An Algebraic Analysis Approach to the Equivalence between Fornasini-Marchesini and Roesser Models

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Algebraic Analysis Approach to Linear Systems Theory: Methodology

1. A linear system is defined by a matrix R with coefficients in a ring D of functional operators:

$$R\,y = 0. \quad (\star)$$

2. To $(\star)$ we associate a left D-module M (finitely presented).
3. There exists a dictionary between the properties of $(\star)$ and M.
4. Homological algebra allows to check the properties of M.
5. Effective algebra (non-commutative Gröbner/Janet bases) gives algorithms.
6. Implementation (Maple, Mathematica, Singular/Plural, Cocoa, GAP4/homalg ...).

Roesser (R) Model (simple case)

$$\text{(R)}: \begin{cases} r(i+1,j) &= a_{11}\, r(i,j) + a_{12}\, s(i,j) \\ s(i,j+1) &= a_{21}\, r(i,j) + a_{22}\, s(i,j) \end{cases}$$

◇ To simplify the coeffs a_{ij} are assumed to be constants in K.

◇ $D = K\langle \sigma_i, \sigma_j \rangle$ (commutative) ring of partial shift operators with constant coefficients in K:

$$\delta \in D,\ \delta = \sum_{k,l} \underbrace{d_{kl}}_{\in K} \sigma_i^k\, \sigma_j^l, \quad \delta\, u(i,j) = \sum_{k,l} d_{kl}\, u(i+k, j+l).$$

◇ The (R) model can then be written $R\, y = 0$ with

$$R = \begin{pmatrix} \sigma_i - a_{11} & -a_{12} \\ -a_{21} & \sigma_j - a_{22} \end{pmatrix} \in D^{2\times 2}, \quad y = \begin{pmatrix} r(i,j) \\ s(i,j) \end{pmatrix}.$$

Fornasini-Marchesini (FM) Model (simple case)

$$\text{(FM)}: y(i+1, j+1) = \alpha\, y(i+1, j) + \beta\, y(i, j+1) + \gamma\, y(i, j)$$

◇ To simplify the coeffs α, β, γ are assumed to be constants in K:

$$\delta \in D,\ \delta = \sum_{k,l} \underbrace{d_{kl}}_{\in K} \sigma_i^k\, \sigma_j^l, \quad \delta\, y(i,j) = \sum_{k,l} d_{kl}\, y(i+k, j+l).$$

◇ $D = K\langle \sigma_i, \sigma_j \rangle$ (commutative) ring of partial shift operators with constant coefficients in K.

◇ The (FM) model can thus we written $F\, y = 0$ with

$$F = (\sigma_i\, \sigma_j - \alpha\, \sigma_i - \beta\, \sigma_j - \gamma) \in D, \quad y = (\ y(x,t)\ ).$$

The left D-module M

$\diamond$ D Ore algebra of functional operators, $R \in D^{q \times p}$ and a left D-module $\mathcal{F}$ (the functional space).

$\diamond$ Consider the linear system (behavior)

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

$\diamond$ To $\ker_{\mathcal{F}}(R.)$ we associate the left D-module:

$$M = D^{1 \times p}/(D^{1 \times q}\, R)$$

given by the finite presentation

$$\begin{array}{ccccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R. & & & & \end{array}$$

Theorem [Malgrange]:

$$\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F}) = \{f : M \to \mathcal{F},\ f \text{ is left } D\text{-linear}\}.$$

Roesser (R) Model (simple case)

$$D = K\langle \sigma_i, \sigma_j \rangle, \quad R = \begin{pmatrix} \sigma_i - a_{11} & -a_{12} \\ -a_{21} & \sigma_j - a_{22} \end{pmatrix} \in D^{2\times 2}.$$

$$\begin{array}{ccc} D^{1\times 2} & \xrightarrow{.R} & D^{1\times 2}, \\ (\delta_1, \delta_2) & \longmapsto & (\delta_1\,(\sigma_i - a_{11}) + \delta_2\,(-a_{21}) \quad \delta_1\,(-a_{12}) + \delta_2\,(\sigma_j - a_{22}))\,. \end{array}$$

