbf{A}(\xi)$ is $\mathcal{D}$-stable for any instance of $\xi \in \Xi$. In the Lyapunov context, such a property is summarized by the following lemma :

Lemma 1 *: $\mathcal{A}$ is robustly $\mathcal{D}$-stable* **if and only if** *:*

$$\forall \xi \in \Xi,\ \exists P(\xi) = P^*(\xi) > 0 \in \mathbb{R}^{n \times n} \,|\, \mathcal{M}_{\mathcal{D}}(\mathbf{A}(\xi), P(\xi)) < 0 \tag{17}$$

It is well known that the previous condition is not tractable from a computational point of view since checking inequality (17) over the whole polytope is known to be an $\mathcal{NP}$-hard problem. However, in [33], a powerful and tractable sufficient condition is proposed.

Théorème 5 *[33, Theorem 4] : Let the polytope $\mathcal{A}$ and the region $\mathcal{D}$ be respectively defined by (9) with (10) and (14) with (13). If there exist matrices $F \in \mathbb{R}^{nd \times nd}$ and $G \in \mathbb{R}^{nd \times nd}$ as well as N symmetric positive definite matrices $P_j \in \mathbb{R}^{n \times n}$, $j \in \{1, ..., N\}$, such that :*

$$\mathcal{N}_{\mathcal{D}}(A_j, P_j, F, G) =$$
$$\begin{bmatrix} R_{00} \otimes P_j + F(\mathbf{I}_d \otimes A_j) + (\mathbf{I}_d \otimes A_j^*)F^* & R_{10} \otimes P_j + (\mathbf{I}_d \otimes A_j^*)G - F \\ R_{10}^* \otimes P_j + G^*(\mathbf{I}_d \otimes A_j) - F^* & R_{11} \otimes P_j - G - G^* \end{bmatrix} < 0, \quad \forall j \in \{1, ..., N\} \tag{18}$$

then $\mathcal{A}$ is robustly $\mathcal{D}$-stable.

The proof is of course achieved in [33] but we must however explain, for clarity, that the previous condition is sufficient for the existence of a parameter-dependent Lyapunov matrix $P(\xi)$ formulated as a convex combination of the various P_j, $j \in \{1, ..., N\}$:

$$P(\xi) = \sum_{j=1}^{N} (\xi_j P_j) \quad \forall \xi \in \Xi \tag{19}$$

It is noticeable that, following the reasonning achieved in the proof of Theorem 5, it appears that $\mathcal{N}_{\mathcal{D}} < 0$ holds over the whole polytope (for some nominal matrix for instance), *i.e.* :

$$\mathcal{N}_{\mathcal{D}}(\mathbf{A}(\xi), P(\xi), F, G) < 0 \quad \forall \xi \in \Xi \tag{20}$$

We insist on the fact that this framework is of high interest. In the most classical conditions relevant to "quadratic stability", the robust $\mathcal{D}$-stability is attested by the existence of a single Lyapunov matrix for the whole polytope while in this work, at each point of the polytope, corresponds a Lyapunov matrix dependent on ξ. This leads to a strong reduction of conservatism. Actually, the flexibility provided by F and G is proved since quadratic $\mathcal{D}$-stability is necessary for condition (18) to hold [36, 43] while the contrary is of course wrong. Numerical examples have highlighted this flexibility in [39, 33, 43]. Besides, other contributions have emphasized the relevence of considering Lyapunov functions of the form (19). The reader is invited to see [35, 34, 40].

3 A new robust $\mathcal{D}$-stability bound

In this section, the main result of the paper is presented. The search for a tight robust $\mathcal{D}$-stability bound $\bar{\delta}^\star$ is formulated as a generalized eigenvalue problem. At last, a first attempt to apply this work to robust control is made.

3.1 The main result

As previously mentionned, we aim to compute $\tilde{\delta}^\star$, *i.e.* the size of the largest Δ in the sense of $\tilde{\delta}$ such that it is sure that $\mathbf{A}(\delta)$ is $\mathcal{D}$-stable, for all the values of $\delta \in \Delta$. Unfortunately, to our knowledge, there is no efficient method to reach the ideal bound $\tilde{\delta}^\star$, that is why we will compute $\overline{\tilde{\delta}}^\star$, which is the size of the largest Δ such that sufficient condition (18) holds (This set can be denoted by $\overline{\Delta}^\star$).

