ADVANCED FEEDBACK CONTROL (MODELS, CONTROL, ROBUSTNESS)

Olivier Bachelier

LIAS-ENSIP

Advanced Control course, MEE3

Outline of the course

- About the various usual structures of feedback control
- About the difficulty to apprehend coupling and the handle MIMO models as SISO ones
- Eigenstructure assignement (strict pole placement)
- Model reduction
- Introduction to LMI-based synthesis

Various structures of feedback

Various structures of feedback

We will study various structures of feedback laws. Those structures depend on whether the feedback is applied

- from the all state vector (if it can be measured),
- or from the ouput vector.

Besides, the feedback itself can be either

- static (the control vector entries are linear combinations of the measurements),
- or dynamic (the control vector becomes the ouput of dynamic system - the controller - from which the measurement vector is the input).

Therefore, three kinds of structures will be considered

- static state feedback (usually simply referred to as state feedback),
- static output feedback,
- dynamic output feedback.

Indeed, dynamic state feedback is rarely used.

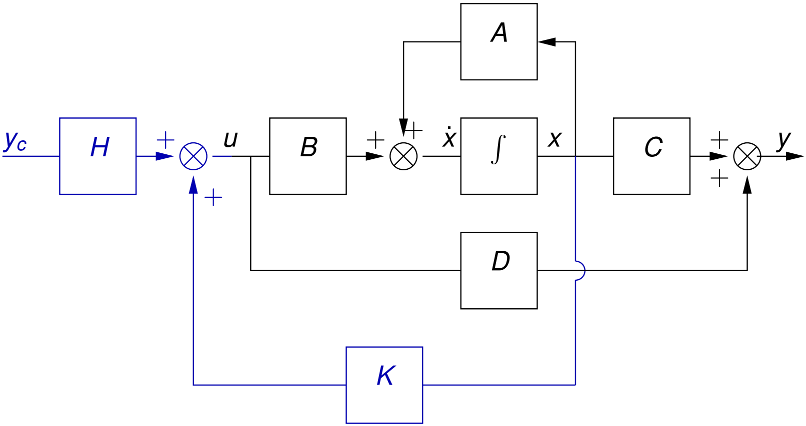
The system model to be considered is simply a realization

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \right. \tag{1}$$

($x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$)

Static state feedback

This is a control law that might have been studied in the case of SISO sytstems



The corresponding mathematical description is

$$u(t) = Hy_c(t) + Kx(t). \tag{2}$$

with

- $K \in \mathbb{R}^{m\times n}$: state feedback matrix ;
- $H \in \mathbb{R}^{m\times p}$: feedforward matrix ;
- $y_c \in \mathbb{R}^{p}$: reference vector.

Static state feedback

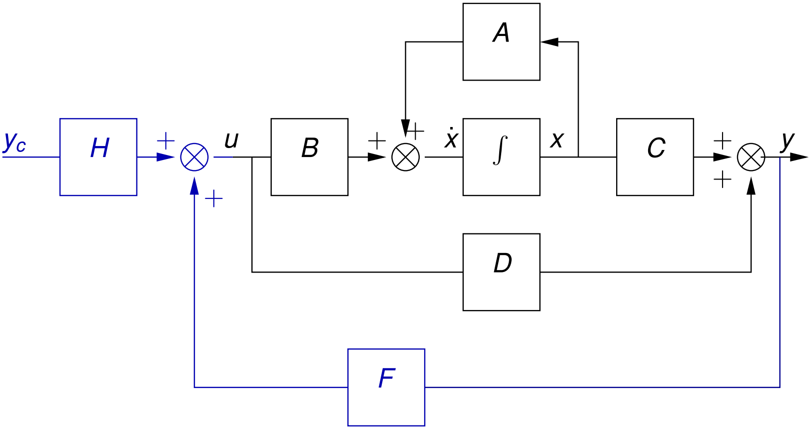
The induced feedback model is given by

$$\left\{\begin{array}{rcrcl} \dot{x} & = & (A+BK)x & + & BHy_c \\ y & = & (C+DK)x & + & DHy_c. \end{array}\right. \tag{3}$$

- K is computed to ensure stability and either to possibly reach transient performances (pole placement) or to minimize some criterion (*e.g.* LQ control, optimal control).
- H, if used, is rather computed to reach static performances.

Static output feedback

Assume that not all the entries of x are measured but only the entries of y.



The corresponding mathematical description is

$$u(t) = Hy_c(t) + Fy(t). \quad (4)$$

with

- $F \in \mathbb{R}^{m\times p}$: output feedback matrix (or gain) ;
- $H \in \mathbb{R}^{m\times p}$: feedforward matrix ;
- $y_c \in \mathbb{R}^{p}$: reference vector.

If $D = \mathbb{O}$ (no direct transmission to make simpler) then the closed-loop model is

$$\left\{ \begin{array}{rcl} \dot{x} & = & (A + BFC)x \quad + \quad BHy_c \\ y & = & Cx \end{array} \right. \tag{5}$$

If $D \neq \mathbb{O}$ then the control vector u complies with

$$u = Hy_c + FCx + FDu$$
$$\Leftrightarrow \ (\mathbb{I}_m - FD)u = Hy_c + FCx$$

$$\begin{array}{rcccccc} \Leftrightarrow & u & = & \underbrace{(\mathbb{I}_m - FD)^{-1}H}_{\hat{H}} & y_c & + & \underbrace{(\mathbb{I}_m - FD)^{-1}F}_{\hat{F}} & Cx \\ \Leftrightarrow & u & = & \hat{H} & y_c & + & \hat{F} & Cx. \end{array} \tag{6}$$

This leads to the following closed-loop model :

$$\left\{ \begin{array}{rcccc} \dot{x} & = & (A + B\hat{F}C)x & + & B\hat{H}y_c \\ y & = & (C + D\hat{F}C)x & + & D\hat{H}y_c. \end{array} \right. \tag{7}$$

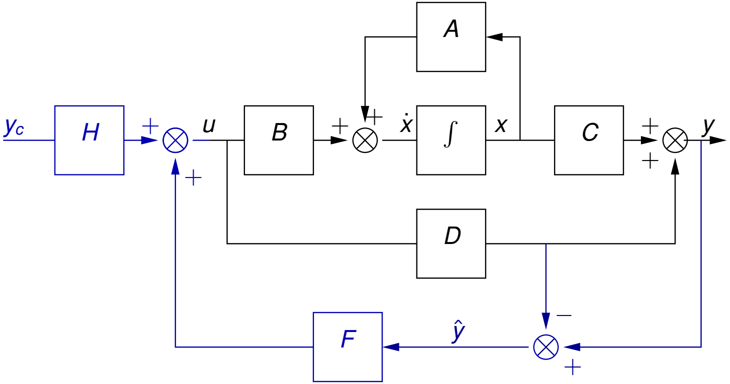
One can compute $\hat{F}$ and $\hat{H}$ for design purpose and deduce F and H which are implemented in practice.

Static output feedback

Another possibility is to modify the control law :

$$u = Hy_c + F(y - Du) = Hy_c + F\hat{y} \tag{8}$$

$\hat{y} = y - Du = Cx \in \mathbb{R}^p$ is the new "measure" one has to built (it's part of the controller) so that one gets

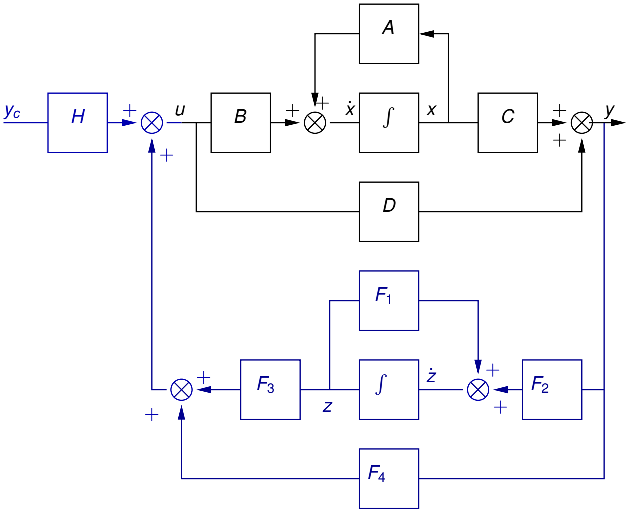


Then, the closed-loop model directly depends on F and H :

$$\left\{\begin{array}{lcccc} \dot{x} & = & (A+BFC)x & + & BHy_c \\ y & = & (C+DFC)x & + & DHy_c \end{array}\right. \tag{9}$$

F and H are computed to get satisfactory performances. Note however that, with this kind of structure, it might be preferable to measure u too.

Dynamic ouput feedack



Dynamic ouput feedack

The control law is given by

$$\begin{cases} \dot{z} = F_1 z + F_2 y \\ u = F_3 z + F_4 y + H y_c, \end{cases} \tag{10}$$

where $z \in \mathbb{R}^l$ is the state vector of the feedback system.

The transfer matrix of this controller is

$$G_F(s) = F_3(s\mathbb{I}_l - F_1)^{-1} F_2 + F_4. \tag{11}$$

Dynamic ouput feedack

Linking controller and process realizations yields

$$(\mathbb{I}_m - F_4 D)u = F_3 z + F_4 C x + H y_c$$

$$u = \underbrace{(\mathbb{I}_m - F_4 D)^{-1} F_4}_{u = \hat{F}_4} Cx + \underbrace{(\mathbb{I}_m - F_4 D)^{-1} F_3}_{\hat{F}_3} z + \underbrace{(\mathbb{I}_m - F_4 D)^{-1} H}_{\hat{H}} y_c,$$

Consider a concatenation of process and controller state vectors $\xi' = [x' \;\; z']'$ to get :

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & \begin{bmatrix} A + B\hat{F}_4 C & B\hat{F}_3 \\ F_2 C + F_2 D \hat{F}_4 C & F_1 + F_2 D \hat{F}_3 \end{bmatrix} \xi + \begin{bmatrix} B\hat{H} \\ F_2 D \hat{H} \end{bmatrix} y_c \\ y & = & \begin{bmatrix} C + D\hat{F}_4 C & D\hat{F}_3 \end{bmatrix} \xi + D\hat{H} y_c. \end{array} \right. \quad (12)$$

Dynamic ouput feedack

Let the next augmented dynamic model be defined :

$$\left\{\begin{array}{ccccc} \dot{\xi} & = & \tilde{A}\xi & + & \tilde{B}\tilde{u} \\ \tilde{y} & = & \tilde{C}\xi & + & \tilde{D}\tilde{u}, \end{array}\right. \tag{13}$$

where $\xi \in \mathbb{R}^{n+l}$ and

$$\tilde{A} = \left[\begin{array}{cc} A & \mathbb{O} \\ \mathbb{O} & \mathbb{O}_l \end{array}\right]; \tilde{B} = \left[\begin{array}{cc} B & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_l \end{array}\right];$$

$$\tilde{C} = \left[\begin{array}{cc} C & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_l \end{array}\right]; \tilde{D} = \left[\begin{array}{cc} D & \mathbb{O} \\ \mathbb{O} & \mathbb{O}_l \end{array}\right].$$

Dynamic ouput feedback

Also let some control law (static ouput feedback) be applied on this model :

$$\tilde{u} = \tilde{F}\tilde{y} + \tilde{H}y_c, \quad \text{with} \tag{14}$$

$$\tilde{F} = \begin{bmatrix} F_4 & F_3 \\ F_2 & F_1 \end{bmatrix} \quad \text{an} \quad \tilde{H} = \begin{bmatrix} H \\ \mathbb{O}_{l,m} \end{bmatrix}, \tag{15}$$

After some few calculation, one gets the same closed-loop model as the one obtained by applying the dynamic feedback on the original process model.

$\Rightarrow$ *Applying a dynamic output feedback controller on a linear model is equivalent to applying a static feedback gain on an augmented system.*

Dynamic ouput feedack

When $D = \mathbb{O}$ the closed-loop model reduces to :

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & \left[\begin{array}{cc} A + BF_4C & BF_3 \\ F_2C & F_1 \end{array} \right] \xi \;+\; \left[\begin{array}{c} BH \\ \mathbb{O} \end{array} \right] y_c \\ \\ y & = & \left[\begin{array}{cc} C & \mathbb{O} \end{array} \right] \xi. \end{array} \right. \quad (16)$$

Note that the feedback matrices can be computed in another basis of the state space *i.e.*, with a full rank T,

$$\check{F} = \left[\begin{array}{cc} F_4 & F_3T^{-1} \\ TF_2 & TF_1T^{-1} \end{array} \right] = \left[\begin{array}{cc} \mathbb{I}_p & \mathbb{O} \\ \mathbb{O} & T \end{array} \right] \tilde{F} \left[\begin{array}{cc} \mathbb{I}_m & \mathbb{O} \\ \mathbb{O} & T^{-1} \end{array} \right], \quad (17)$$

since F and $\check{F}$ correspond to the same transfer matrix.

Dynamic or static ?

Just a little question : Assume one simply wants to stabilize a realization (A, B, C, D). What is usually the easiest way,

- dynamic ouput feedback,
- or static ouput feedback (on the original model) ?