$\rightsquigarrow$ Associated left D-module $M_R = D^{1\times 2}/D^{1\times 2}\,R$.

$$\begin{array}{ccc} D^{1\times 2} & \xrightarrow{\pi_R} & M_R, \\ \delta = (\delta_1, \delta_2) & \longmapsto & \pi_R(\delta). \end{array}$$

$\diamond$ $\pi_R(\delta)$ residue class of δ in M_R, i.e.,

$$\pi_R(\delta) = \pi_R(\delta') \Longleftrightarrow \exists\, \mu \in D^{1\times 2};\ \delta = \delta' + \mu\, R.$$

In particular, if $\delta = \mu\, R$, then $\pi_R(\delta) = \pi_R(0) = 0$.

Roesser (R) Model (simple case)

◇ $f_1 = (1 \quad 0)$, $f_2 = (0 \quad 1)$ standard basis of $D^{1\times 2}$.

◇ $y_1 = \pi_R(f_1)$, $y_2 = \pi_R(f_2)$ are generators of M_R: indeed $m \in M_R$,

$$m = \pi_R(\delta) = \pi_R(\delta_1\, f_1 + \delta_2\, f_2) = \delta_1\, \pi_R(f_1) + \delta_2\, \pi_R(f_2) = \delta_1\, y_1 + \delta_2\, y_2.$$

◇ These generators satisfy D-linear relations:

$$\begin{aligned}(\sigma_i - a_{11})\, y_1 + (-a_{12})\, y_2 &= (\sigma_i - a_{11})\, \pi_R(f_1) + (-a_{12})\, \pi_R(f_2),\\ &= \pi_R\left((\sigma_i - a_{11})\, f_1 + (-a_{12})\, f_2\right),\\ &= \pi_R\left((\sigma_i - a_{11} \quad - a_{12})\right),\\ &= \pi_R((1 \quad 0)\, R),\\ &= 0.\end{aligned}$$

Similarly, $(-a_{21})\, y_1 + (\sigma_j - a_{22})\, y_2 \;=\; \pi_R((0 \quad 1)\, R) = 0.$

⇝ If $y = (y_1 \quad y_2)^T$, then it yields $R\, y = 0$.

Equivalence of systems / Isomorphism of modules

◇ D Ore algebra, $R \in D^{q \times p}$, $R' \in D^{q' \times p'}$ and $\mathcal{F}$ a left D-module.

◇ Consider $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ and their associated D-modules $M = D^{1 \times p}/D^{1 \times q}\, R$ and $M' = D^{1 \times p'}/D^{1 \times q'}\, R'$.

◇ A D-(homo)morphism f from M to M' is a D-linear map s.t.:

$$\forall\, \delta_1,\, \delta_2 \in D,\ \forall\, m_1,\, m_2 \in M,\ f(\delta_1\, m_1 + \delta_2\, m_2) = \delta_1\, f(m_1) + \delta_2\, f(m_2).$$

◇ A D-morphism is an isomorphism if it is a bijective map.

◇ The systems $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$ are equivalent iff there exists an isomorphism from M to M', i.e., $M \cong M'$.

$\rightsquigarrow$ Given two systems, a way to prove that they are equivalent is to exhibit an isomorphism between their associated D-module.