We first assume that A (the nominal matrix) is $\mathcal{D}$-stable. We also assume that $\mathbf{A}(\delta)$ is robustly $\mathcal{D}$-stable for any δ of Δ corresponding to some value of $\tilde{\delta}$. So, the corresponding polytope $\mathcal{A}$ is robustly $\mathcal{D}$-stable and there exist N symmetric positive definite matrices P_j, $j \in \{1, ..., N\}$, as well as a couple of matrices $\{F; G\} \in \{\mathbb{R}^{nd \times nd}\}^2$ such that :

$$\mathcal{N}_{\mathcal{D}}(A_j, P_j, F, G) < 0 \qquad \forall j \in \{1, ..., N\} \tag{21}$$

Taking into account the underlying structure of polytopic uncertainty defined by equations (9)-(12) yields, $\forall j \in \{1, ..., N\}$:

$$\begin{bmatrix} R_{00} \otimes P_j + F(\mathbf{I}_d \otimes (A + \tilde{\delta}\mathcal{E}_j)) + (\mathbf{I}_d \otimes (A^* + \tilde{\delta}\mathcal{E}_j^*))F^* & R_{10} \otimes P_j + (\mathbf{I}_d \otimes (A^* + \tilde{\delta}\mathcal{E}_j^*))G - F \\ R_{10}^* \otimes P_j + G^*(\mathbf{I}_d \otimes (A + \tilde{\delta}\mathcal{E}_j)) - F^* & R_{11} \otimes P_j - G - G^* \end{bmatrix} < 0 \tag{22}$$

We define :

$$\mathcal{P}_j = \frac{P_j - P}{\tilde{\delta}} \Leftrightarrow P_j = \tilde{\delta}\mathcal{P}_j + P \qquad \forall j \in \{1, ..., N\} \tag{23}$$

where P is any Lyapunov matrix proving the stability of the nominal matrix A. Indeed, $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix satisfying :

$$\mathcal{N}_{\mathcal{D}}(A,P,F,G) < 0 \tag{24}$$

It comes, from (22), $\forall j \in \{1,...,N\}$:

$$\mathcal{N}_{\mathcal{D}}(A,P,F,G) + \tilde{\delta}\begin{bmatrix} R_{00}\otimes\mathcal{P}_j + F(\mathbf{I}_d\otimes\mathcal{E}_j) + (\mathbf{I}_d\otimes\mathcal{E}_j^*)F^* & R_{10}\otimes\mathcal{P}_j + (\mathbf{I}_d\otimes\mathcal{E}_j^*)G \\ R_{10}^*\otimes\mathcal{P}_j + G^*(\mathbf{I}_d\otimes\mathcal{E}_j) & R_{11}\otimes\mathcal{P}_j \end{bmatrix} < 0 \tag{25}$$

that can be written :

$$\mathcal{N}_{\mathcal{D}}(A,P,F,G) + \tilde{\delta}\mathcal{L}_{\mathcal{D}}(\mathcal{E}_j,\mathcal{P}_j,F,G) < 0 \quad \forall j \in \{1,...,N\} \tag{26}$$

Let λ be defined as $\lambda = \tilde{\delta}^{-1}$. It is easy to see that finding the maximal $\overline{\delta}^\star$ of $\tilde{\delta}$ such that (22) holds is equivalent to find the minimal λ such that :

$$\lambda^\star = \min_{P,\{\mathcal{P}_j\},F,G} \lambda \quad \text{under the constraints :}$$

$$\left\{\begin{array}{ll} \mathcal{L}_{\mathcal{D}}(\mathcal{E}_j,\mathcal{P}_j,F,G) < -\lambda\mathcal{N}_{\mathcal{D}}(A,P,F,G) & \forall j \in \{1,...,N\} \\ P_j > 0 \Leftrightarrow -\mathcal{P}_j < \lambda P & \forall j \in \{1,...,N\} \\ \mathcal{N}_{\mathcal{D}}(A,P,F,G) < 0 & \\ P > 0 & \end{array}\right. \tag{27}$$

As $\mathcal{N}_{\mathcal{D}}(A,P,F,G) < 0$ and $P > 0$, problem (27) is typically a generalized eigenvalue problem that can be solved, for example, by the function `gevp` of the LMI Toolbox of Matlab.

Remarque 6 *: At this stage, we shall notice that all this work can be extended to non-connected regions $\mathcal{D}$ resulting from the union of possibly disjoint convex subregions. This can be achieved following about the same reasonning as the one developped in [43]. We have not considered this generalization here in order to simplify the notations and the calculation. There is however absolutely no obstacle to achieve such a generalization.*

Remarque 7 *: Assume that :*

$$\underline{\psi}_i = \overline{\psi}_i = 1 \quad \forall i \in \{1,...,p\} \tag{28}$$

In this very usual case, Δ is an hypercube and looking for $\tilde{\delta}^\star$ is looking for the maximal hypercube for which $\mathcal{D}$-stability holds. In fact, $\tilde{\delta}^\star$ is some kind of multivariable $\mathcal{D}$-stability margin generalizing k_m (see [17]). Indeed, if $\mathcal{D}$ is just the open left half complex plane corresponding to asymptotic stability for continuous-time systems, $\tilde{\delta}^\star$ is k_m so $\overline{\delta}^{\star -1}$ is an upper bound of μ [18].