Answer : Dynamic output feedback because one can exploit a greater number of degrees of freedom since there are more entries in $\tilde{F} \in \mathbb{R}^{(m+l)\times(p+l)}$ than in $F \in \mathbb{R}^{m\times p}$.

Actually, the problem of stabilization by static output feedback control is still an open problem !

Reference

The developments in this part are actually very easy to produce whith quite simple calculation and matrix manipulations. Only one reference might deserve to be cited, where the dynamic controller is formulated as a static one applied on an augmented system :

P. Hippe and J. O'Reilly.
Parametric compensator design.
International Journal of Control, Vol 45(4), p. 1455-1468, 1987.

About coupling

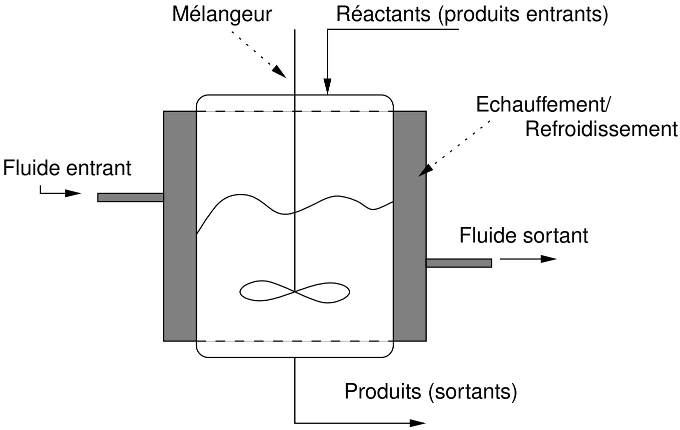
Coupling between channels

The purpose in this part is to highlight the inherent difficulty of controlling MIMO models due to coupling between the various *inputs to outputs channels*.

For example, consider a process with two ouputs y_1 and y_2 and two control inputs u_1 and u_2.
It is interesting to control y_1 that should track some reference y_{c_1} as well as to control y_2 that should track some reference y_{c_2}.

Unfortunately, in most cases, those control laws cannot be designed independently. An action on y_{c_1}, and thus on u_1 has an influence on y_2 and the other way around.

An example : chemical reactor



An example : chemical reactor

The temperature of the reactor is highly influent on the quality of the reaction which is itself influent on the temperature of the environment.

The process includes two inputs :

- the rate (concentration) of entering chemicals,
- the temperature of the heating/cooling fluid,

and two ouputs :

- the rate (concentration) of outgoing chemicals,
- the temperature inside the reactor.

An example : chemical reactor

So two input/ouput channels :

- one for the chemical rates ;
- the other one for the temperature.

Why is there a couplig between the two channels ?

- If the temperature of the outside fluid changes (in order to control that of the reactor), then the quality of the reaction is modified and the rates of the products are changed.
- If the rates of the entering chemicals are changed (in order to control the rates of products), then the reaction is of course more or less important inducing a change of temperature because the reaction either provides or absorbs heat.

Another (numerical) example

$$A = \begin{bmatrix} -1 & \mathbf{0} \\ \mathbf{0} & 0,5 \end{bmatrix};\ B = \begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix};\ D = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}.$$

It is an unstable square system (see the poles). The *emphasized* entries are those responsible for the coupling. The corresponding transfer matrix is $G(s) =$

$$\frac{1}{s^2 + 0,5s - 0.5} \begin{bmatrix} s - 0,5 & \mathbf{s - 0,5} \\ \mathbf{s + 1} & s + 1 \end{bmatrix} = \begin{bmatrix} G_{11}(s) & \mathbf{G_{12}(s)} \\ \mathbf{G_{21}(s)} & G_{22}(s) \end{bmatrix}.$$

Another (numerical) example

Assume that one *ignores (!!!)* the coupling transfers $G_{12}(s)$ and $G_{22}(s)$ and that one designs some controllers only for diagonal transfers.

From u_1 to y_1 :

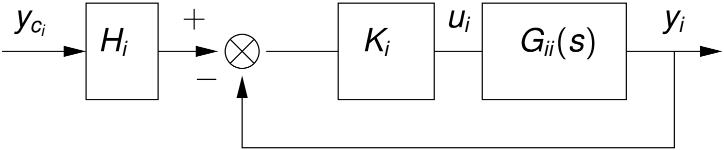
$$G_{11}(s) = \frac{s-0,5}{s^2+0,5s-0,5} = \frac{1}{s+1},$$

From u_2 to y_2 :

$$G_{22}(s) = \frac{s+1}{s^2+0,5s-0,5} = \frac{1}{s-0,5}.$$

Another (numerical) example

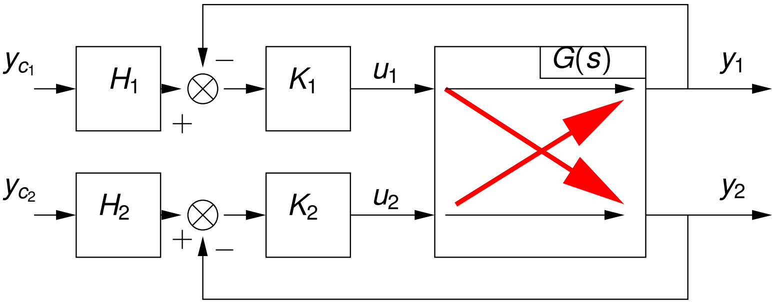
For each first order channel, ones applies



With $H_1 = -1$, $K_1 = -0.5$, $H_2 = 0.5$ and $K_2 = 1$, one gets the two following closed-loop models :

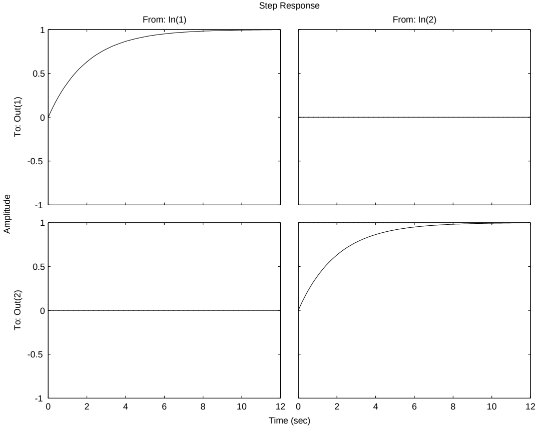
$$\bar{G}_{11} = \bar{G}_{22} = \frac{1}{1+2s}.$$

The global control structure is then as follows :

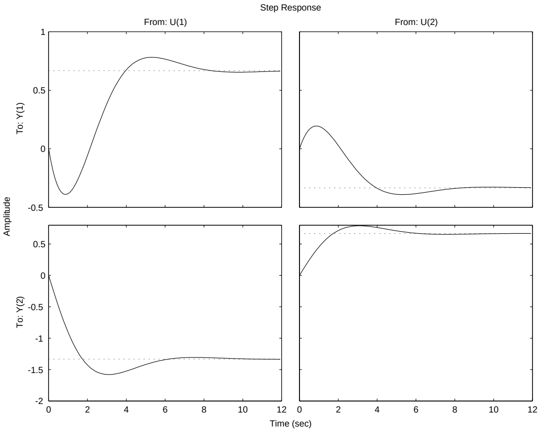


The arrows represent the ignored transfers.

With such a simple (and false) reasoning, one should get the next step response :



...whereas one actually gets



Hence...

... from these examples, one can conclude that :

- The coupling cannot always be neglected ;
- The responses can be drastically distorted.

Indeed, some models can even be unstable due to couplings...

Thus...

One can formulate several problems :

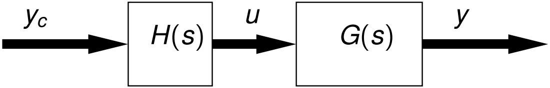
- Static decoupling (only for steady-state response),
- Tansient decoupling (also for the transient response).

Those problems can be handled

- either from a frequency point of view (frequency decoupling),
- or from a time point of view (state-space approach).

Freq. app./static decoupling

Some possibility is to use feedforward control :



$$Y(s) = G(s)U(s) = G(s)H(s)Y_c(s)$$

$$\Rightarrow y_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0}(sY(s)) = \lim_{s\to 0}(sG(s)H(s)Y_c(s)).$$

If ones considers that all the reference entries y_{c_i} are steps of magnitude α_i, one has to satisfy :

$$
\begin{aligned}
y_\infty = \lim_{s\to 0}\left(sG(s)H(s)\frac{1}{s}\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix}\right) &= \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} \\
\Leftrightarrow\ G(0)H(0)\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} &= \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_p\end{bmatrix} \\
\Leftrightarrow\ G(0)H(0) &= \mathbb{I}_p. \qquad (18)
\end{aligned}
$$

So $H(s) = H(0) = H$ (constant feedforward matrix) has to check (18).

Freq. app./static decoupling

- If $m = p$ (square model) then $H = G(0)^{-1}$;
- If $m > p$ then H can be a pseudo-inverse of $G(0)$ (for example, the Moore-Penrose one) ;
- If $m < p$ then no generic solution : not enough actuators compared with the number of outputs.

So the limits are :

- $m \geq p$;
- $G(0)$ must be of full rank ;
- The process must be stable or be stabilized first because this is only a feedforward control.

Example

$$G(s) = \begin{bmatrix} \dfrac{20(s+1)}{(s^2+3s+12)(s+2)} & \dfrac{-130(s-0,3)}{s^2+2s+80} & \dfrac{-10(s-3)}{(s^2+3s+12)(s+8)} \\ \dfrac{15(s-1)}{(s^2+4s+12)(s+2)} & \dfrac{43(s+1}{(s^2+2s+32)(s+2)} & \dfrac{30(s+1)}{s^2+2s+122} \\ \dfrac{-9(s-4)}{s^2+2s+52} & \dfrac{30(0,5s+4)}{s^2+2s+412} & \dfrac{3,2}{s+2} \end{bmatrix}.$$

This is a square stable model $\Rightarrow$

$$H = G(0)^{-1} = \begin{bmatrix} 0,833 & -0,572 & -0,075 \\ 0,972 & 0,927 & -0,332 \\ -0,537 & 0,079 & 0,718 \end{bmatrix}.$$

Just some idea that can sometimes be used !

The idea is to compute $H(s)$ such that $Q(s) = G(s)H(s)$ checks

$$\begin{cases} q_{ii}(s) \neq 0 & \forall i \in \{1, ..., p\} \\ q_{ij}(s) = 0 & \forall \{i, j \neq i\} \in \{1, ..., p\}^2 \end{cases}$$

But it is illusory to solve such constraints so one can simply try to reach

$$|q_{ij}(\mathbf{i}\omega)| << 1 \quad \forall \{i, j \neq i\} \in \{1, ..., p\}^2$$

There are several techniques in the literature based on that simple idea (whose efficiency has still to be proved (author's note)). With those techniques, one has to check that the useful transfers $q_{ii}(s)$

- have no instable zeros ;
- are strictly proper ;
- should be preferably of weak order.

In any case, one has to keep in mind that a decoupling procedure does not ensure other performances and should be accompanied by other control laws to guarantee stability, transient behaviour, and so on.

Decoupling with time approach

Also very difficult but let us have a look to this very particular case where $m = p = n$ (yes, it can exist ! *e.g.* some printers).

Assume one wants to satisfy :

$$\dot{y} = Q(y - y_c) \text{ with } Q \text{ diagonal}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_p \end{bmatrix} = \begin{bmatrix} q_{11} & 0 & \dots & 0 \\ 0 & q_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_{pp} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} - \begin{bmatrix} q_{11} & 0 & \dots & 0 \\ 0 & q_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_{pp} \end{bmatrix}$$

Decoupling with time approach

These p independent linear 1st order differential equations would correspond, in Laplace's domain, to :

$$\frac{Y_i(s)}{Y_{c_i}(s)} = \frac{-q_{ii}}{s - q_{ii}} \quad \forall i \in \{1, ..., p\}.$$

that is to some transfers with unit static gain and one pole q_{ii}.

Decoupling with time approach

If one looks for a state feedback control law such that these transfers are obtained, ones can write

$$\dot{x} = Ax + Bu = Ax + B(Hy_c + Kx) = (A + BK)x + BHy_c$$

$$\Leftrightarrow \; C\dot{x} = \dot{y} = (CA + CBK)x + CBHy_c.$$

to be identified to

$$\dot{y} = QCx - Qy_c,$$

leading to (assuming that $W = (CB)^{-1}$ exists)

$$\left\{ \begin{array}{l} K = W(QC - CA) \\ H = -WQ. \end{array} \right. \tag{19}$$

Decoupling with time approach

Thus a very simple technique that is unfortunately only useful when $m = p = n$.

Indeed,

- it cannot be extended to static output feedback ;
- It cannot be used when $D \neq \mathbb{O}$.

... so very restrictive !

Decoupling : Some conclusion

- Frequential approach and feedforward sometimes efficient (not alone) for static decoupling.
- Time approach rarely used (except under drastic constraints) but see the next part for some attempt to transient decoupling.
- There exist other methods of decoupling such as the "relative gain" method whose efficiency has not convinced the author of these frames.
- Other techniques consists in tracking a reference model which is usually chosen with no coupling... but it is not a decoupling approach in itself. It is rather connected to some further issues in these frames.