D-morphisms between f.p. D-modules

◇ Let $M = D^{1\times p}/(D^{1\times q}\, R)$ and $M' = D^{1\times p'}/(D^{1\times q'}\, R')$

◇ $\exists\, f \in \hom_D(M, M') \Longleftrightarrow \exists\, P \in D^{p\times p'},\; Q \in D^{q\times q'}$ s.t.

$$R\, P = Q\, R'.$$

◇ Hence, we have the following commutative exact diagram

$$\begin{array}{ccccccc}
D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow .Q & & \downarrow .P & & \downarrow f & & \\
D^{1\times q'} & \xrightarrow{.R'} & D^{1\times p'} & \xrightarrow{\pi'} & M' & \longrightarrow & 0.
\end{array}$$

◇ $f \in \hom_D(M, M')$ is defined by:

$$\forall\, \lambda \in D^{1\times p}, \quad f(\pi(\lambda)) = \pi'(\lambda\, P).$$

Computing morphisms between f.p. D-modules

◇ Given R, R' as before, we must solve the equation $R\,P = Q\,R'$

◇ For (R) and (FM) models, D is commutative

◇ In this case:

1. $\hom_D(M, M')$ inherits a D-module structure,
2. we have algorithms for computing generators and relations,
3. we have implementations in Maple (OreMorphisms) and Mathematica (OreAlgebraicAnalysis).

(based on Kronecker product and Gröbner bases computations)

⇝ We can thus compute (a representation of) all D-morphisms

Computing isomorphisms between f.p. D-modules

$\diamond$ Given a D-morphism f, i.e., given P and Q, we can compute:

1. $S \in D^{r \times p}$ and $T \in D^{r \times q'}$ such that:

$$\ker_D\left(.(P^T \quad R'^T)\right) = D^{1 \times r}\,(S \;\; -T),$$

2. $L \in D^{q \times r}$ such that $R = L\,S$,
3. $S_2 \in D^{r_2 \times r}$ such that $\ker_D(.S) = D^{1 \times r_2}\,S_2$.

(Gröbner bases computations $\rightarrow$ syzygies, factorizations, ...)

$\diamond$ f isomorphism iff $\left(L^T \quad S_2^T\right)^T$ and $\left(P^T \quad R'^T\right)^T$ admit left inverses over D which can be checked effectively (Gröbner bases)

$\rightsquigarrow$ We have algorithms and implementations to check if a given morphism is an isomorphism

(FM) $\to$ (R) (simple case)

(FM): $y(i+1,j+1) = \alpha\, y(i+1,j) + \beta\, y(i,j+1) + \gamma\, y(i,j)$

$\diamond$ If we define $r(i,j) := y(i,j+1) - \alpha\, y(i,j)$, then we get:

$$r(i+1,j) = \beta\, r(i,j) + (\beta\,\alpha + \gamma)\, y(i,j).$$

$$\rightsquigarrow \text{(R)}: \left\{ \begin{array}{rcl} r(i+1,j) & = & \beta\, r(i,j) + (\beta\,\alpha + \gamma)\, y(i,j) \\ y(i,j+1) & = & r(i,j) + \alpha\, y(i,j) \end{array} \right.$$

Let us try to prove that these two models (systems) are equivalent.

(FM) $\to$ (R) (simple case)

◇ Let $D = K\langle \sigma_i, \sigma_j \rangle$ with $K = \mathbb{Q}(\alpha, \beta, \gamma)$

◇ The matrices corresponding to (FM) and (R) are resp. given by:

$$F = (\sigma_i \, \sigma_j - \alpha \, \sigma_i - \beta \, \sigma_j - \gamma) \in D,$$

$$R = \begin{pmatrix} \sigma_i - \beta & -(\beta \, \alpha + \gamma) \\ -1 & \sigma_j - \alpha \end{pmatrix} \in D^{2 \times 2}.$$

◇ The corresponding D-modules are resp. given by:

$$M_F = D/(D \, F), \quad M_R = D^{1 \times 2}/(D^{1 \times 2} \, R).$$

Let us exhibit an isomorphism from M_F to M_R.