Remarque 8 *: From a theoretical point of view, we are sure that $\overline{\delta}^\star$ is the maximal size of Δ such that (22) (or(18)) holds. But, as $N = 2^p$, the computation becomes harder when p and n increase. Even if the $\mathcal{LMI}$ solvers are quite efficient, heavy computations may sometimes lead to suboptimal values of $\overline{\delta}^\star$ for some examples. It may be required to change some optimization parameters to improve the process result.*

Compared with many other robustness bounds, no expression of $\overline{\delta}^\star$ is given, what may appear as a drawback in some control methods such as the one proposed in [31]. However, for most of those bounds, their expressions are derived from the application of triangular inequality on the 2-norm of matrices (see the references given in introduction). Besides, they are obtained using NSC for nominal matrix $\mathcal{D}$-stability relying on quadratic $\mathcal{D}$-stability. More precisely, considering an uncertainty domain bounded by such an expression is only sufficient for the existence of a single Lyapunov function attesting $\mathcal{D}$-stability over the whole domain. Moreover, the existence of such a single Lyapunov function, *i.e.* quadratic $\mathcal{D}$-stability, is also sufficient for robust $\mathcal{D}$-stability. To reduce this conservatism, some authors use a similarity matrix that can appear in the bound expression [11, 44, 24, 8]. The bound value is maximized with respect to the similarity matrix entries. Nevertheless, even if a significant improvement is brought, the optimization problem is nonlinear and the global optimality is then not guaranteed.

In our approach, bounding the domain by $\overline{\delta}^\star$ is necessary and sufficient for the existence of an affine parameter-dependent Lyapunov function of the form (19) ensuring $\mathcal{D}$-stability over the whole domain $\overline{\Delta}^\star$. Besides, even if the existence of such functions is sufficient for robust $\mathcal{D}$-stability, it is less conservative than quadratic $\mathcal{D}$-stability. Considering these reasons, we claim that our bound may turn to be one of the least conservative in the literature, from a statistical point fo view.

3.2 A step towards robust synthesis

In this paragraph, we are interested in the robust control of the uncertain system described by :

$$\dot{x}(t) \;=\; \mathbf{A}x(t) \,+\, \mathbf{B}u(t) \tag{29}$$

where t denotes time, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control signal vector and where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ are such that a matrix $\mathbf{S}$ can be defined as follows :

$$\mathbf{S} \;= \mathbf{S}(\delta) \;=\; \begin{bmatrix} A+E & B+H \end{bmatrix} \in \mathbb{R}^{n\times(n+m)} \tag{30}$$

with :

$$E \;=\; \sum_{i=1}^{p}(\delta_i E_i) \quad \& \quad H \;=\; \sum_{i=1}^{p}(\delta_i H_i) \tag{31}$$

As in the above reasonnings, the parameter vector $\delta = [\delta_1 \cdots \delta_i \cdots \delta_p]^* \in \mathbb{R}^p$ is assumed to belong to Δ defined by (3). The matrices H_i, $i = 1,...,p$ are known matrices. Also notice that $\mathbf{S}(\mathbf{0}_{p,1}) = S = [A\ B]$ corresponds to the nominal plant.

At this stage, we can follow exactly the same reasonning as in paragraph 2.1. Hence, when δ varies in the convex hull Δ defined in (5), the matrix $\mathbf{S}(\delta)$ describes a polytope of matrices $\mathcal{S}$ defined by :

$$\mathcal{S} \;=\; \{\mathbf{S}(\xi) = \begin{bmatrix} \mathbf{A}(\xi) & \mathbf{B}(\xi) \end{bmatrix} \,|\, \mathbf{S}(\xi) = \sum_{j=1}^{N}(\xi_j S_j); \xi \in \Xi\} \tag{32}$$

The extreme matrices S_j can be expressed as follows :

$$S_j \;=\; \begin{bmatrix} A_j & B_j \end{bmatrix} \;=\; S \,+\, \tilde{\delta}\mathbb{S}_j \;=\; \begin{bmatrix} A & B \end{bmatrix} + \tilde{\delta}\begin{bmatrix} \mathcal{E}_j & \mathcal{H}_j \end{bmatrix} \;\forall j \in \{1,...,N\} \tag{33}$$

where the matrices $\mathcal{E}_j$ are given by (12) and where the matrices $\mathcal{H}_j$ can be defined by :

$$\mathcal{H}_j \;=\; \sum_{i=1}^{p}((-\phi_{j_i}\underline{\psi}_i + (1-\phi_{j_i})\overline{\psi}_i)H_i) \qquad \forall j \in \{1,...,N\} \tag{34}$$