As a conclusion, decoupling is fundamental but so difficult !

References

- P. T. Tham
  Notes - An introduction to Decoupling control.
  Department of Chemical and Process Engineering, University of Newcastle upon Tyne, England... for the example of chemical reactor
- *Course Notes, Chapter 6 : Analysis and Design of Multivariable Control Systems.*
  Electrical Engineering Department, State University of Binghamton, New-York, USA... for decoupling by time approach and other insights.
- E. H. Bristol
  On a new measure of interaction in multivariable process control.
  IEEE Transactions on Automatic Control, Vol 11, p. 133-134... for the reader interested in "relative gain approach".
- J. P. Corriou.
  Commande des procédés.
  Lavoisier Editions, TEC&DOC Collection, 1996 (in French, sorry !), for connected information.

Eigenstructure assignment

Eigenstructure and its influence

Eigenstructure assignment

Motivation : assigning the poles and possibly the associated eigenvectors in order to try to shape the transient response of the closed-loop system.

It can be way to obtain some transient input/ouput decoupling.

Techniques based upon eigenstructure placement are also called *Modal Control*.

Matrix eigenstructure

λ is an *eigenvalue* of $A \in \mathbb{C}^{n\times n}$ iff

$$P(\lambda) = \det(\lambda \mathbb{I}_n - A) = 0. \quad (20)$$

A owns n eigenvalues λ_i (which will be assumed distinct for the sake of conciseness). This set is referred to as the *spectrum* $\lambda(A)$.

$A \in \mathbb{R}^{n\times n} \Rightarrow \lambda(A)$ is closed under conjugation.

There exists n non zero vectors $v_i \in \mathbb{C}^n$, called *right eigenvectors*, such that

$$Av_i = \lambda_i v_i \quad \forall i \in \{1, ..., n\}. \quad (21)$$

One should talk about *eigendirections* since they can be multiplied by any non zero scalar.

Matrix eigenstructure

$$V = [v_1, \cdots, v_n] \tag{22}$$

is called the *modal matrix*.

$$\Rightarrow \Lambda = \text{diag}\{\lambda_1, \cdots, \lambda_n\} = V^{-1}AV \tag{23}$$

One can define, by duality, *left eigenvectors* $u_i \in \mathbb{C}^n$ such that

$$u_i'A = \lambda_i u_i' \quad \forall i \in \{1, ..., n\} \Rightarrow U = [u_1, \cdots, u_n]. \tag{24}$$

u_i and v_i can be scaled so that

$$U'V = \mathbb{I}_n \quad \text{(orthogonality condition)}. \tag{25}$$

The eigenvectors v_i (or u_i) make a basis of $\mathbb{C}^n$.

Feedback model eigenstructure

Closed-loop model eigenstructure=eigenstructure of its state matrix

$$A_c = A + BFC.$$

$$\Rightarrow \begin{cases} A_c v_i = (A + BFC)v_i = \lambda_i v_i \quad \forall i \in \{1, ..., n\} \\ u_i' A = u_i'(A + BFC) = \lambda_i u_i' \quad \forall i \in \{1, ..., n\} \\ U' V = \mathbb{I}_n \\ A_c = A + BFC = V \Lambda U'. \end{cases}$$

Feedback model eigenstructure

Remark : In practice, the matrices are real meaning that not only $\lambda(A_c)$ but also the sets of eigenvectors are closed under conjugation.

- Input directions :

$$w_i = FCv_i \quad \forall i \in \{1, ...n\}.$$

- Output directions :

$$l_i' = u_i' BF \quad \forall i \in \{1, ...n\}.$$

Influence of the eigenvalues

It can be easily proved that the free response of a model to an initial condition is

$$x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i. \quad (26)$$

- $Re(\lambda_i) < 0\,\forall i$ otherwise there are non vanishing terms (instability).
- $|Re(\lambda_i)| \nearrow \Rightarrow$ the term *(mode)* reduces faster.
- $|Im(\lambda_i)| \nearrow \Rightarrow$ the term induces stronger oscillation (none if λ_i is real).

So $\lambda(A)$ has an influence on stability, settling time, oscillations, characterizing the transient behaviour.

Influence of the eigenvectors

Consider the perturbed closed-loop model

$$\left\{ \begin{array}{l} \dot{x} = (A + BFC)x + BHy_c + \bar{B}d \\ y = Cx. \end{array} \right. \tag{27}$$

With the basis change $x = V\xi$, V being the modal matrix of $A_c = A + BFC$:

$$\left\{ \begin{array}{l} \dot{\xi} = \Lambda\xi + U'BHy_c + U'\bar{B}d \\ y = CV\xi. \end{array} \right. \tag{28}$$

Also consider the identity matrix :

$$\mathbb{I}_n = \left[\begin{array}{ccc} e_1 & \dots & e_n \end{array} \right], \tag{29}$$

Influence of the eigenvectors

- y_{c_i} has no effect on λ_j iff

$$u_j' B H e_i = 0.$$

$\Rightarrow$ left eigenvectors distribute the effects of the references on the eigenvalues

- λ_i has no effect on x_j (resp. y_j) iff

$$e_j' v_i = 0.$$

$\Rightarrow$ right eigenvectors v_i (resp. Cv_i) distribute the effects of the eigenvalues on the state entries (resp. outputs).

Influence of the eigenvectors

- d_i has no effect on λ_j iff

$$u_j'\bar{B}e_i = 0.$$

$\Rightarrow$ left eigenvectors distribute the effects of some disturbances on the eigenvalues

- λ_j has no effect on u_j iff (less obvious)

$$e_j'w_i = 0.$$

$\Rightarrow$ input directions distribute the effects of the eigenvalues on the control entries.

Influence of the eigenstructure

The effect of the environment on the system dynamics is mainly described by the left eigenstructure whereas the effect of these dynamics on the system outputs is mainly described by the right eigenstructure (Ibrahim Chouaib).

<u>Remark :</u> It can also be proved that eigenvectors have an influence on the local sensitivity of eigenvalues with respect to additive unstructured uncertainty affecting the state matrix (not detailed here).

Eigenstructure assignment by state feedback

State feedback assignment

Pole Placement Problem : find $K \in \mathbb{R}^{m\times n}$ such that $\lambda(A_c = A + BK)$ equals some specified set.

The computation of feedforward matrix H will be considered later.

There is always some solution provided the pair (A, B) is controllable.

State feedback assignment

- At first sight, one needs n degrees of freedom (dof) to place n poles. It remains $n(m-1)$ to place right eigenvectors (because of the orthogonality condition, a choice of V implies a choice of U).
- However, an eigenvector is characterized by $(n-1)$ entries (not n since it can be scaled).
- So, not enough parameters for an arbitrary choice of eigenvectors.
- Indeed, each v_i belongs to some *characteristic subspace*.

State feedback assignment

Characteristic subspaces

Because for one λ, the associated v and $w = Kv$ comply with

$$(A - \lambda \mathbb{I}_n)v + Bw = \mathbb{O},$$

then $v \in S(\lambda)$ where

$$S(\lambda) = \{v \in \mathbb{C}^n \mid \exists w \in \mathbb{C}^m \mid (A - \lambda \mathbb{I}_n)v + Bw = \mathbb{O}\}$$

(A, B) controllable $\Rightarrow \dim(S(\lambda)) = m$.

$\Rightarrow$ Only $(m-1)$ should be exploited to assign $v \in S(\lambda)$, exactly what is offered by K.

State feedback assignment

Define :

$$T_\lambda = \begin{bmatrix} A - \lambda \mathbb{I}_n & B \end{bmatrix} \in \mathbb{C}^{n\times(n+m)},$$

$$R_\lambda = \begin{bmatrix} N_\lambda \\ M_\lambda \end{bmatrix} = \mathrm{Ker}(T_\lambda) \quad \text{with} \quad N_\lambda \in \mathbb{C}^{n\times m},\ M_\lambda \in \mathbb{C}^{m\times m}.$$

and with some parameter vector $z \in \mathbb{C}^m$, it comes

$$\pi = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} N_\lambda z \\ M_\lambda z \end{bmatrix},$$

leading to *admissible* eigenvector $v \in S(\lambda)$ and associated input direction w.

Choice of z

Assume v_d is some desired eigenvector (with for instance zero entries to try to reach some decoupling properties). One has to assign an *admissible* v as close as possible to v_d. Solving a classical least square problem leads to

$$z = (N'_\lambda N_\lambda)^{-1} N'_\lambda v_d. \tag{30}$$

Remark : It is possible to rather give specifications on various u_i and then to deduce suitable v_{d_i}.

State feedback assignment

Theorem

There exists $K \in \mathbb{R}^{m\times n}$ solving the problem iff

(i) *vectors v_i are linearly independent ;*

(ii) $v_i = \tilde{v}_j$ *when* $\lambda_i = \tilde{\lambda}_j$;

(iii) $v_i \in S(\lambda_i)$.

In this event the unique solution is given by

$$K = WV^{-1} \tag{31}$$

where

$$W = \left[\ w_1 \quad \ldots w_n\ \right].$$

State feedback assignment

Algorithm :

1. Choose a desired spectrum $\{\lambda_i\}$ and some desired eigenvectors v_{d_i} (do not forget about the conjugation)
2. Compute matrices T_{λ_i} and then R_{λ_i} (*i.e.* N_{λ_i} and M_{λ_i}) ;
3. Compute parameter vectors z_i so that each v_i is admissible and as close as possible to v_{d_i} (note that $v_j = \tilde{v}_i \Leftrightarrow z_j = \tilde{z}_i$) ;
4. $$\left\{ \begin{array}{l} v_i = N_{\lambda_i} z_i \\ w_i = M_{\lambda_i} z_i \end{array} \right. \quad \forall i \in \{1, ..., n\};$$
5. Check the independence of v_i (otherwise go back to step 1 or 3) ;
6. Compute V, W and K according to the previous theorem.

Eigenstructure assignment by output feedback

Pole Placement Problem : find $F \in \mathbb{R}^{m \times n}$ such that $\lambda(A_c = A + BFC)$ equals some specified set.

Remark : It is possible but dangerous to assign only part of the spectrum following the same kind of reasoning as for state feedback.

Necessary condition for solving the problem : (A, B, C) minimal.

Output feedback assignment

About the dof :

- At first fight, $\exists m \times p$ entries in F so the problem can be solved if $mp \geq n$ but not so simple.
- In 1975, Kimura proved that $m + p > n \Rightarrow$ generic assignability.
- Later (1981), it was proved that the condition is $mp \geq n$ but in the field of *complex* matrices... but no need for a complex F !
- In 1996, Wang proved that a sufficient condition in the field of real matrices is $mp > n$ but the associated design method is not very tractable.
- In practice, the tractable (*e.g.* non iterative) techniques require that Kimura's condition holds.

What to do if Kimura's condition does not hold ?

- Assign only part of the spectrum (dangerous !),
- or apply a dynamic feedback.

If it holds, there are several techniques available with different restrictions, *e.g.* :

- "Polynomial" design ;
- Parametric approach ;
- Geometric approach (very elegant) ;
- Coupled Sylvester Equations (my favourite !),
- and many others I may not know or that still have to be found.

Principle of the "Sylvester approach" :

Solve the system :

$$AV - V\Lambda = -BW \quad \text{(right eigenstructure)} \tag{32}$$

$$U'A - \Lambda U' = -L'C \quad \text{(left eigenstructure)} \tag{33}$$

$$\mathrm{Ker}(U') = \mathrm{Im}(V) \quad \text{(orthogonality)} \tag{34}$$

The main idea : assign $\{\lambda_i,\ i \in \{1, ..., p\}\}$ and the associated v_i as well as $\{\lambda_i,\ i \in \{p+1, ..., n\}\}$ and the associated u_i, while respecting the three above equations.

Simplified algorithm :

- Choose $\Lambda_{n-p} = \text{diag}\{\lambda_i,\ i \in \{p+1, ..., n\}\}$ (subspectrum closed under conjugation) and $L_{n-p} = [l_{p+1}, \ldots, l_n] \in \mathbb{C}^{p\times(n-p)}$ and solve

$$U'_{n-p}A - \Lambda_{n-p}U'_{n-p} = -L'_{n-p}C; \tag{35}$$

in $U_{n-p} = [u_{p+1}, \ldots, u_n] \in \mathbb{C}^{n\times(n-p)}$.