(FM) $\to$ (R) (simple case)

$\rightsquigarrow$ A D-morphism $f \in \hom_D(M_F, M_R)$ is given by

$$\forall\, \lambda \in D, \quad f(\pi_F(\lambda)) = \pi_R(\lambda\, P),$$

where $P \in D^{1\times 2}$ is such that $\exists\, Q \in D^{1\times 2}$ with $F\,P = Q\,R$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{.F} & D & \xrightarrow{\pi_F} & M_F & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
0 & \longrightarrow & D^{1\times 2} & \xrightarrow{.R} & D^{1\times 2} & \xrightarrow{\pi_R} & M_R & \longrightarrow & 0.
\end{array}$$

$\diamond$ Using OreMorphisms, we find that $\hom_D(M_F, M_R)$ is generated by f_1 and f_2 resp. defined by:

$$f_1(\pi_F(\lambda)) = \pi_R(\lambda \underbrace{(1 \quad 0)}_{P_1}), \quad f_2(\pi_F(\lambda)) = \pi_R(\lambda \underbrace{(0 \quad 1)}_{P_2}).$$

(FM) $\rightarrow$ (R) (simple case)

◇ Consider the second generator f_2 defined by $P_2 = (0 \quad 1) \in D^{1\times 2}$.

◇ We have $F\,P_2 = Q_2\,R$ with $Q_2 = (1 \quad \sigma_1 - \beta)$.

◇ We find:

$$\ker_D\left(.\underbrace{\begin{pmatrix} 0 & 1 \\ \sigma_1-\beta & -\beta\,\alpha-\gamma \\ -1 & \sigma_2-\alpha \end{pmatrix}}_{(P_2^T \quad R^T)^T}\right) = D\left(\underbrace{-\alpha\,\sigma_1-\beta\,\sigma_2+\sigma_1\,\sigma_2-\gamma}_{S} \quad \underbrace{-1 \quad -\sigma_1+\beta}_{-T}\right)$$

◇ $F = \underbrace{1}_{L}\,S$, $\ker_D(.S) = 0$, $1.L = 1 \Rightarrow$ f_2 is injective.

$$\diamond \begin{pmatrix} \sigma_2-\alpha & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \sigma_1-\beta & -\beta\,\alpha-\gamma \\ -1 & \sigma_2-\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow f_2 \text{ is surjective.}$$

$\rightsquigarrow$ f_2 isomorphism from M_F to M_R so that $M_F \cong M_R$, i.e., the systems (FM) and (R) are equivalent

(R) $\to$ (FM) (simple case)

$$\text{(R)}: \begin{cases} r(i+1,j) &= a_{11}\, r(i,j) + a_{12}\, s(i,j) \\ s(i,j+1) &= a_{21}\, r(i,j) + a_{22}\, s(i,j) \end{cases}$$

◇ Assuming that a_{21} admits a left inverse a_{21}^{-1}, we have:

$$r(i,j) = a_{21}^{-1}\, \left(s(i,j+1) - a_{22}\, s(i,j)\right),$$

so that

$$\begin{aligned} s(i+1,j+1) &= a_{21}\, r(i+1,j) + a_{22}\, s(i+1,j), \\ &= a_{21}\, \left(a_{11}\, r(i,j) + a_{12}\, s(i,j)\right) + a_{22}\, s(i+1,j), \end{aligned}$$

$$\rightsquigarrow \text{(FM)}: \begin{cases} s(i+1,j+1) = a_{22}\, s(i+1,j) + a_{21}\, a_{11}\, a_{21}^{-1}\, s(i,j+1) \\ \qquad\qquad + \left(a_{21}\, a_{12} - a_{21}\, a_{11}\, a_{21}^{-1}\, a_{22}\right) s(i,j) \end{cases}$$

Let us try to prove that these two models (systems) are equivalent.

(R) → (FM) (simple case)

◇ To simplify, we suppose that the a_{ij}'s are scalars with $a_{21} \neq 0$.