Assume that the control law is $x(t) = Kx(t)$ where $K \in \mathbb{R}^{m\times n}$ is a state feedback matrix. The uncertain closed-loop system behaviour is described by :

$$\dot{x}(t) \;=\; \overline{\mathbf{A}}x(t) \tag{35}$$

where the closed-loop state matrix $\overline{\mathbf{A}}$ is :

$$\overline{\mathbf{A}} \;=\; \overline{\mathbf{A}}(\xi) \;=\; \mathbf{A}(\xi) \;+\; \mathbf{B}(\xi)K \qquad \forall \xi \in \Xi \tag{36}$$

$\overline{\mathbf{A}}$ also belongs to a polytope $\overline{\mathcal{A}}$ defined by :

$$\overline{\mathcal{A}} \;=\; \{\overline{\mathbf{A}}(\xi) = \mathbf{A}(\xi) + \mathbf{B}(\xi)K \,|\, \overline{\mathbf{A}}(\xi) = \sum_{j=1}^{N}(\xi_j\overline{A}_j); \xi \in \Xi\} \tag{37}$$

The vertices $\overline{A}_j$ are of course defined by :

$$\overline{A}_j \;=\; A_j+B_jK \;=\; (A+\tilde{\delta}\mathcal{E}_j)+(B+\tilde{\delta}\mathcal{H}_j)K \;=\; A+BK+\tilde{\delta}(\mathcal{E}_j+\mathcal{H}_j)K \;=\; \overline{A}+\tilde{\delta}\overline{\mathcal{A}}_j \quad \forall j \in \{1,...,N\} \tag{38}$$

Let us now assume that K $\mathcal{D}$-stabilizes the uncertain plant for all $\delta \in \Delta$. It means that the polytope $\overline{\mathcal{A}}$ is robustly $\mathcal{D}$-stable. As already mentionned in the previous paragraphs, it is very hard to find the size $\tilde{\delta}^\star$ of the largest Δ such that $\overline{\mathcal{A}}$ remains $\mathcal{D}$-stable but, following the same reasonning as in paragraph 3.1, we can compute a lower bound $\bar{\tilde{\delta}}^\star$ of $\tilde{\delta}^\star$, as tight as possible to $\tilde{\delta}^\star$. This bound value is the greatest value of $\tilde{\delta}$ ensuring that condition (21) holds but with substituting the various A_j by $\overline{A}_j$, $j = 1, ..., N$, *i.e.* :

$$\exists\{\{P_j = P_j^* > 0\}, F, G\} \quad | \quad \mathcal{N}_{\mathcal{D}}(\overline{A}_j, P_j, F, G) \;<\; 0 \tag{39}$$

Still following the same reasonning, it is equivalent to find the greatest value of $\tilde{\delta}$ satisfying :

$$\mathcal{N}_{\mathcal{D}}(\overline{A}, P, F, G) \;+\; \tilde{\delta}\mathcal{L}_{\mathcal{D}}(\overline{\mathcal{A}}_j, \mathcal{P}_j, F, G) \;<\; 0 \quad \forall j \in \{1, ..., N\} \tag{40}$$

that can also be written :

$$\mathcal{N}_{\mathcal{D}}(S, K, P, F, G) \;+\; \tilde{\delta}\mathcal{L}_{\mathcal{D}}(\mathcal{S}_j, K, \mathcal{P}_j, F, G) \;<\; 0 \quad \forall j \in \{1, ..., N\} \tag{41}$$

For a specified K, this problem is equivalent to a generalized eigenvalue problem (see §3.1). Besides, it is very important to notice that both $\mathcal{N}_{\mathcal{D}}$ and $\mathcal{L}_{\mathcal{D}}$ are linear with respect to P and $\mathcal{P}_j$ ($j = 1, ..., N$) and bilinear with respect to F, G and K. Hence, K can not be an $\mathcal{LMI}$ variable when solving the optimization problem. However, with fixing F and G, both $\mathcal{N}_{\mathcal{D}}$ and $\mathcal{L}_{\mathcal{D}}$ become linear with respect to K and $\tilde{\delta}$ can be maximized by solving another generalized eigenvalue problem where K is a decision variable (instead of F and G).