- Choose the self-conjugate set $\{\lambda_i, i \in \{1, ..., p\}\}$ and compute

$$\mathcal{N}_{\lambda_i} = \begin{bmatrix} A - \lambda_i \mathbb{I}_n & B \\ U'_{n-p} & \mathbb{O}_{n-p,m} \end{bmatrix} \forall i \in \{1, ..., p\}; \tag{36}$$

Output feedback assignment

Simplified algorithm (cont'd) :

- Compute

$$\begin{bmatrix} N_{\lambda_i} \\ M_{\lambda_i} \end{bmatrix} = \mathrm{Ker}(\mathcal{N}_{\lambda_i}) \in \mathbb{C}^{(n+m)\times r_i)}$$

(it generically exists when $m + p > n$) ;

- Choose p parameter vectors $z_i \in \mathbb{C}^{r_i}$ such that

$$V_p = [Cv_1, ..., Cv_p] = [CN_{\lambda_1} z_1, \ldots, CN_{\lambda_p} z_p] \tag{37}$$

is a full rank matrix ;

Simplified algorithm (cont'd) :

- Compute

$$W_p = [w_1, \ldots, w_p] = [M_{\lambda_1} z_1, \ldots, M_{\lambda_p} z_p]; \tag{38}$$

- The feedback matrix is given by

$$F = W_p (V_p)^{-1}. \tag{39}$$

⋄ The *dof* are on the entries of L_{n-p} and $z_i \, \forall i \in \{1, ..., p\}$. It can be shown that this flexibility corresponds to the flexibility brought by F. It can be used to assign part of the eigenstructure.

With direct transmission D :

Just find $\hat{F}$ by the above technique to assign the spectrum of

$$A_c = A + B\hat{F}C$$

and then deduce

$$F = \hat{F}(\mathbb{I}_p + D\hat{F})^{-1}.$$

Output feedback assignment

$\underline{m + p \leq n}$

It is possible to design a dynamic feedback or order l in order to assign $n + l$ poles but one has to satisfy

$$l \geq n - m - p + 1. \tag{40}$$

Hence, Kimura's condition holds for the "augmented system" (see part on the various feedback structures) and then one computes a static gain for this augmented system which corresponds to a dynamic gain for the original system.

Some special cases can also be handled with static gain (since 2006 !)

Pole placement and feedforward

It might be possible (depending on dimensions) to compute $\hat{H}$ and $H = (\mathbb{I}_m - FD)\hat{H}$ such that

$$(-(C + D\hat{F}C)(A + B\hat{F}C)^{-1}B + D)\hat{H} = \mathbb{I}_p. \qquad (41)$$

to ensure a unit static gain otherwise add integrators before to solve the problem... but with integrators, Kimura's condition is harder to satisfy.

Simple example with MATLAB

Model and desired spectrum :

```
» A=[1 4 5 ;0 2 6 ;1 0 3] ;
» B=[1 1 ;1 0 ;0 0] ;
» C=[1 0 0 ;0 1 0] ;
» D=eye(2) ;
» n=3 ;m=2 ;p=2 ;
» lambda=[-1 -2 -3] ;
```

Choice of L_{n-p}, solution to "left" Sylvester equation :

```
» Lam_nmoinsp=diag(lambda(p+1 :n))

Lam_nmoinsp =

        -3
» L_nmoinsp=[1 ;1] ;
» U_nmoinsp=sylv(A',-Lam_nmoinsp',-C'*L_nmoinsp)

U_nmoinsp =

       -0.3025
        0.0420
        0.2101
```

Computation of $\mathcal{N}_1$, its kernel, v_1 and w_1 :

» NN1=[A-lambda(1)*eye(3) B ;U_nmoinsp' zeros(n-p,m)]
NN1 =

2.0000	4.0000	5.0000	1.0000	1.0000
0	3.0000	6.0000	1.0000	0
1.0000	0	4.0000	0	0
-0.3025	0.0420	0.2101	0	0

» R1=null(NN1)
R1 =

-0.0364
-0.3074
0.0091
0.8675
0.3892

» v1=R1(1 :3) ;w1=R1(4 :5) ;

The same for λ_2.

```
» NN2=[A-lambda(2)*eye(3) B ;U_nmoinsp' zeros(n-p,m)]

NN2=
      3.0000     4.0000     5.0000     1.0000     1.0000
      0          4.0000     6.0000     1.0000     0
      1.0000     0          5.0000     0          0
     -0.3025     0.0420     0.2101     0          0

» R2=null(NN2)
R2 =

     -0.0305
     -0.2499
      0.0061
      0.9629
      0.0975

» v2=R2(1 :3) ;w2=R2(4 :5) ;
```

Computation of W_p, V_p and $\hat{F}$:

```
» Wp=[w1 w2] ;Vp=C*[v1 v2] ;

» F_hat=Wp*inv(Vp)
F_hat =
        -285.8000        31.0000
         242.8000       -30.0000
```

Verification of the cloded-loop spectrum :

```
» eig(A+B*F_hat*C)
ans =

        -3.0000
        -2.0000
        -1.0000
```

Deduction of F and computation of H :

```
» F=F_hat*inv(eye(p)-D*F_hat)
F =

   -0.9773    0.0227
    0.1780   -0.7897

» H_hat=inv(-(C+D*F_hat*C)*inv(A+B*F_hat*C)*B+D)
H_hat =

   58.5529   -84.1059
  -49.6706    71.3412

» H=(eye(p)-F*D)*H_hat
H =

   -0.2161    0.3106

   -0.0962    0.1406
```

Output feedback assignment

Construction of the closed-loop model and verification of the static gain

```
» Ac=(A+B*inv(eye(m)-F*D)*F*C) ;
» Bc=B*inv(eye(m)-F*D)*H ;
» Cc=(C+D*inv(eye(m)-F*D)*F*C) ;
» Dc=D*inv(eye(m)-F*D)*H ;
» -Cc*inv(Ac)*Bc+Dc

ans =

    1.0000    0.0000
    0.0000    1.0000
```

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... and all the references therein.

Model reduction

Model reduction

The main idea : Approximate a high order (n) linear model by a reduced order (r) model to make the design simpler.

$$\mathbf{S} = \left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array} \right. \rightarrow \mathbf{S}_r = \left\{ \begin{array}{l} \dot{x}_r = A_r x_r + B_r u \\ y_r = C_r x_r + D_r u \end{array} \right. \quad (42)$$

Quality of $\mathbf{S}_r$

Several criteria can be considered :

- Preserve the dominant poles (*i.e.* neglect the fast (high frequency) dynamics) ;
- Approximate the input/ouput behaviour : for a same input vector, y_r should be as close as possible to y.

Some existing methods

According to these criteria, a non complete list of existing techniques is as follows :

- By modal approach ;
- By "agregation" ;
- By Schur decomposition ;
- By minimization of norm (ex : $\mathcal{H}_\infty$-norm) ;
- By balancing transformation (the only one presented here and maybe the most known !).

Balanced reduction

Only valid for asymptotically stable *minimal* models but there exists a (not well known) extension to unstable models.

The idea : neglect the dynamics of the state entries that are the less controllable and observable in **S**.

But how to quantify controllability and observability ?

Answer : through the grammians (or Gramm matrices).

Balanced reduction

Controllability grammian

$$W_c = \int_0^\infty e^{A\tau} BB' e^{A'\tau} d\tau. \tag{43}$$

which solves Lyapunov equation

$$AW_c + W_c A' = -BB'. \tag{44}$$

Observability grammian

$$W_o = \int_0^\infty e^{A'\tau} C' C e^{A\tau} d\tau \Rightarrow \tag{45}$$

$$A' W_o + W_o A = -C' C. \tag{46}$$

(hence the stability assumption).

Balanced reduction

The controllability grammian can be interpreted in terms of energy.

There exists a basis in $\mathbb{R}^n$ in which W_c is diagonal. In this basis each diagonal entry w_{c_i} of W_c is the reciprocal of the minimum energy required to (asymptotically) bring the state vector to $[0, \ldots, 0, 1, 0 \ldots, 0]'$. Thus, it can be seen as a controllability index of x_i.

For observability, the reasoning is based upon duality to conclude that in the "diagonal" basis, w_{o_i} is an observability index of x_i.

Balanced reduction

Let **S** be decribed by the triplet of matrices (A, B, C), assuming $D = 0$ for the sake of conciseness.

Theorem

There exists a full rank matrix T such that the realisation $\mathbf{S} = (T^{-1}AT, T^{-1}B, CT) = (\bar{A}, \bar{B}, \bar{C})$ *is* balanced i.e. *both grammians equal to the same diagonal matrix*

$$\bar{W}_o = \bar{W}_c = \Sigma.$$

It means that in this basis, for each entry $\bar{x}_i$, the controllability and observability indices are the same $\Rightarrow$ one has to neglect the dynamics of the less controllable and observable $\bar{x}_i$.

In the balanced basis, the model is (here, D is kept)

$$\left\{\begin{array}{rcccccc} \dot{\bar{x}}_1 & = & \bar{A}_{11}\bar{x}_1 & + & \bar{A}_{12}\bar{x}_2 & + & \bar{B}_1 u \\ \dot{\bar{x}}_2 & = & \bar{A}_{21}\bar{x}_1 & + & \bar{A}_{22}\bar{x}_2 & + & \bar{B}_2 u \\ y & = & \bar{C}_1\bar{x}_1 & + & \bar{C}_2\bar{x}_2 & + & Du. \end{array}\right. \tag{47}$$

The limit between the preserved dynamics (that of $\bar{x}_1$ and the neglected ones (that of $\bar{x}_2$) depends on a possible gap in the diagonal entries of Σ.

The idea is then to neglect the dynamics of $\bar{x}_2$, the less controllable and observable part of $\bar{x}$ (thus the less influent on the input/ouput behaviour) by imposing $\dot{\bar{x}}_2 = 0$. This technique is sometimes called the "singular perturbations approximation".

It is also possible to simply truncate $\bar{x}$ by imposing $\bar{x}_2 = 0$ but it does not preserve the static gain so it is rather rough as a reduction.

Balanced reduction

So, the reduced model $\mathbf{S}_r$ is given by

$$\mathbf{S}_r = \left\{ \begin{array}{ccll} \dot{x}_r & = & A_r x_r & +B_r u \\ \\ y_r & = & C_r x_r & + \quad D_r u. \end{array} \right. \tag{48}$$

where $x_r = x_1$ and

$$\left\{ \begin{array}{ccc} A_r & = & \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21} \\ B_r & = & B_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2 \\ C_r & = & \bar{C}_1 - \bar{C}_2\bar{A}_{22}^{-1}\bar{A}_{21} \\ D_r & = & D - \bar{C}_2\bar{A}_{22}^{-1}\bar{B}_2. \end{array} \right. \tag{49}$$

Balanced reduction

MATLAB corresponding functions

- `minreal` : Compute the minimal realization of original system $\mathbf{S}$;
- `balreal` : Compute the balanced realization $(\bar{A}, \bar{B}, \bar{C}, D)$ of $\mathbf{S}$;
- `modred` : Compute the final realization (A_r, B_r, C_r, D_r) of reduced system $\mathbf{S}_r$.

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Towards LMI-based synthesis

About norms

Definition and properties

A norm enables ones to compare an element with another in a set which does not necessarily own a relation of order (here a vector space on $\mathbb{R}$ or $\mathbb{C}$). It is usually denoted by $||u||_{\bullet}$ where u is the concerned element and $\bullet$ stands for the considered norm.

$$\begin{array}{rl} \text{(i)} & ||u||_{\bullet} \geq 0 \\ \text{(ii)} & ||u||_{\bullet} = 0 \Leftrightarrow u = 0 \\ \text{(iii)} & ||au||_{\bullet} = |a|.||u||_{\bullet}, \quad \forall a \in \mathbb{C} \\ \text{(iv)} & ||u+v||_{\bullet} \leq ||u||_{\bullet} + ||v||_{\bullet} \text{ (triangular inequality)} \end{array} \tag{50}$$

Euclidean norm

Inner product of a couple $\{x; y\} \in \{\mathbb{C}^n\}^2$:

$$< x, y > = \sum_{i=1}^{n} x_i' y_i \; = \; x' y \tag{51}$$

From this inner product, one can define the Euclidean norm of 2-norm (the most natural) :

$$||x||_2 \; = \; \sqrt{< x, x >} \; = \; \sqrt{\sum_{i=1}^{n} x_i^2} \; = \; \sqrt{x' x}. \tag{52}$$

There are many other vector norms not detailed here.

Vector function norms

Now the vectors depend on some real or complex variable (t or s).

$\mathcal{L}_2$ and $\mathcal{H}_2$-norms

Let $\mathcal{L}_2^n$ be the set of vector functions $X(s) \in \mathbb{C}^n$, with $s \in \mathbb{C}$ whose square can be "summed" along the imaginary axis :

$$||X||_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X'(\mathbf{i}\omega)X(\mathbf{i}\omega)d\omega \right)^{1/2} < \infty. \qquad (53)$$

$||X||_2$ is called the $\mathcal{L}_2$-norm of$X$ ($\mathcal{L}$ for Lebesgue).
It can be shown that $\mathcal{L}_2^n$ is an Hilbert space.

Vector function norms

$\mathcal{H}_2^n \subset \mathcal{L}_2^n$ is the restriction to analytic functions over $\mathbb{C}^+$ (owning Taylor's expansion in every points). Then the $\mathcal{L}_2$-norm is called $\mathcal{H}_2$-norm ($\mathcal{H}$ for Hardy).