◇ Let $D = K\langle \sigma_i, \sigma_j \rangle$ with $K = \mathbb{Q}(a_{11}, a_{12}, a_{21}, a_{22})$

◇ The matrices corresponding to (R) and (FM) are resp. given by:

$$R = \begin{pmatrix} \sigma_i - a_{11} & -a_{12} \\ -a_{21} & \sigma_j - a_{22} \end{pmatrix} \in D^{2\times 2}.$$

$$F = (\sigma_i\,\sigma_j - a_{22}\,\sigma_i - a_{11}\,\sigma_j - (a_{21}\,a_{12} - a_{11}\,a_{22})) \in D,$$

◇ The corresponding D-modules are resp. given by:

$$M_R = D^{1\times 2}/(D^{1\times 2}\,R), \quad M_F = D/(D\,F).$$

Let us exhibit an isomorphism from M_R and M_F.

(R) → (FM) (simple case)

⇝ A D-morphism $f \in \hom_D(M_R, M_F)$ is given by

$$\forall\, \lambda \in D^{1\times 2}, \quad f(\pi_R(\lambda)) = \pi_F(\lambda\, P),$$

where $P \in D^2$ is such that $\exists\, Q \in D^2$ with $R\, P = Q\, F$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D^{1\times 2} & \xrightarrow{.R} & D^{1\times 2} & \xrightarrow{\pi_R} & M_R & \longrightarrow & 0 \\
 & & \downarrow .Q & & \downarrow .P & & \downarrow f & & \\
0 & \longrightarrow & D & \xrightarrow{.F} & D & \xrightarrow{\pi_F} & M_F & \longrightarrow & 0.
\end{array}$$

⋄ Using OREMORPHISMS, we find that $\hom_D(M_F, M_R)$ is generated by f_1 and f_2 resp. defined by: $\forall\, \lambda = (\lambda_1 \quad \lambda_2) \in D^{1\times 2}$:

$$f_1(\pi_R(\lambda)) = \pi_F(\lambda\, \underbrace{(\sigma_2 - a_{22} \quad a_{21})^T}_{P_1}) = \pi_F(\lambda_1\, (\sigma_2 - a_{22}) + \lambda_2\, a_{21}),$$

$$f_2(\pi_R(\lambda)) = \pi_F(\lambda\, \underbrace{(a_{12} \quad \sigma_1 - a_{11})^T}_{P_2}) = \pi_F(\lambda_1\, (a_{12}) + \lambda_2\, (\sigma_1 - a_{11})).$$

(R) $\rightarrow$ (FM) (simple case)

$\diamond$ Consider the 1st gen. f_1 given by $P_1 = (\sigma_2 - a_{22} \quad a_{21})^T \in D^2$.

$$\ker_D \left(. \underbrace{\begin{pmatrix} \sigma_2 - a_{22} \\ a_{21} \\ a_{22}\, a11 - a_{11}\, \sigma_2 - a_{12}\, a_{21} - a_{22}\, \sigma_1 + \sigma_1\, \sigma_2 \end{pmatrix}}_{(P_1^T \quad R^T)^T} \right) = D \left(\left(\underbrace{\begin{matrix} -a_{21} & \sigma_2 - a_{22} \\ \sigma_1 - a_{11} & -a_{12} \end{matrix}}_{S} \quad \underbrace{\begin{matrix} 0 \\ -1 \end{matrix}}_{-T} \right) \right)$$

$$\diamond\ R = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{L} S,\ \ker_D(.S) = 0,\ L.L = I_2 \Rightarrow f_1 \text{ is injective.}$$

$$\begin{pmatrix} 0 & {a_{21}}^{-1} & 0 \end{pmatrix} \begin{pmatrix} \sigma_2 - a_{22} \\ a_{21} \\ a_{22}\, a11 - a_{11}\, \sigma_2 - a_{12}\, a_{21} - a_{22}\, \sigma_1 + \sigma_1\, \sigma_2 \end{pmatrix} = 1 \Rightarrow f_1 \text{ is surjective}$$

$\rightsquigarrow$ f_1 isomorphism from M_R to M_F so that $M_R \cong M_F$, i.e., the systems (R) and (FM) are equivalent

(R) $\rightarrow$ (FM) (simple case) : some remarks

$\diamond$ The morphism f_2 also defines an isomorphism.