Those remarks enable us to express a relaxation scheme in order to make a static state feedback control law more robust in the sense of $\tilde{\delta}$. However, it is well known that there is no guarantee of optimality nor convergency for relaxation schemes and we have encountered numerical problems applying those techniques to some examples. More precisely, the previous scheme, which seems to be the most natural one, is not the most tractable from a computational point of view. Among many other possible relaxations schemes, we here introduce the one that has often enabled us to go round some local minima and which seems to be the most efficient :

Algorithm 1

1. *Compute a nominal initial state feedback matrix* $K_{\{0\}}$ *such that* $\overline{A}_{\{0\}} = A + BK_{\{0\}}$ *is* $\mathcal{D}$*-stable (any pole assignment technique can be used).*
2. *Choose natural parameters* q *and* $\mathbf{q}$ *as well as a real accuracy* ϵ*.*

3. $k \leftarrow 0$
4. $l \leftarrow 0$
5. *Sarting from the current set* $\{P, \{\mathcal{P}_j\}, F, G, K\}_{\{k\}}$ *(except for* $k = 0$ *where only* K *is initialized), solve the following generalized eigenvalue problem :*

$$\lambda^\star = \min_{P,\{\mathcal{P}_j\},F,G} \lambda \quad \text{under the constraints :}$$

$$\begin{cases} \mathcal{L}_\mathcal{D} < -\lambda \mathcal{N}_\mathcal{D} & \forall j \in \{1, ..., N\} \\ -\mathcal{P}_j < \lambda P & \forall j \in \{1, ..., N\} \\ \mathcal{N}_\mathcal{D} < 0 \\ P > 0 \end{cases} \tag{42}$$

6. *Starting from solution got in step 5 or step 11, solve the following generalized eigenvalue problem :*

$$\lambda^\star = \min_{K} \lambda \quad \text{under the constraints (42)} \tag{43}$$

and get $\lambda^\star_{\{k\}}$ *and* $K_{\{k\}}$.

7. *From the solution got in step 6, solve :*

$$\lambda^\star = \min_{P,\{\mathcal{P}_j\}} \lambda \quad \text{under the constraints (42)} \tag{44}$$

8. *From the solution got in step 7, solve (43).*
9. *From the solution got in step 8, solve :*

$$\lambda^\star = \min_{F} \lambda \quad \text{under the constraints (42)} \tag{45}$$

10. *From the solution got in step 9, solve (43) and get* $K_{\{k+1\}}$.
11. *From the solution got in step 10, solve :*

$$\lambda^\star = \min_{G} \lambda \quad \text{under the constraints (42)} \tag{46}$$

and get $\lambda^\star_{\{k+1\}}$.

12. $k \leftarrow k + 1$; $l \leftarrow l + 1$.
13. *if* $(\lambda^\star_{\{k+1\}} - \lambda^\star_{\{k\}})/\lambda^\star_{\{k\}}$ *is less than* ϵ *or if* $\mathbf{q}$, *go to step 15, else go to step 14.*
14. *If* $l = q$, *come back to step 4, else come back to step 6.*
15. *Set* $\overline{\delta}^\star = \lambda^{\star^{-1}}_{\{k\}}$ *and* $K^\star = K_{\{k\}}$.

This relaxation scheme has led to interesting results despite two main problems :

- It often leads to a local minimum.
- Although each optimization step starts from a feasible solution given by the previous step in theory, the process might sometimes not be able to accept this initial argument because it is just near the frontier of the feasibility domain. Nevertheless, the process can go on and in practice, this numerical phenomenum tends to make the process a bit slower.

Remarque 9 *: coming back to step 4 every* q *iterations helps the process avoiding local minima. Of course, the improvement of the bound value might depend on the choice of* q.

We insist on the fact that this heuristic algorithm cannot be considered as a robust **synthesis** technique since the uncertainty domain for which $\mathcal{D}$-stability must be ensured is not known *a priori* but it can be considered as a robust **control** technique anyway. Indeed, this is a way to make a control law a bit more robust in the sense of $\mathcal{D}$-stability.

Remarque 10 *: in the paper, we only present the static state feedback case but this technique can easily be extended to static output feedback provided that the uncertainty only affects one factor in the term BKC (either B or C (or even K if fragility is concerned)) in order to preserve the linearity with respect to $\tilde{\delta}$.*

4 Numerical illustrations

In this section, we illustrate the previous analytical results by some simple numerical extensions. All the computation is achieved on a PC Pentium 166MHz with MATLAB and its LMITOOLBOX [45].

4.1 First example

This example is borrowed from [12] :

$$\mathbf{A} = \mathbf{A}(\delta) = A + \sum_{i=1}^{2}(\delta_i E_i) \tag{47}$$

with :

$$A = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} ; E_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} ; E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{48}$$

We first consider the classical problem of computing a robust stability bound with :

$$\underline{\Psi} = \overline{\Psi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{49}$$

The authors of [12] found $\bar{\delta}^\star = 1.5533$. In [7], the proposed condition leads to the computation of a lower bound $\bar{\delta}^\star = 1.7499$. This is in fact the necessary and sufficient condition but the author of [7] has a priori knowledge of this value and he tests it instead of computing it. For this example, the most tractable technique is the one proposed in [15] that gives the good value with a little computation time. However, let us notice that the method of [15] is only available for stability bounds while our method is available for other regions and for this example, it leads to the same value in only `18.3s`.