Parserval's theorem

$$||X||_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X'(\mathbf{i}\omega)X(\mathbf{i}\omega)d\omega \right)^{1/2} =$$

$$\left(\int_0^\infty x'(t)x(t)dt \right)^{1/2} = \left(\int_0^\infty ||x(t)||_2^2 dt \right)^{1/2} = ||x||_2.$$

Beware of the fooling notation : $||x(t)||_2$ is the 2 (Euclidean)-norm of vector x at time t (*i.e.* reflects the instantaneous energy) whereas $||x||_2$ (or $||X||_2$) is the $\mathcal{H}_2$-norm of signal vector x which depends on time (resp. of its Laplace transform, *i.e.* reflects the signal energy over and an infinite horizon).

$\mathcal{L}_\infty$ and $\mathcal{H}_\infty$-norms

Let $\mathcal{L}_\infty^n$ be the set of vector functions $X(s) \in \mathbb{C}^n$, with $s \in \mathbb{C}$ bounded along the imaginary axis *i.e.* :

$$||X||_\infty = \sup_\omega ||X(\mathbf{i}\omega)||_2 < +\infty. \tag{54}$$

$||X||_\infty$ is called the $\mathcal{L}_\infty$-norm of X.
$\mathcal{L}_\infty^n$ is not an Hilbert space.

$\mathcal{H}_\infty^n \subset \mathcal{L}_\infty^n$ contains only analytic functions over $\mathbb{C}^+$ and one defines the $\mathcal{H}_\infty$-norm (still from Hardy).

$\mathcal{L}_2$-gain

Let a mathematical operator $\mathcal{R}$ be defined over the following sets :

$$\begin{array}{rcl} \mathcal{R} : \mathcal{L}_2^{n_w} & \rightarrow & \mathcal{L}_2^{n_w} \\ w(t) & \mapsto & e(t) \end{array}$$

Then,

$$\mathcal{G}_{\mathcal{L}_2}(\mathcal{R}) = \sup_{w \in \mathcal{H}_2^{n_w}} \frac{||e||_2}{||w||_2} \tag{55}$$

is the $\mathcal{L}_2$-gain of $\mathcal{R}$ which corresponds to the highest energy gain associated with $\mathcal{R}$.

Matrix norm

There are so many but one is particularly interesting.

Singular values of a matrix

Any matrix $M \in \mathbb{C}^{m\times n}$ can be factorized as follows (singular value decomposition) :

$$M = U\Sigma W'. \tag{56}$$

$U \in \mathbb{C}^{m\times m}$ et $W \in \mathbb{C}^{n\times n}$ are such that

$$UU' = \mathbb{I}_m \quad \text{et} \quad WW' = \mathbb{I}_n, \tag{57}$$

Matrix norm

and Σ, if $q = \min\{m, n\}$, complies with

$$\left\{\begin{array}{lll} \Sigma = \left[\begin{array}{cccc|c} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_q & 0 \end{array}\right] & \text{si} & q = m, \\ \Sigma = \left[\begin{array}{cccc} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_q \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] & \text{si} & q = n, \\ \Sigma = \operatorname{diag}\{\sigma_1, \cdots, \sigma_q\} & \text{si} & q = m = n. \end{array}\right. \tag{58}$$

Matrix norm

σ_i are the singular values of M :

$$\bar{\sigma}(M) = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q = \underline{\sigma}(M) \geq 0. \qquad (59)$$

- rank(M) $=$ number of non-zero singular values $=$ number of linearly independant rows or columns.
- M such that rank(M) $< \min\{m; n\}$ is rank deficient, otherwise it is full rank.
- M square and rank deficient cannot be inverted and owns $n - r$ zero singular values.
- $\sigma_i =$ eigenvalues of MM' (if $m \leq n$) or $M'M$ (if $n \leq m$).
- M Hermitian $\Rightarrow \sigma_i = |\lambda_i|$.

Matrix norm

$\bar{\sigma}(M)$ is a norm called 2-norm because it is induced by the Euclidean vector norm in the following way :

$$\bar{\sigma}(M) = ||M||_2 = \max_{x \neq 0 \in \mathbb{C}^n} \left(\frac{||Mx||_2}{||x||_2} \right) = \max_{x \neq 0 \in \mathbb{C}^n} \sqrt{\frac{x' M' Mx}{x' x}}. \tag{60}$$

Besides

$$\underline{\sigma}(M) \leq \frac{||Mx||_2}{||x||_2} \leq \bar{\sigma}(M). \tag{61}$$

shows that the gain from x to (Mx) lies in the range $[\underline{\sigma}(M) ; \bar{\sigma}(M)]$

Transfer matrix norm

In this part, the matrices whose norm is defined depend on s.

Singular value of transfer matrix

If w is an input harmonic signal vector of a plant $G(s) \in \mathbb{C}^{n_e \times n_w}$ and e is the output harmonic signal vector. Then, at a given frequency ω, the gain from w to e complies with

$$\underline{\sigma}(G(\mathbf{i}\omega)) \leq \frac{||e(\mathbf{i}\omega)||_2}{||w(\mathbf{i}\omega)||_2} = \frac{||G(\mathbf{i}\omega)w(\mathbf{i}\omega)||_2}{||w(\mathbf{i}\omega)||_2} \leq \bar{\sigma}(G(\mathbf{i}\omega)). \qquad (62)$$

Lower and upper bounds of the gain (in the sense of the 2-norm) are given by the minimum and maximum singular values of $G(\mathbf{i}\omega)$. These bounds also depend on ω.

$\mathcal{L}_\infty - \mathcal{H}_\infty$-norm of a transfer

Let $\mathcal{RL}_\infty^{n_e \times n_w}$ (resp. $\mathcal{RH}_\infty^{n_e \times n_w}$), be the set of *proper* transfer matrices $G(s) \in \mathbb{C}^{n_e \times n_w}$ (*i.e.* with finite direct transmission) and with no pole on the imaginary axis $\mathcal{I}$ (resp. with no pole over $\mathbb{C}^+ \cup \mathcal{I}$). Then the $\mathcal{L}_\infty$ (resp. $\mathcal{H}_\infty$-norm) simply corresponds to the frequency for which the transfer is the highest in the sense of the 2-norm. Hence :

$$||G||_\infty = \sup_\omega ||G(\mathbf{i}\omega)||_2 = \sup_\omega \bar{\sigma}(G(\mathbf{i}\omega)). \tag{63}$$

Transfer matrix norm

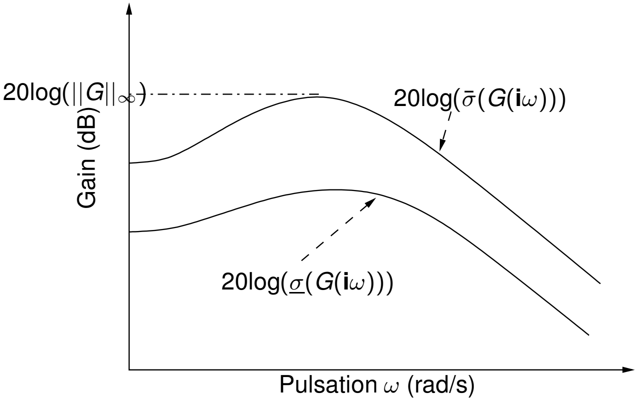


FIGURE: Gain Bode diagramm in the MIMO case

The actual transfer lies somewhere between both curves.

$\mathcal{H}_\infty$-norm

Energetic interpretation

Let S be a stable plant whith transfer matrix $G(s)$.

$$||G||_\infty = \mathcal{G}_{\mathcal{L}_2}(S) = \sup_{w \in \mathcal{H}_2(t)^{n_w}} \frac{||e||_2}{||w||_2}.$$

meaning that the $\mathcal{H}_\infty$-norm is the $\mathcal{L}_2$-gain

$\mathcal{L}_2/\mathcal{H}_2$-norm of a transfer

Let $\mathcal{RL}_2^{n_e\times n_w}$ (resp. $\mathcal{RH}_2^{n_e\times n_w}$), be the set of *strictly proper* transfer matrices $G(s) \in \mathbb{C}^{n_e\times n_w}$ (*i.e.* with no direct transmission) and with no pole on the imaginary axis $\mathcal{I}$ (resp. with no pole over $\mathbb{C}^+ \cup \mathcal{I}$). Then the $\mathcal{L}_2$ (resp. $\mathcal{H}_2$-norm) is defined by

$$||G||_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(G'(\mathbf{i}\omega)G(\mathbf{i}\omega))d\omega \right)^{1/2} \Leftrightarrow \quad (64)$$

$$||G||_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^{\min\{n_w;n_e\}} (\sigma_i(G(\mathbf{i}\omega)))^2 d\omega \right)^{1/2}. \quad (65)$$

Energetic interpretation

Assume that $\hat{e}_i(t) \in \mathcal{L}_2^{n_e}$ is the response to (only) a Dirac impulse on the i^{th} entry in w. One can prove that

$$\sum_{i=1}^{n_w} ||\hat{e}_i||_2^2 = ||G||_2^2. \tag{66}$$

The $\mathcal{H}_2$-norm is related to the sum off all input energies induced by these impulses.
In the SISO case, it means that the $\mathcal{H}_2$ norm if the energy of the impulse response.

$$\text{Time domain : } ||G||_2 = \sqrt{\int_0^\infty \text{trace}(e(t)e'(t))dt}. \tag{67}$$

Stochastic interpretation

If the w_i are white noises scaled such that $W(\mathbf{i}\omega)W'(\mathbf{i}\omega) = \mathbb{I}_{n_w}$, then the expectation of the inner product of the induced input vector checks

$$\sum_{i=1}^{n_e} \mathcal{E}(e_i'(t)e_i(t)) = ||G||_2^2. \quad (68)$$

Remark : For this reason the so-called $\mathcal{H}_2$-problem can be related to the celebrated LQG-problem

$\mathcal{H}_2$-norms in terms of gain

Reminding that the $\mathcal{H}_\infty$-norm is the the $\mathcal{L}_2$-gain then the $\mathcal{H}_2$-norm checks

$$||G||_2 = \sup_{W(s)\in\mathcal{H}_\infty^{n_W}} \frac{||E||_2}{||W||_\infty}. \tag{69}$$

which, unlike for the $\mathcal{H}_\infty$-norm, is not very meaningful.

$\mathcal{H}_2$-norms and grammians

Remember the controllability and observability grammians :

$$W_c = \int_0^\infty e^{At} B B^T e^{A' t} dt \quad ; \quad W_o = \int_0^\infty e^{A' t} C^T C e^{At} dt. \tag{70}$$

that satisfy

$$\begin{cases} AW_c + W_c A' = -BB', \\ A' W_o + W_o A = -C' C. \end{cases} \tag{71}$$

$\mathcal{H}_2$-norms and grammians (cont'd)

Then

$$||G||_2 = \sqrt{\text{trace}(B'W_oB)} = \sqrt{\text{trace}(CW_cC')}. \qquad (72)$$

This provides an analytical expression of the $\mathcal{H}_2$-norm which is valid for calculation.

Unlike the $\mathcal{H}_\infty$-norm, the $\mathcal{H}_2$-norm can be computed directly through its analytic expression.

Matrix inequalities

For simplicity, only real matrices are considered.

Matrix inequalities

This is related to the notion of sign definition (partial order of Löwner).

$M \in \mathbb{R}^{n\times n}$ is positive definite $(M > 0)$ (resp. semi-positive definite $(M \geq 0)$) iff

$$x^T Mx > 0 \text{ (resp.} \geq 0) \quad \forall x \neq 0 \in \mathbb{R}^n. \tag{73}$$

M is negative definite $(M < 0)$ (resp. semi-negative definite $(M \leq 0)$) iff $(-M) > 0$ (resp. $(-M) \geq 0$).

Matrix inequalities (MI)

In practice mostly symmetric matrices are handled so sign definition is now considered only for those matrices.
With this assumption, one gets

$$\left\{\begin{array}{l} M < (\leq)\, 0 \Leftrightarrow \lambda_{\max}(M) < (\leq)\, 0 \\ \\ M > (\geq)\, 0 \Leftrightarrow \lambda_{\min}(M) > (\geq)\, 0 \end{array}\right. \tag{74}$$

A straightforward notation is

$$\left\{\begin{array}{l} M > (\geq)\, N \Leftrightarrow M - N > (\geq)\, 0 \\ M < (\leq)\, N \Leftrightarrow M - N < (\leq) 0. \end{array}\right. \tag{75}$$

LMI bases

Example of MI :

$$M = M^T = AX^3 + (X^3)^T A^T + e^B YY^T (e^B)^T < 0$$

with, for instance, X and Y that are unknown.

Among all possible MI only two will be considered because they are often encountered :

- LMI : Linear matrix inequalities, that can be solved,
- BMI : Bilinear matrix inequalities.

Properties of MI

- If $M_1 < 0$ and $M_2 < 0$ then one can stack these properties in one single MI :

$$\begin{bmatrix} M_1 & \mathbb{O} \\ \mathbb{O} & M_2 \end{bmatrix} < 0. \tag{76}$$

- If $M = M^T$ is such that

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} < 0, \tag{77}$$

then $M_1 < 0$ and $M_3 < 0$ but the reverse may be false.

LMI

LMI are interesting because they can be solved !

The most famous LMI come from...

Theorem

Let the autonomous continuous (resp. discrete) model

$$\dot{x} = Ax \quad (resp. \quad x_{k+1} = Ax_k)$$

This model is asymptotically stable iff $\exists P = P^T > 0$ such that

$$A^T P + PA < 0, \quad (resp. \quad -P + A^T PA < 0).$$

(Lyapunov's inequality and its discrete counterpart due to Stein).