$\diamond$ If $a_{21} = 0$, but $a_{12} \neq 0$, then a similar process can be applied.

$\diamond$ If a_{21} and a_{12} both do not admit a left inverse, then we can always consider the following (FM) model associated with (R):

$$u(i+1, j+1) = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix} u(i, j+1) + \begin{pmatrix} 0 & 0 \\ a_{21} & a_{22} \end{pmatrix} u(i+1, j),$$

with $u = (r \quad s)^T$.

However, the two models do not seem to be equivalent: intuitively, we would have to use the inverses of the σ_i's that are not in D.

General case (with control)

◇ We add control terms u:

$$(\mathrm{FM}): \begin{cases} y(i+1,j+1) = \alpha\, y(i+1,j) + \beta\, y(i,j+1) + a\, y(i,j) \\ \qquad\qquad\qquad + \gamma\, u(i+1,j) + \delta\, u(i,j+1) + b\, u(i,j) \end{cases}$$

$$(\mathrm{R}): \begin{cases} r(i+1,j) &= a_{11}\, r(i,j) + a_{12}\, s(i,j) + b_1\, u(i,j) \\ s(i,j+1) &= a_{21}\, r(i,j) + a_{22}\, s(i,j) + b_2\, u(i,j) \end{cases}$$

(FM) $\to$ (R) (general case)

$$\text{(FM)}: \begin{cases} y(i+1,j+1) = \alpha\, y(i+1,j) + \beta\, y(i,j+1) + a\, y(i,j) \\ \qquad\qquad\qquad + \gamma\, u(i+1,j) + \delta\, u(i,j+1) + b\, u(i,j) \end{cases}$$

$\diamond$ If we define

$$r(i,j) := y(i,j+1) - \alpha\, y(i,j) - \gamma\, u(i,j), \quad v(i,j) = u(i,j+1) - \gamma\, u(i,j),$$

then we get:

$$r(i+1,j) = \beta\, r(i,j) + (\beta\, \alpha + a)\, y(i,j) + (\delta\, \gamma + \beta\, \gamma + b)\, u(i,j) + \delta\, v(i,j).$$

$\rightsquigarrow$ (R):

$$\begin{cases} r(i+1,j) = \beta\, r(i,j) + (\beta\, \alpha + a)\, y(i,j) + (\delta\, \gamma + \beta\, \gamma + b)\, u(i,j) + \delta\, v(i,j) \\ y(i,j+1) = r(i,j) + \alpha\, y(i,j) + \gamma\, u(i,j) \\ u(i,j+1) = \gamma\, u(i,j) + v(i,j) \end{cases}$$

Let us try to prove that these two models (systems) are equivalent.

(FM) $\to$ (R) (general case)

◇ Let $D = K\langle \sigma_i, \sigma_j \rangle$ with $K = \mathbb{Q}(\alpha, \beta, \gamma, \delta, a, b)$

◇ The matrices corresponding to (FM) and (R) are resp. given by:

$$F = (\sigma_1\,\sigma_2 - \alpha\,\sigma_1 - \beta\,\sigma_2 - a \quad -\gamma\,\sigma_1 - \delta\,\sigma_2 - b) \in D^{1\times 2}.$$

$$R = \begin{pmatrix} \sigma_1 - \beta & -(\beta\,\alpha + a) & -(\delta\,\gamma + \beta\,\gamma + b) & -\delta \\ -1 & \sigma_2 - \alpha & -\gamma & 0 \\ 0 & 0 & \sigma_2 - \gamma & -1 \end{pmatrix} \in D^{3\times 4},$$

◇ The corresponding D-modules are resp. given by:

$$M_F = D^{1\times 2}/(D\,F), \quad M_R = D^{1\times 4}/(D^{1\times 3}\,R).$$

Let us exhibit an isomorphism from M_F and M_R.