• But for this example, we want to insist on the interest of using the vectors $\underline{\Psi}$ and $\overline{\Psi}$. Assume that the designer expects a greater variation of δ_2, compared to the variation of δ_1. It is more appropriate to consider that δ varies in a hyperrectangle rather than in an hypercube. For instance, assume now that :

$$\underline{\Psi}_2 = \overline{\Psi}_2 = 2 \tag{50}$$

For this new instance, we get $\bar{\delta}^\star = 1.5$. It is of course less than 1.7499 but let us now compare the size of the corresponding square and rectangle in the parameter space. This comparison is graphically expressed on figure 1.

• A step further can be made owing to those specified vectors $\underline{\Psi}$ and $\overline{\Psi}$. Not only they easily enable to specify symmetric hyperrectangles (what is often less easy in other techniques) but they also allow to consider nonsymmetric hyperrectangles (what is very hard, even impossible in other techniques dealing with affine parametric uncertainty). For instance, assume that the designer rather thinks that δ_1 and δ_2 are more likely to be negative. He can specify, for instance :

$$\underline{\Psi} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad ; \quad \overline{\Psi} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{51}$$

The computed value of $\overline{\delta}^\star$ is now 1.4052.

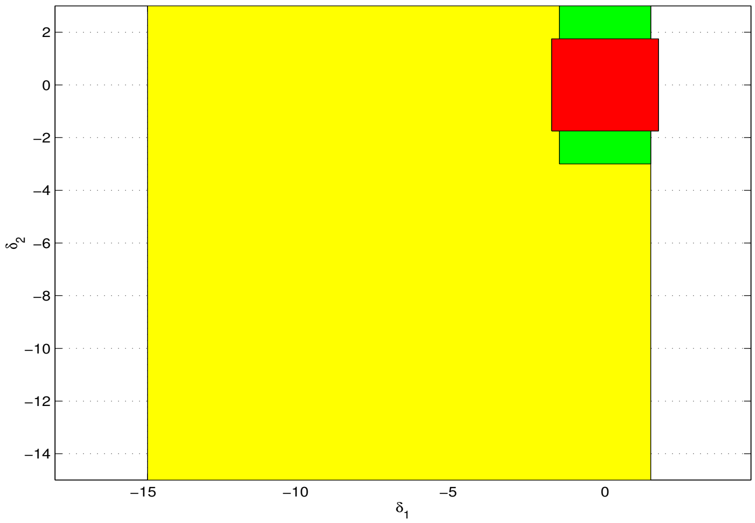

FIG. 1 – Centred cube, centred rectangle and nonsymmetric rectangle in the parameter space

This time again, the comparison of the size of the various domains is significant on figure 1 where it is easy to see that $\underline{\Psi}$ and $\overline{\Psi}$ can be very efficiently specified for robust analysis.

4.2 Second example

This example is borrowed from [15].

Consider the uncertain matrix $\mathbf{A} = A + E$ where :

$$A = \begin{bmatrix} -1.50 & -12.06 & -0.06 & 0 \\ -0.25 & 0.03 & 1.00 & 0.50 \\ 0.25 & -4.00 & -1.03 & 0 \\ 0 & 0.50 & 0 & -4.00 \end{bmatrix} \tag{52}$$

and where :

$$E = \sum_{i=1}^{2}(\delta_i E_i) \tag{53}$$

with :

$$E_1 = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

We assume that δ belongs to an hypercube i.e. :

$$\underline{\Psi} = \overline{\Psi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{55}$$

Thanks to the so-called skew-symmetric method (bialternate sum), the robust stability bound proposed in [15] is 0.439. This method is improved in [16] where the bound value increases to 0.956. Solving our generalized eigenvalue problem leads to $\bar{\delta}^\star = 1.8439$ which is far less conservative (`Computation time : 7min59s`)

4.3 Third example

This last example consists in improving the robustness of a control law. It is extracted from [31, page 128]. The open-loop system is described by :

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \mathbf{B}u(t) = (A_0 + \sum_{i=1}^{2}(\delta_i E_i))x(t) + (B + \delta_3 H_3)u(t) \tag{56}$$

where :

$$A_0 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3229 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix} ; B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0477 & -7.5922 \\ -5.5200 & 4.4900 \\ 0.0000 & 0.0000 \end{bmatrix} \tag{57}$$

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.1603 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5820 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

$$H_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^* \tag{59}$$

In [31, page132], the following static state feedback control law is computed :

$$K_{\{0\}} = \begin{bmatrix} -89.7223 & -8.8292 & -8.3635 & 22.3756 \\ -31.1773 & -2.5363 & -3.3678 & 4.8087 \end{bmatrix} \tag{60}$$

This control law assigns the following spectrum to the closed-loop state matrix :

$$\{-6 \pm 3\mathbf{i}\,;\, -7\,;\, -4.5\} \tag{61}$$

As a region $\mathcal{D}$, we choose the disk of center $(-6;0)$ and of radius 5.5. It can be checked that the eigenvalues given in (61) lie in this disk.