BMI

Consider the next MI with respect to X and Y :

$$AX + X^T A^T + XBY + Y^T B^T X^T > 0$$

Because of the two last terms, it is bilinear.

Unfortunately those BMI are very difficult to solve in spite of some existing software.

Some crucial control problems are unfortunately very easily formulated as BMI, not as LMI (*e.g.* static output feedback stabilization).

A useful tool !

Schur's lemma

Let S, $Q = Q^T$ and $R = R^T$ be matrices.

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0 \quad \Leftrightarrow \quad \begin{cases} R < 0 \\ Q - SR^{-1}S^T < 0 \end{cases} \tag{78}$$

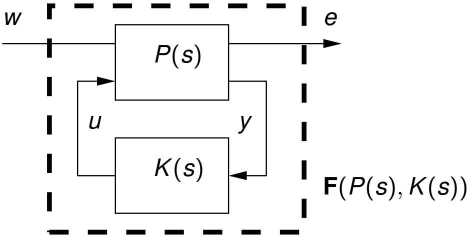
This lemma enables to handle Stein's inequality as an LMI wrt A :

$$\begin{bmatrix} -P + A^T PA & \mathbb{O} \\ \mathbb{O} & -P \end{bmatrix} < 0 \Leftrightarrow \begin{bmatrix} -P & A^T P \\ PA & -P \end{bmatrix} < 0.$$

$\mathcal{H}_\bullet$-design

Standard $\mathcal{H}_\bullet$-problem

The $\mathcal{H}_\bullet$-problem is basically a disturbance rejection problem !
The studied feedback system matches the next figure :



where $P(s)$ is the process model, $K(s)$ is the controller model and $\mathbf{F}(P(s), K(s))$ is the closed-loop model.

The process

- u : control vector issued from control law ;
- w : disturbance to be rejected (in practice, not always actual exogeneous signals) ;
- y : measured ouput for the purpose of control ;
- e : vector of signals to be controlled.

$$\begin{bmatrix} E(s) \\ Y(s) \end{bmatrix} = P(s) \begin{bmatrix} W(s) \\ U(s) \end{bmatrix}, \quad \text{with} \quad P(s) = \begin{bmatrix} P_{ew}(s) & P_{eu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix}.$$

The idea is to reduce the transfer from w to e.

The process

$$P(s) = D + C(s\mathbb{I} - A)^{-1}B, \text{ with} \tag{79}$$

$$B = \begin{bmatrix} B_w & B_u \end{bmatrix} \quad ; \quad C = \begin{bmatrix} C_e \\ C_y \end{bmatrix} \quad ; \quad D = \begin{bmatrix} D_{ew} & D_{eu} \\ D_{yw} & D_{yu} \end{bmatrix}. \tag{80}$$

Or in other words

$$\left\{ \begin{array}{rcccccc} \dot{x}(t) & = & Ax(t) & + & B_w w(t) & + & B_u u(t) \\ e(t) & = & C_e x(t) & + & D_{ew} w(t) & + & D_{eu} u(t) \\ y(t) & = & C_y x(t) & + & D_{yw} w(t) & + & D_{yu} u(t) \end{array} \right.$$

$x(t) \in \mathbb{R}^{n}$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $e(t) \in \mathbb{R}^{n_e}$ et $y(t) \in \mathbb{R}^{n_y}$.

Assumptions

- **A1** : $(A; B_u)$ and $(A; C_y)$ are respectively stabilisable and detectable ;
- **A2** : $D_{yu} = \mathbb{O}_{n_y,n_u}$;
- **A3** : $D_{ew} = \mathbb{O}_{n_e,n_w}$ (only for $\mathcal{H}_2$-problem).

A1 is rather classical and completely compulsory.
A2 is just technical and induces no loss of generality. **A3** is necessary for the $\mathcal{H}_2$-norm to be defined.

One considers a static state feedback controller

$$u = Kx. \tag{81}$$

In such a case, since $y = x$, the process model reduces to

$$\left\{ \begin{array}{rclclcl} \dot{x}(t) & = & Ax(t) & + & B_w w(t) & + & B_u u(t) \\ e(t) & = & C_e x(t) & + & D_{ew} w(t) & + & D_{eu} u(t). \end{array} \right. \tag{82}$$

Dynamic controller

One considers a dynamic output feedback controller

$$\left\{\begin{array}{rclcl} \dot{x}_c(t) & = & A_c x_c(t) & + & B_c y(t) \\ u(t) & = & C_c x_c(t) & + & D_c y(t) \end{array}\right. \quad (83)$$

where $x_c(t) \in \mathbb{R}^n$. Thus,

$$K(s) = D_c + C_c(s\mathbb{I}_n - A_c)^{-1}B_c. \quad (84)$$

In the $\mathcal{H}_2$-case (not studied in these frames), one has to consider a strictly proper controller *i.e.* $D_c = \mathbb{O}_{n_u,n_y}$.

Closed-loop model

What is the state-space model of $\mathbf{F}(P(s), K(s))$?

With static controller

Under Assumption A_2 :

$$\left[\begin{array}{c}\dot{x}\\ \hline e\end{array}\right] = \left[\begin{array}{c|c}A_f & B_f\\ \hline C_f & D_f\end{array}\right]\left[\begin{array}{c}x\\ \hline w\end{array}\right] = \left[\begin{array}{c|c}A+B_uK & B_w\\ \hline C_e+D_{eu}K & D_{ew}\end{array}\right]\left[\begin{array}{c}x\\ \hline w\end{array}\right].$$

In the $\mathcal{H}_2$-case, $D_f = 0$.

Closed-loop model

With dynamic controller

Still under Assumption A_2 :

$$\left[\begin{array}{c} \dot{x} \\ \dot{x}_c \\ \hline e \end{array}\right] = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right] \left[\begin{array}{c} x \\ x_c \\ \hline w \end{array}\right] =$$

$$\left[\begin{array}{cc|c} A + B_u D_c C_y & B_u C_c & B_w + B_u D_c D_{yw} \\ B_c C_y & A_c & B_c D_{yw} \\ \hline C_e + D_{eu} D_c C_y & D_{eu} C_c & D_{ew} + D_{eu} D_c D_{yw} \end{array}\right] \left[\begin{array}{c} x \\ x_c \\ \hline w \end{array}\right].$$

In the $\mathcal{H}_2$-case, $D_f = 0$.

$\mathcal{H}_\bullet$-problem

Problem

Let $P(s)$ and $\gamma_\bullet > 0$ be given. Also let assumptions $\mathbf{A_1}$ to $\mathbf{A_3}$ hold. Find a stabilizing (static or dynamic) feedback such that $||\mathbf{F}(P(s), K(s))||_\bullet < \gamma_\bullet$.

If $\bullet = \infty$, then Assumption $\mathbf{A_3}$ can be omitted.

With no additional constraints (such as weighting matrices), the problem is referred to as *standard*.

$\mathcal{H}_\bullet$-problem

$\mathcal{H}_2$ or $\mathcal{H}_\infty$?

... not exactly the same philosophy.

In the $\mathcal{H}_\infty$-case, one looks after the $\mathcal{L}_2$-gain *i.e.* the highest possible energy transfer or, from the frequency viewpoint, the energy transfer at the worst frequency.

In the $\mathcal{H}_2$-case, one considers energy transfer over the whole frequency range, not focusing on the worst one.

$\mathcal{H}_\bullet$-synthesis

In the following frames, some solutions are given for

- the $\mathcal{H}_2$-design by state static feedback,
- the $\mathcal{H}_\infty$-design by state static feedback,
- the $\mathcal{H}_\infty$-design by output dynamic feedback,

Property of the closed-loop model

Lemme

Under assumptions **A1**-**A3**, *the* $\mathcal{H}_2$-*norm of* $\mathbf{F}(P(s), K(s))$ *is less than* $\gamma_2 > 0$ *iff there exist two symmetric positive definite matrices* $\{X_2; T\} \in \{\mathbb{R}^{n\times n}\}^2$, *such that (primal and dual versions)*

$$\left\{\begin{array}{l} B_f^T X_2 B_f < T, \\ \begin{bmatrix} A_f^T X_2 + X_2 A_f & C_f^T \\ C_f & -\mathbb{I}_{n_e} \end{bmatrix} < 0, \\ \mathrm{trace}(T) = \gamma_2{}^2, \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{l} C_f X_2 C_f^T < T, \\ \begin{bmatrix} A_f X_2 + X_2 A_f^T & B_f \\ B_f^T & -\mathbb{I}_{n_w} \end{bmatrix} < 0, \\ \mathrm{trace}(T) = \gamma_2{}^2. \end{array}\right.$$

“$\mathcal{H}_2$ static design”

The idea is that X_2 is a matrix upper bound of either the observability grammian ($X_2 > W_o$: primal version) or of the controllability grammian ($X_2 > W_c$: dual version). So the Lyapunov equations used to calculate the grammians are here replaced by LMIs.

The dual version enables ones to derive some K.

"$\mathcal{H}_2$ static design"

Theorem

There exists $u = Kx$, $K \in \mathbb{R}^{n_u \times n}$ such that the $\mathcal{H}_2$-norm of $\mathbf{F}(P(s), K(s))$ is less than γ_2 iff there exist two symmetric positive definite matrices $\{X_2; T\} \in \{\mathbb{R}^{n \times n}\}^2$, and a matrix $L \in \mathbb{R}^{n_u \times n}$ such that

$$\left\{ \begin{array}{l} \begin{bmatrix} AX_2 + B_u L + X_2 A^T + L^T B_u^T & B_w \\ B_w^T & -\mathbb{I}_{n_w} \end{bmatrix} < 0, \\[2ex] \begin{bmatrix} -T & C_e X_2 + D_{eu} L \\ X_2 C_e^T + L^T D_{eu}^T & -X_2 \end{bmatrix} < 0, \\[2ex] \mathrm{trace}(T) = \gamma_2{}^2. \end{array} \right.$$

"$\mathcal{H}_2$ static design"

In this event, K is given by

$$K = LX_2^{-1}.$$

- Notice that with various LMI solvers, it is possible to minimize γ_2 while satisfying the LMI constraints.
- Also notice that X_2 can be inverted since it is positive definite.

"$\mathcal{H}_\infty$ static design"

Property of the closed-loop model

Lemme

(Bounded real lemma) Under assumptions **A1**-**A2**, *the $\mathcal{H}_\infty$-norm of* $\mathbf{F}(P(s), K(s))$ *is less than* $\gamma_\infty > 0$ *iff there exists a symmetric positive definite matrix* $X_\infty \in \mathbb{R}^{n\times n}$, *such that (primal and dual versions)*

$$\begin{bmatrix} A_f^T X_\infty + X_\infty A_f & X_\infty B_f & C_f^T \\ B_f^T X_\infty & -\gamma_\infty \mathbb{I}_{n_w} & D_f^T \\ C_f & D_f & -\gamma_\infty \mathbb{I}_{n_e} \end{bmatrix} < 0, \qquad (85)$$

"$\mathcal{H}_\infty$ static design"

or in dual version :

$$\begin{bmatrix} A_f X_\infty + X_\infty A_f^T & B_f & X_\infty C_f^T \\ B_f^T & -\gamma_\infty \mathbb{I}_{n_w} & D_f^T \\ C_f X_\infty & D_f & -\gamma_\infty \mathbb{I}_{n_e} \end{bmatrix} < 0. \qquad (86)$$

Once again, the dual version is useful to derive a static state feedback control law.

Theorem

There exists $u = Kx$, $K \in \mathbb{R}^{n_u \times n}$ *such that the* $\mathcal{H}_\infty$*-norm of* $\mathbf{F}(P(s), K(s))$ *is less than* $\gamma_\infty > 0$ *iff there exist a symmetric positive definite matrix* $X_\infty \in \mathbb{R}^{n \times n}$*, and a matrix* $L \in \mathbb{R}^{n_u \times n}$ *such that*

$$\begin{bmatrix} AX_\infty + B_u L + X_\infty A^T + L^T B_u^T & (\bullet) & (\bullet) \\ B_w^T & -\gamma_\infty \mathbb{I}_{n_w} & (\bullet) \\ C_e X_\infty + D_{eu} L & D_{ew} & -\gamma_\infty \mathbb{I}_{n_e} \end{bmatrix} < 0. \qquad (87)$$

"$\mathcal{H}_\infty$ static design"

In this event, K is given by

$$K = LX_\infty{}^{-1}.$$

- Notice that with various LMI solvers, it is possible to minimize γ_∞ while satisfying the LMI constraints.
- Also notice that X_∞ can be inverted since it is positive definite.