(FM) $\to$ (R) (general case)

$\diamond$ Proceeding as before, we find that the morphism given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in D^{2\times 4},$$

defines an isomorphism from M_F to M_R.

$\rightsquigarrow$ The two systems (FM) and (R) are equivalent

(R) → (FM) (general case)

$$\text{(R):} \left\{ \begin{array}{lcl} r(i+1,j) & = & a_{11}\, r(i,j) + a_{12}\, s(i,j) + b_1\, u(i,j) \\ s(i,j+1) & = & a_{21}\, r(i,j) + a_{22}\, s(i,j) + b_2\, u(i,j) \end{array} \right.$$

◇ Assuming the coefficient a_{21} admits a left inverse a_{21}^{-1}, we get:

$$\rightsquigarrow \text{(FM):} \left\{ \begin{array}{l} s(i+1,j+1) = a_{22}\, s(i+1,j) \\ \qquad + a_{21}\, a_{11}\, a_{21}^{-1}\, s(i,j+1) \\ \qquad + (a_{21}\, a_{12} - a_{21}\, a_{11}\, a_{21}^{-1}\, a_{22})\, s(i,j) \\ \qquad + b_2\, u(i+1,j) \\ \qquad + (a_{21}\, b_1 - a_{21}\, a_{11}\, a_{21}^{-1}\, b_2)\, u(i,j) \end{array} \right.$$

Let us try to prove that these two models (systems) are equivalent.

(R) $\to$ (FM) (general case)

◇ To simplify, we suppose that the a_{ij}'s are scalars with $a_{21} \neq 0$.

◇ Let $D = K\langle \sigma_i, \sigma_j \rangle$ with $K = \mathbb{Q}(a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2)$

◇ The matrices corresponding to (R) and (FM) are resp. given by:

$$R = \begin{pmatrix} \sigma_1 - a_{11} & -a_{12} & -b_1 \\ -a_{21} & \sigma_2 - a_{22} & -b_2 \end{pmatrix} \in D^{2\times 3},$$

$$F = \begin{pmatrix} \sigma_1\,\sigma_2 - a_{22}\,\sigma_1 - a_{11}\,\sigma_2 - (a_{21}\,a_{12} - a_{11}\,a_{22}) \\ -b_2\,\sigma_1 - (a_{21}\,b_1 - a_{11}\,b_2) \end{pmatrix}^T \in D^{1\times 2}.$$

◇ The corresponding D-modules are resp. given by:

$$M_R = D^{1\times 3}/(D^{1\times 2}\,R), \quad M_F = D^{1\times 2}/(D\,F).$$

Let us exhibit an isomorphism from M_R and M_F.

(R) → (FM) (general case)

◇ Proceeding as before, we find that the morphism given by

$$P = \begin{pmatrix} \sigma_2 - a_{22} & -b_2 \\ a_{21} & 0 \\ 0 & a_{21} \end{pmatrix} \in D^{3\times 2},$$

defines an isomorphism from M_R to M_F.

⇝ The two systems (R) and (FM) are equivalent

◇ If $a_{21} = 0$, but $a_{12} \neq 0$, then a similar process can be applied.

Conclusions

◇ We illustrate the use of the algebraic analysis approach to linear systems theory to prove the equivalence of (FM) and (R) models.

◇ Computations performed without dividing by the coeffs a_{ij}'s, ...
$\Rightarrow$ This can be generalized to matrix coefficients.

◇ We prove:

- (FM) can always be studied by means of an equivalent (R),
- (R) can be studied by means of an equivalent (FM) if we assume that one coeff. (A_{12} or A_{21}) admits a left inverse.

◇ Can we find a (FM) model equivalent to a (R) model where A_{12} and A_{21} both do not admit a left-inverse?

$$\text{(R)}: \begin{cases} r(i+1,j) & = & A_{11}\, r(i,j) + A_{12}\, s(i,j) \\ s(i,j+1) & = & A_{21}\, r(i,j) + A_{22}\, s(i,j) \end{cases}$$