With such a closed-loop model, our analytical method yields :

$$\bar{\delta}^\star{}_{\{0\}} = 0.1574 \tag{62}$$

If we apply algorithm 1 starting from $K_{\{0\}}$ (`computation time : 2h38min27s`), we get :

$$K^\star = \begin{bmatrix} -47.3573 & -7.1178 & 6.6308 & 19.0024 \\ -69.7598 & -6.8339 & -7.7187 & 13.2792 \end{bmatrix} \tag{63}$$

which assigns the following spectrum to the closed-loop state matrix :

$$\{-4.9414 \pm 3.2573\mathbf{i}\,;\, -2.4393 \pm 0.7431\mathbf{i}\} \tag{64}$$

This control law leads to a better value of the robut $\mathcal{D}$-stability bound :

$$\overline{\delta}^{\star} = 0.7301 \tag{65}$$

The bound value is really increased even if the solution is only a local minimum (a better result was obtained in [31] through a completely different manner). The improvement in robust $\mathcal{D}$-stability is highlighted in figure 2 where the pertubred spectra of both matrices $\bar{A}_{\{0\}} = A + BK_{\{0\}}$ and $\bar{A}^{\star} = A + BK^{\star}$ are drawn while those matrices are subject to the same uncertainty affecting A and B and such that $\tilde{\delta} \leq \overline{\delta}^{\star}$.
Not only it is clear that the robustness of the control law has been stronlgy improved but the one can have an idea of the weak conservatism induced by our analytical method.

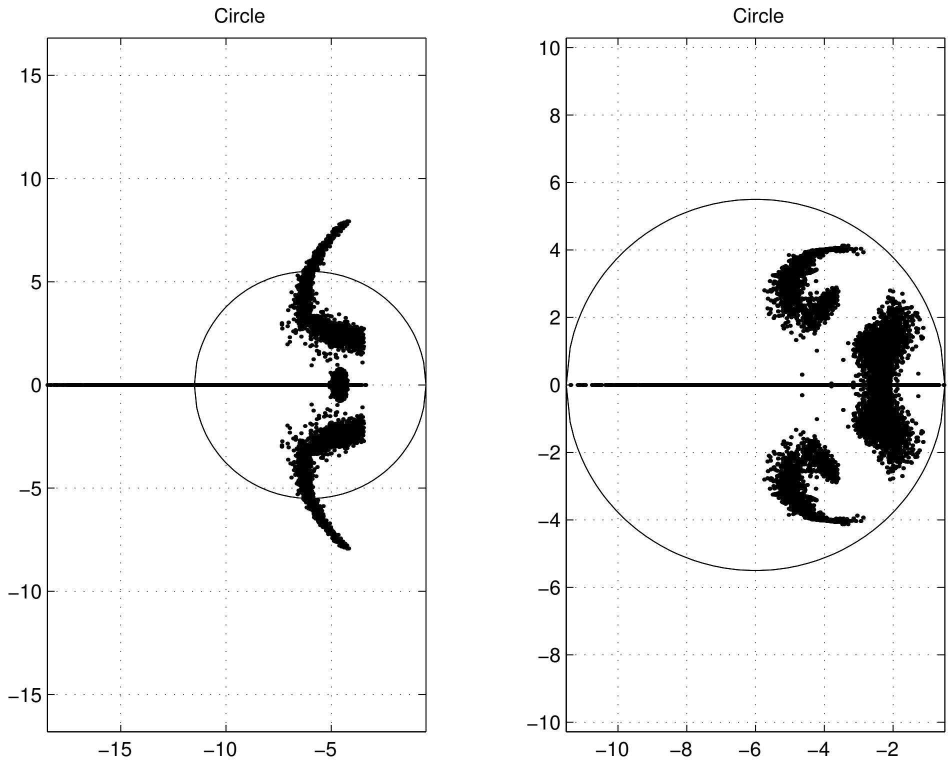

FIG. 2 – Perturbed spectra of $\bar{A}_{\{0\}}$ and $\bar{A}^{\star}$ in $\mathcal{D}$ when $\tilde{\delta} \leq 0.7301$

Assume now that the spectrum is assigned by an output feedback control law and that the output matrix is :

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{66}$$

Following the pole assignment technique proposed in [46], a feedback matrix is computed :

$$K'_{\{0\}} = \begin{bmatrix} -185.6881 & -56.2467 & -38.1567 \\ -109.5754 & -32.8515 & -22.1296 \end{bmatrix} \tag{67}$$