A first procedure

Condition for solvability

Under assumptions $\mathbf{A_1}$-$\mathbf{A_2}$, the $\mathcal{H}_\infty$ dynamic problem can be solved iff there exist $R = R^T$ and $S = S^T$ such that

$$\left\{\begin{array}{l}
\begin{bmatrix} N_R & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{n_w} \end{bmatrix}^T \begin{bmatrix} AR + RA^T & RC_e^T & B_w \\ C_e R & -\gamma_\infty \mathbb{I}_{n_e} & D_{ew} \\ B_w^T & D_{ew}^T & -\gamma_\infty \mathbb{I}_{n_w} \end{bmatrix} \begin{bmatrix} N_R & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{n_w} \end{bmatrix} < 0 \\
\begin{bmatrix} N_S & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{n_e} \end{bmatrix}^T \begin{bmatrix} A^T S + SA & SB_w & C_e^T \\ B_w^T S & -\gamma_\infty \mathbb{I}_{n_w} & D_{ew}^T \\ C_e & D_{ew} & -\gamma_\infty \mathbb{I}_{n_e} \end{bmatrix} \begin{bmatrix} N_S & \mathbb{O} \\ \mathbb{O} & \mathbb{I}_{n_e} \end{bmatrix} < 0 \\
\begin{bmatrix} R & \mathbb{I}_n \\ \mathbb{I}_n & S \end{bmatrix} \geq 0
\end{array}\right. \tag{88}$$

where $\mathrm{Span}(N_R) = \mathrm{Ker}([B_u^T \;\; D_{eu}^T])$ and $\mathrm{Span}(N_S) = \mathrm{Ker}([C_y \;\; D_{yw}])$.

Moreover, a *n*th-order controller exists <u>iff</u>

$$\mathrm{rang}(\mathbb{I}_n - RS) = n. \tag{89}$$

- It is also possible to achieve

$$\min_{R=R^T;S=S^T} \gamma_\infty \quad \text{under the LMI constraints.}$$

How to recover $K(s)$?

Achieve the singular value decomposition of $(\mathbb{I}_n - RS)$ in order to obtain $\{M; N\} \in \{\mathbb{R}^{n\times n}\}^2$ such that

$$MN^T = \mathbb{I}_n - RS.$$

$$\text{Then } X_\infty = \begin{bmatrix} S & N \\ N^T & -M^{-1}RN \end{bmatrix}$$

is solution to the condition of the bounded real lemma (primal version) which therefore becomes an LMI (thus solvable) w.r.t. (A_c, B_c, C_c, D_c).

A second procedure

Assume that X_∞ and its inverse are patitionned as follows

$$X_\infty = \begin{bmatrix} R & M \\ M^T & U \end{bmatrix}, \quad X_\infty^{-1} = \begin{bmatrix} S & N \\ N^T & V \end{bmatrix}$$

with $R \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$.

From $X_\infty X_\infty^{-1} = \mathbb{I}_{2n}$, it comes

$$MN^T = I_n - RS.$$

"$\mathcal{H}_\infty$ dynamic design"

Also define the new "controller variables" according to the following system :

$$\begin{cases} \mathcal{B} & = & NB_c + SB_uD_c, \\ \mathcal{C} & = & C_cM^T + D_cC_yR, \\ \mathcal{A} & = & NA_cM^T + NB_cC_yR + \\ & & SB_uC_cM^T + S(A + B_uD_cC_y)R. \end{cases}$$

This system is such that given matrices

- $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,
- R, S, M and N,
- D_c (direct transfer of the controller to be found),

then A_c, B_c and C_c can always be computed and even uniquely determined.

Condition for solvability

Under assumptions $\mathbf{A_1}$-$\mathbf{A_2}$, the $\mathcal{H}_\infty$ dynamic problem can be solved iff there exist $R = R^T$, $S = S^T$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and D_c such that

$$\begin{bmatrix} R & \mathbb{I}_n \\ \mathbb{I}_n & S \end{bmatrix} > 0,$$

$$\begin{bmatrix} \Phi_{11} & \Phi_{21}^T \\ \Phi_{21} & \Phi_{22} \end{bmatrix} < 0.$$

"$\mathcal{H}_\infty$ dynamic design"

with

$$\Phi_{11} = \left[\begin{array}{cc} AR + RA^T + B_u\mathcal{C} + \mathcal{C}^T B_u^T & B_w + B_u D_c D_{yw} \\ (B_w + B_u D_c D_{yw})^T & -\gamma_\infty \mathbb{I}_{n_w} \end{array} \right],$$

$$\Phi_{21} = \left[\begin{array}{cc} \mathcal{A} + (A + B_u D_c C_y)^T & SB_w + \mathcal{B} D_{yw} \\ C_e R + D_{eu}\mathcal{C} & D_{ew} + D_{eu} D_c D_{yw} \end{array} \right],$$

$$\Phi_{22} = \left[\begin{array}{cc} A^T S + SA + \mathcal{B} C_y + C_y^T \mathcal{B}^T & (C_e + D_{eu} D_c C_y)^T \\ C_e + D_{eu} D_c C_y & -\gamma_\infty \mathbb{I}_{n_e} \end{array} \right].$$

How to recover $K(s)$?

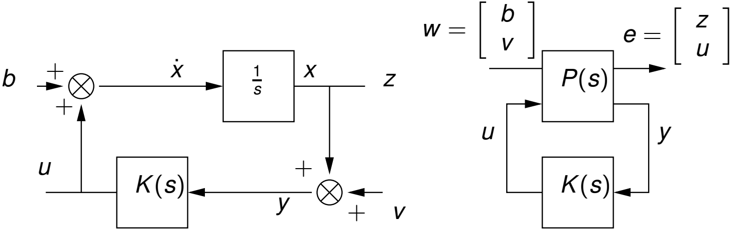
Given a solution to the previous LMI system, one has to compute :

$$MN^T = \mathbb{I}_n - RS,$$

for example by using a SVD factorization, and

$$\left\{\begin{array}{rcl} B_c & = & N^{-1}(\mathcal{B} - SB_uD_c), \\ C_c & = & (\mathcal{C} - D_cC_yR)M^{-T}, \\ A_c & = & N^{-1}(\mathcal{A} - NB_cC_yR - \\ & & SB_uC_cM^T - S(A + B_uD_cC_y)R)M^{-T}. \end{array}\right.$$

Example



Find the state-space model, write the LMI system, deduce the minimum value of γ_∞ and explain how to recover $K(s)$.

"$\mathcal{H}_\infty$ dynamic design"

State-space model :

$$
\left\{
\begin{array}{l}
\dot{x} = [0]x + \left[\begin{array}{cc} 1 & 0 \end{array}\right] \underbrace{\left[\begin{array}{c} b \\ v \end{array}\right]}_{w} + [1]u \\
\underbrace{\left[\begin{array}{c} z \\ u \end{array}\right]}_{e} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} b \\ v \end{array}\right] + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u \\
y = [1]x + \left[\begin{array}{cc} 0 & 1 \end{array}\right] \left[\begin{array}{c} b \\ v \end{array}\right] + [0]u.
\end{array}
\right.
\tag{90}
$$

which correspond to these matrices :

$$A = 0 \qquad B_w = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad B_u = 1$$

$$C_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad D_{ew} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D_{eu} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_y = 1 \qquad D_{yw} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D_{yu} = 0.$$

Assumptions $\mathbf{A_1}$ and $\mathbf{A_2}$ are easily verified. The first procedure is now applied.

"$\mathcal{H}_\infty$ dynamic design"

$$[B_u^T \;\; D_{eu}^T] = [1 \,|\, 0 \;\, 1] = [C_y \;\; D_{yw}] \Rightarrow$$

$$N_R = N_S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

A is scalar then so are R and S. The first LMI to be solved is

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]^T \left[\begin{array}{c|cc|cc} 0 & R & 0 & 1 & 0 \\ \hline R & -\gamma_\infty & 0 & 0 & 0 \\ 0 & 0 & -\gamma_\infty & 0 & 0 \\ \hline 1 & 0 & 0 & -\gamma_\infty & 0 \\ 0 & 0 & 0 & 0 & -\gamma_\infty \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$< \; 0$$

which becomes

$$\Leftrightarrow \begin{bmatrix} -\gamma_\infty & R & 1 & 0 \\ R & -\gamma_\infty & 0 & 0 \\ 1 & 0 & -\gamma_\infty & 0 \\ 0 & 0 & 0 & -\gamma_\infty \end{bmatrix} < 0.$$

and, by Schur's lemma, is equivalent to

$$\begin{cases} \gamma_\infty > 0, \\ {\gamma_\infty}^2 - 1 - R^2 > 0. \end{cases}$$

In a totally similar way, the 2nd inequality reduces to

$$\gamma_\infty{}^2 - 1 - S^2 > 0.$$

whereas the 3rd one , *i.e.* $\begin{bmatrix} R & 1 \\ 1 & S \end{bmatrix} \geq 0$ leads to

$$\left\{ \begin{array}{l} R \geq 0 \\ \\ RS - 1 \geq 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} S \geq 0 \\ \\ RS - 1 \geq 0. \end{array} \right.$$

“$\mathcal{H}_\infty$ dynamic design”

The whole of constraints yields

$$\gamma_\infty{}^2 - 1 \quad > \quad \min_{R,S} \quad (\max\{R^2; S^2\}).$$

which shows that the optimum is reached for

$$R = S = 1 \Rightarrow \gamma_\infty = \sqrt{2}.$$

But in this case, the optimal controller is not of order $n = 1$.
Anyway, for a suboptimal case $RS \neq 1$, one gets

$$M = -N = \sqrt{RS - 1},$$

“$\mathcal{H}_\infty$ dynamic design”

$$\Rightarrow X_\infty = \begin{bmatrix} S & -\sqrt{RS-1} \\ -\sqrt{RS-1} & R \end{bmatrix}.$$

Once X_∞, it suffices to use its value in the condition of the bounded real lemma which becomes an LMI that can be solved by any LMI software.

Pole placement

LMI region

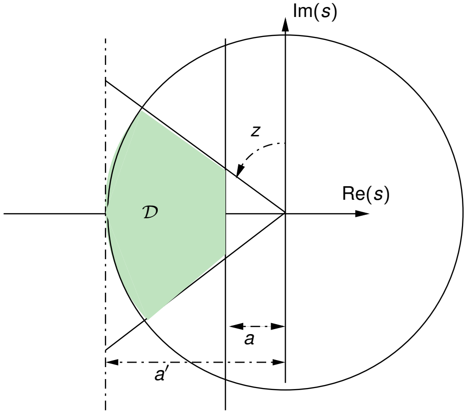
Any set $\mathcal{D} \subset \mathbb{C}$ defined by

$$\mathcal{D} = \{z \in \mathbb{C} \mid \alpha + \beta z + \beta^T \tilde{z} < 0\} \tag{91}$$

where $\alpha = \alpha^T \in \mathbb{R}^{l \times l}$ and $\beta \in \mathbb{R}^{l \times l}$ is an open LMI-region of order l.

These regions are always convex and, if α and β are real (as in the above definition), and symmetric w.r.t. the real axis.

Intersection of LMI regions... is an LMI-region.



LMI formulation of a disc

... of center ρ and radius r.

$$|z-\rho| < r \Leftrightarrow (z-\rho)(\tilde{z}-\rho) - r^2 < 0$$
$$\Leftrightarrow -r + (z-\rho)\frac{1}{r}(\tilde{z}-\rho) < 0.$$

Applying Schur's lemma, it comes

$$\begin{bmatrix} -r & z-\rho \\ \tilde{z}-\rho & -r \end{bmatrix} = \alpha + \beta z + \beta^T \tilde{z} < 0 \Leftrightarrow$$

$$\alpha = \begin{bmatrix} -r & -\rho \\ -\rho & -r \end{bmatrix} < 0 \quad ; \quad \beta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$\mathcal{D}$-stability

Various structures of feedback
About coupling
Eigenstructure assignment
Eigenstructure and its influence
State feedback
Output feedback
Model reduction
Towards LMI-based synthesis
About norms
Matrix inequalities
$\mathcal{H}_\bullet$-design
Pole placement
Mixt synthesis
Insights into robustness

A matrix (or by extension a model) is said $\mathcal{D}$-stable when its eigenvalues lie inside $\mathcal{D}$.

Theorem

Let $\mathcal{D}$ be an LMI-region. A matrix A is $\mathcal{D}$-stable iff $\exists X_{\mathcal{D}} = X_{\mathcal{D}}^T > 0$ such that

$$M_{\mathcal{D}}(A, X_{\mathcal{D}}) = \alpha \otimes X_{\mathcal{D}} + \beta \otimes (AX_{\mathcal{D}}) + \beta^T \otimes (X_{\mathcal{D}}A^T) < 0.$$

This is an LMI w.r.t. $X_{\mathcal{D}}$ or w.r.t. A.

$\mathcal{D}$-stabilization by state feedback

One considers a static state feedback control law.

Theorem

Let $\mathcal{D}$ be an LMI-region. $P(s)$ is $\mathcal{D}$-stabilizable by state feedback iff $\exists X_\mathcal{D} = X_\mathcal{D}{}^T > 0$ and L such that

$$M_\mathcal{D}(A, B_u, X_\mathcal{D}, L) = \alpha \otimes X_\mathcal{D} + \beta \otimes (AX_\mathcal{D}) + \beta^T \otimes (X_\mathcal{D} A^T) +$$

$$\beta \otimes (B_u L) + \beta^T \otimes (L^T B_u^T) < 0.$$

In this event, K is given by $K = LX_\mathcal{D}{}^{-1}$.