It is determined so that the closed-loop spectrum is near the center of $\mathcal{D}$. This spectrum is :

$$\{-7\,;\,-6.5\,;\,-6\,;\,-5.5\} \tag{68}$$

With such a closed-loop model, our analytical method yields :

$$\overline{\delta}'^{\star}{}_0 \;=\; 0.0096 \tag{69}$$

Applying algorithm 1 leads to a new output feedback matrix :

$$K'^{\star} \;=\; \begin{bmatrix} -130.4885 & -37.6007 & -20.0956 \\ -78.4682 & -22.3890 & -12.1831 \end{bmatrix} \tag{70}$$

which assigns the following spectrum to the closed-loop state matrix :

$$\{-6.9523 \pm 3.1227\mathbf{i}\,;\; -1.9353 \pm 3.2050\mathbf{i}\} \tag{71}$$

and leads to a less conservative bound value :

$$\overline{\delta}'^{\star} \;=\; 0.0590 \tag{72}$$

This is a local minimum since a better result is obtained in [31] with a different method. However, the bound value is strongly improved as shown on figure 3 where the pertubred spectra of both matrices $\bar{A}'_{\{0\}} = A + BK'_{\{0\}}C$ and $\bar{A}'^{\star} = A + BK'^{\star}C$ are drawn while those matrices are subject to the same uncertainty affecting A and B and such that $\tilde{\delta} \leq \overline{\delta}^{\star}$.
Once again, it can be seen that our analytical method does not leads to a pessimistic bound value.

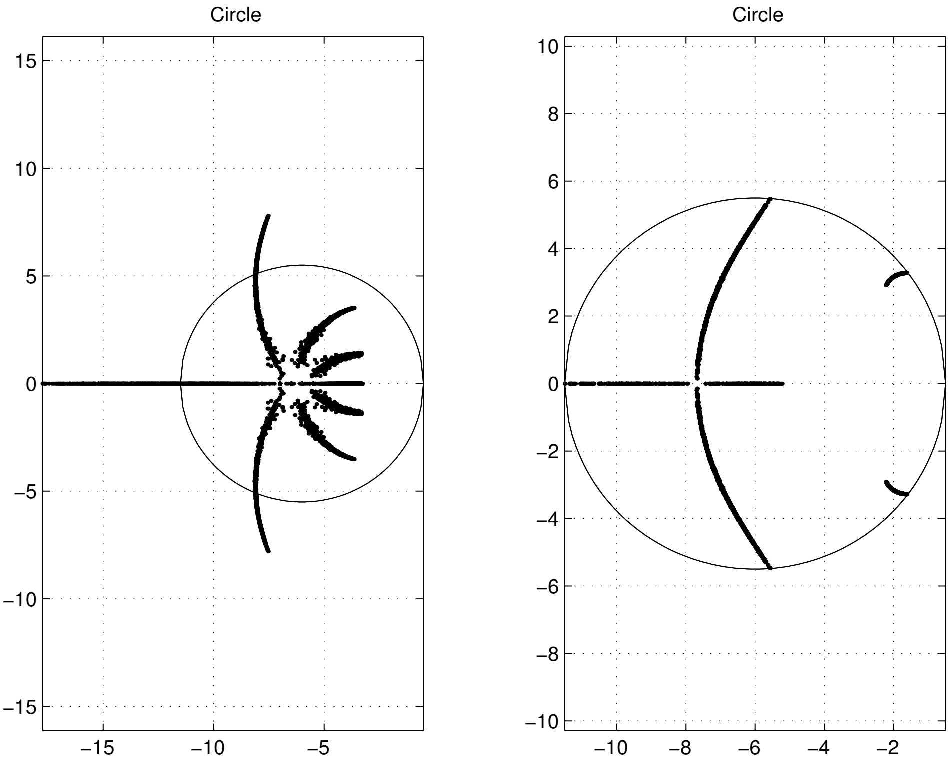

FIG. 3 – Perturbed spectra of $\bar{A}'_{\{0\}}$ and $\bar{A}'^{\star}$ in $\mathcal{D}$ when $\tilde{\delta} \leq 0.0590$

5 Conclusion

In this paper, we have proposed a new method to compute robust $\mathcal{D}$-stability bounds in the case of structured parametric uncertainties. No expression of the bound is given. Its value is just the solution of a generalized eigenvalue problem that can be solved with the various LMI tools. Although this method may not be tractable for a high number of uncertain parameters, numerical illustrations highlighted that it leads to less conservative results than lots of other techniques. This improvement is due to the implicit use of parameter-dependent Lyapunov functions whereas previous bounds were often obtained via a quadratic approach where only a single Lyapunov function attests the $\mathcal{D}$-stability over the whole uncertainty domain.

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