$\mathcal{D}$-stabilization by dynamic output feedback

Let us forget about disturbance rejection an assume that the plant $P(s)$ is restricted to the classic state-space model

$$\left\{\begin{array}{rcrcl} \dot{x} & = & Ax & + & B_u u, \\ y & = & C_y x, & & \end{array}\right.$$

(*i.e.* with $D_{yu} = 0$) controlled by a dynamic *n*th-order output feedback control law

$$\left\{\begin{array}{rcrcl} \dot{x}_c & = & A_c x & + & B_c y, \\ u & = & C_c x & + & Dc_y. \end{array}\right.$$

$\mathcal{D}$-stabilization by dynamic output feedback

Then as seen in the first part of these slides, the closed-loop model is

$$\dot{\xi} = A_f \xi$$

with $\xi = \left[\begin{array}{cc} x' & x_c' \end{array}\right]'$ and

$$A_f = \left[\begin{array}{c|c} A + B_u D_c C_y & B_u C_c \\ \hline B_c C_y & A_c \end{array}\right].$$

$\mathcal{D}$-stabilization by dynamic output feedback

Given an LMI region characterized by $\alpha = \alpha^T$ and β, the purpose is then to find $X_\mathcal{D} = X_\mathcal{D} > 0$ and A_c, B_c , C_c, D_c such that

$$M_\mathcal{D}(A_f, X_\mathcal{D}) = \alpha \otimes X_\mathcal{D} + \beta \otimes (A_f X_\mathcal{D}) + \beta^T \otimes (X_\mathcal{D} A_f^T) < 0.$$

Unfortunately, it is a BMI which is not so easy to transform into an LMI... but it is possible.

$\mathcal{D}$-stabilization by dynamic output feedback

The idea is the same as for the second procedure solving the $\mathcal{H}_\infty$-problem. Assume that $X_\mathcal{D}$ and its inverse are patitionned as follows

$$X_\mathcal{D} = \begin{bmatrix} R & M \\ M^T & U \end{bmatrix}, \quad X_\mathcal{D}^{-1} = \begin{bmatrix} S & N \\ N^T & V \end{bmatrix}$$

with $R \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$.

From $X_\mathcal{D} X_\mathcal{D}^{-1} = \mathbb{I}_{2n}$, it comes

$$MN^T = I_n - RS.$$

$\mathcal{D}$-stabilization by dynamic output feedback

Once again, define the new “controller variables” according to the following system :

$$\left\{\begin{array}{lcl} \mathcal{B} & = & NB_c + SB_uD_c, \\ \mathcal{C} & = & C_cM^T + D_cC_yR, \\ \mathcal{A} & = & NA_cM^T + NB_cC_yR+ \\ & & SB_uC_cM^T + S(A + B_uD_cC_y)R. \end{array}\right.$$

This system is such that given matrices

- $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,
- R, S, M and N,
- D_c (direct transfer of the controller to be found),

then A_c, B_c and C_c can always be computed and even uniquely determined.

$\mathcal{D}$-stabilization by dynamic output feedback

Theorem

Let $\mathcal{D}$ be an LMI-region. $P(s)$ is $\mathcal{D}$-stabilizable by dynamic output feedback iff $\exists$, $R = R^T$, $S = S^T$ $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and D_c such that

$$\begin{bmatrix} R & \mathbb{I}_n \\ \mathbb{I}_n & S \end{bmatrix} > 0,$$

$$\alpha \otimes \begin{bmatrix} R & \mathbb{I}_n \\ \mathbb{I}_n & S \end{bmatrix} + \beta \otimes \Phi + \beta^T \otimes \Phi^T < 0,$$

with

$$\Phi = \begin{bmatrix} AR + B_u\mathcal{C} & A + B_u D_c C_y \\ \mathcal{A} & SA + \mathcal{B}C_y. \end{bmatrix}.$$

$\mathcal{D}$-stabilization by dynamic output feedback

If the LMI system is found feasible, then D_c is found and the other matrices of the controller are obtained by

$$MN^T = \mathbb{I}_n - RS.$$

$$\left\{\begin{array}{rcl} B_c & = & N^{-1}(\mathcal{B} - SB_uD_c), \\ C_c & = & (\mathcal{C} - D_cC_yR)M^{-T}, \\ A_c & = & N^{-1}(\mathcal{A} - NB_cC_yR - \\ & & SB_uC_cM^T - S(A + B_uD_cC_y)R)M^{-T}. \end{array}\right.$$

Mixt synthesis

Mixt synthesis

One still considers a static state feedback control law.

Theorem

Let $\mathcal{D}$ be an LMI-region. $P(s)$ is $\mathcal{D}$-stabilizable by state feedback that ensures $||\mathbf{F}(P(s),K)||_\bullet < \gamma_\bullet \ \forall \bullet \in \{\infty; 2\}$ **if** $\exists\{X = X^T > 0; T = T^T; L\}$ *such that*

$$\mathcal{Z}_\infty(P(s), X, L, \gamma_\infty) = \begin{bmatrix} AX + B_uL + XA^T + L^TB_u^T & (\bullet) & (\bullet) \\ B_w^T & -\gamma_\infty \mathbb{I}_{n_w} & (\bullet) \\ C_eX + D_{eu}L & D_{ew} & -\gamma_\infty \mathbb{I}_{n_e} \end{bmatrix} < 0$$

$$\mathcal{Z}_{2_1}(P(s), X, L) = \begin{bmatrix} AX + B_uL + XA' + L'B_u' & B_w \\ B_w^T & -\mathbb{I}_{n_w} \end{bmatrix} < 0.$$

Mixt synthesis

$$\mathcal{Z}_{2_2}(P(s), X, L, T) = \begin{bmatrix} -T & C_e X + D_{eu} L \\ X C_e^T + L^T D_{eu}^T & -\mathbb{I}_n \end{bmatrix} < 0$$

$$\text{trace}(T) < {\gamma_2}^2$$

$$M_{\mathcal{D}}(A, B_u, X, L) = \alpha \otimes X + \beta \otimes (AX) + \beta^T \otimes (XA^T) + \beta \otimes (B_u L) + \beta^T \otimes (L^T B_u^T) < 0.$$

In this event, K is given by $K = LX^{-1}$.

- A great interest in LMI is that one can stack several LMI constraints with preserving the LMI nature of the problem but...

Mixt synthesis

- ...The condition is only sufficient because one imposes the constraint

$$X = X_2 = X_\infty = X_{\mathcal{D}},$$

with also a single L. This is referred to as the *Lyapunov Shaping Paradigm*.

- In such a mixt synthesis only one criterion is minimized (either γ_2 or γ_∞) and the other one is arbitrarily chosen. Another possibility is to minimize a weighted objective function.
- The design of dynamic controllers is also possible but harder and not detailed here. Nevertheless, it relies on previously notions here introduced here.

Insights into robustness

Polytopic uncertainty

(...still considering state feedback)

The process model is assumed to be uncertain (not precisely known) but also assumed to belong to a family of models defined by

$$M = M(\tau) = \begin{bmatrix} A(\tau) & B_w(\tau) & B_u(\tau) \\ C_e(\tau) & D_{ew}(\tau) & D_{eu}(\tau) \end{bmatrix} = \sum_{j=1}^{N} (\tau_j M_j).$$

Polytopic uncertainty

in which $\tau = [\tau_1, ..., \tau_N]^T$ contains the coefficients of a convex combination (*i.e.* $\tau_j \geq 0$ and $\sum_{j=1}^{N}(\tau_j) = 1$) and M_j are the vertices of a so-called polytope of matrices :

$$M_j = \begin{bmatrix} A_j & B_{j_w} & B_{j_u} \\ C_{j_e} & D_{j_{ew}} & D_{j_{eu}} \end{bmatrix}. \tag{92}$$

Remarque

There are many possible descriptions of uncertainties either in the frequency domain (i.e. *affecting the transfer matrix) or in the time domain* (i.e. *affecting the state-space model). Here only few time-domain uncertainties are introduced.*

Why polytopic uncertainty ?

It has a very interesting special case *i.e* the affine parametric uncertainty :

$$M = M_0 + \sum_{i=1}^{p}(\delta_i N_i), \qquad (93)$$

M_0 is the nominal part, the matrices N_i are known and the δ_i are unknown parameters obeying

$$\delta_{i_{min}} \leq \delta_i \leq \delta_{i_{max}} \quad \forall i \in \{1, ..., p\}. \qquad (94)$$

In this case, the polytope has $N = 2^p$ vertices.

Polytopic uncertainty

Example :

$$M = \begin{bmatrix} -1+\delta_1 & \delta_2 & 3 \\ 1 & -2+2\delta_1 & 0 \end{bmatrix} \text{ where } \begin{cases} |\delta_1| \leq 0,5 \\ |\delta_2| \leq 0,2. \end{cases} \Leftrightarrow$$

$$M_0 = \begin{bmatrix} -1 & 0 & 3 \\ 1 & -2 & 0 \end{bmatrix} ; N_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} ; N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which corresponds to the polytopic description :

$$[M_1|M_2|M_3|M_4] =$$

$$\left[\begin{array}{ccc|ccc|ccc|ccc} -0,5 & 0,2 & 3 & -0,5 & -0,2 & 3 & -1,5 & 0,2 & 3 & -1,5 & -0,2 & 3 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -3 & 0 & 1 & -3 & 0 \end{array}\right]$$

Yes, but, once again, why polytopic uncertainty ?

Simply because it is easily handled through LMI machinery.

For example, assume one wants to analyze the robust stability of a polytopic matrix A with two vertices A_1 and A_2. This matrix is robustly stable **if** $\exists P = P^T > 0$ such that

$$A_1^T P + P A_1 < 0 \quad \& \quad A_2^T P + P A_2 < 0.$$

The condition is only sufficient since only one unique P is considered and it is not a function of the uncertainty (one talks about *quadratic stability*).

Robust mixt synthesis

Theorem

Consider an uncertain process model $P(s,\tau)$ and some LMI-region $\mathcal{D}$. There exists a $\mathcal{D}$-stabilizing state feedback K that ensures $||\mathbf{F}(P(s),K)||_\bullet < \gamma_\bullet \ \forall \bullet \in \{\infty;2\}$ **if** *$\exists\{X = X^T > 0; T = T^T; L\}$ such that*

$$\left\{\begin{array}{lll} \mathcal{Z}_{\infty_j} & = \mathcal{Z}(P_j(s),X,L,\gamma_\infty) & < 0 \\ \mathcal{Z}_{2_{1_j}} & = \mathcal{Z}(\mathbf{F}(P_j(s),X,L)) & < 0 \\ \mathcal{Z}_{2_{2_j}} & = \mathcal{Z}(\mathbf{F}(P_j(s),X,L,T)) & < 0 \qquad \forall j \in \{1,\dots,N\}, \\ \mathrm{trace}(T) & < {\gamma_2}^2 & \\ M_{\mathcal{D}_j} & = M_{\mathcal{D}}(A_j + B_{u_j},X,L) & < 0 \end{array}\right.$$

In this event, K is given by $K = LX^{-1}$.

Robust mixt synthesis

There are two reasons why the condition is conservative :

- Lyapunov shaping paradigm,
- Quadratic stability.

Avoiding the shaping paradigm is quite difficult but there exist techniques that consider a matrix $X(\tau)$ (*i.e.* which is dependent on the uncertainty).

Robustness and $\mathcal{H}_\infty$

Many authors consider that $\mathcal{H}_\infty$ approach belongs to the realm of robust control whereas it is simply disturbance rejection.

The reason is as follows :
Consider the uncertain matrix *(Linear Fractional Transform (LFT)-based uncertainty)* :

$$\mathbf{A} = A + B\bar{\Delta}C \tag{95}$$

where $\bar{\Delta} = \Delta(\mathbb{I} - D\Delta)^{-1}$ and $||\Delta|| \leq \rho$, with Δ complex (this special LFT is called *norm-bounded uncertainty*).

Problem : find the largest value of ρ such that $\mathbf{A}$ is Hurwitz for all Δ. This value is the so-called *complex stability radius*.

Robustness and $\mathcal{H}_\infty$

Surprisingly, it has been proved that the complex stability radius is exactly the reciprocal of the $\mathcal{H}_\infty$-norm of the realization (A, B, C, D).

$\Rightarrow$ The bounded real lemma enables ones to compute this radius with no conservativeness.

- However, one shall mention that it is anyway conservative in pratice since the actual realness of the uncertainty is not taken into account.
- Notice that, in this case, quadratic stability is not pessimistic ($P(\Delta)$ is not needed but P suffices).
- It is also possible to consider static or dynamic synthesis even when the uncertanty is polytopic LFT-based.
- There exist discrete counterparts to all these results... See all the possibilities !

References

There are so many (I used quite many in French but...)

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  Lectures Notes DISC Course on Linear Matrix Inequalities in Control
  available for downloading from the web, very general and elegant approach covering many of the aspects of these slides and much more.

... and all the references therein.

See also the very good frames proposed by D. Henrion on his web page :

http://www.laas.fr/~henrion

Hoping you enjoyed these frames, the control community now needs you to investigate many of the problems that are still unsolved !