MS-DOS MEETING
OVERVIEW OF THE COLLABORATION BETWEEN LIAS AND XLIM-DMI

Xlim & LIAS-ENSIP, University of Aquitaine (or nearly!)

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Outline of the talk

What is done

- Reminder of the two studied models (Roesser (R) and Fornasini-Marchesini (FM)) ;
- The algebraic approach - Notion of equivalence ;
- Structural stability and algebraic approach ;
- Control laws and algebraic equivalence of models
- How to compute a stabilizing state feedback control law for a R model ;
- How deduce a stabilizing control law for FM model.

What is to be done

- A way to compute a dynamic feedback control law for R model ;
- How to deduce a control law for FM model.

What is done

- IFAC SSSC'16 (Istanbul)
- Submission to MSSP

Roesser and Fornasini models

Open-loop Roesser model

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} u(i,j), \qquad (1)$$

Autonomous (or control-free) Roesser model

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} \qquad (2)$$

we will further see that stability might be defined from the autonomous model.

Roesser and Fornasini-Marchesini models

Open-loop FM model

$$x(i+1,j+1) = F_1\,x(i+1,j) + F_2\,x(i,j+1) + F_3\,x(i,j) + G_1\,u(i+1,j) + G_2\,u(i,j+1) + G_3\,u(i,j), \quad (3)$$

Autonomous (or control-free) FM model

$$x(i+1,j+1) = F_1\,x(i+1,j) + F_2\,x(i,j+1) + F_3\,x(i,j) \quad (4)$$

Once again, stability might be defined from the autonomous model.

Closed-loop Roesser model

If the static state feedback control law

$$u(i,j) = \underbrace{(K_1 \quad K_2)}_{K} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix}, \qquad (5)$$

is applied to the open-loop R model, one gets the closed-loop autonomous R model

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = (A + B\,K) \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix}.$$

Closed-loop FM model

Similarly, if the static state feedback control law

$$u(i,j) = K\,x(i,j), \qquad (6)$$

is applied to the open-loop FM model, one gets the closed-loop autonomous FM model

$$x(i+1,j+1) = (F_1 + G_1\,K)\,x(i+1,j) + (F_2 + G_2\,K)\,x(i,j+1).$$

We are interested in the properties of the closed-loop models... especially the structural stability.

Linear system

A linear system can always be written as

$$R\,\eta = 0,$$

where $R \in D^{q \times p}$ is a $q \times p$ matrix with entries in a (noncommutative) ring D of functional operators and η is a vector of p unknown functions which belongs to a functional space. In the present work, $D = \mathbb{Q}\langle \sigma_i, \sigma_j \rangle$, where σ_i and σ_j are the shift operators along both directions.

The Roesser model (1) is written as $R' \, \eta' = 0$ with

$$R' = \begin{pmatrix} I_{d_h} \sigma_i - A_{11} & -A_{12} & -B_1 \\ -A_{21} & I_{d_v} \sigma_j - A_{22} & -B_2 \end{pmatrix} \in D^{(d_h+d_v)\times(d_h+d_v+d_u)},$$

$$\eta' = \begin{pmatrix} x^h \\ x^v \\ u' \end{pmatrix} = \begin{pmatrix} x' \\ u' \end{pmatrix},$$

and is studied by means of the D-module $M = D^{1\times p}/(D^{1\times q} \, R)$, where $p = d_h + d_v + d_u$ and $q = d_h + d_v$,

Models in the algebraic framework

The Fornasini model (33) is written as $R\,\eta = 0$ with

$$R = \begin{pmatrix} I_{d_x}\,\sigma_i\,\sigma_j - F_1\,\sigma_i - F_2\,\sigma_j - F_3 & -G_1\,\sigma_i - G_2\,\sigma_j - G_3 \end{pmatrix} \in D^{d_x \times (d_x + d_u)},$$

$$\eta = \begin{pmatrix} x \\ u \end{pmatrix},$$

and is studied by means of the D-module $M = D^{1\times p}/(D^{1\times q}\,R)$, where $p = d_x + d_u$ and $q = d_x$.

Equivalence between models

Two linear models $R\eta = 0$ and $R'\eta' = 0$ are said *equivalent* in the sense of algebraic approach when there exists an isomorphism from M to M' (the associated modules).

Such an isomorphism exists if and only if matrices $P \in D^{p\times p'}$, $Q \in D^{q\times q'}$, $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$, and $Z' \in D^{p'\times q'}$ exist and satisfy

$$R\,P = Q\,R',$$

$$R'\,P' = Q'\,R, \quad P\,P' + Z\,R = I_p, \quad P'\,P + Z'\,R' = I_{p'}.$$

On a alors

$$\eta = P\eta', \quad \eta' = P'\eta.$$

Equivalence between R and FM

In the paper he presented at nDS'15, Thomas investigated the possible equivalence between R and FM.

While most of researchers in automatic control claim that R model is a special case of so-called 2nd FM model (a particular instance of FM model where $F_3 = 0$ and $G_3 = 0$), meaning that one can always transform a R model into a peculiar FM model, thus letting think that the FM model is more general, the work by Thomas undermines this preconceived idea and proves that in the sense of algebraic approach, one can always transform a FM model into a R model *by an equivalent mapping*!!!
The other way around is possible under some restrictive conditions and may lead to implicit models. This yields more interest in R model.

Equivalence between R and FM

Thomas also proposed explicit expressions of the equivalent tranformations *i.e.* expressions of matrices P, P', Q, Q', Z and Z'.

This result gave us a new tool to inteprete one of our result dedicated to R models when faced to FM models... as now explained.

Structural stability of an R model

An R model is said structurally stable if the associated autonomous model described by

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix}$$

is structurally stable *i.e.* if

$$\forall (\lambda_1, \lambda_2) \in \mathbb{S}, \quad \det \begin{pmatrix} \lambda_1 \, I_{d_h} - A_{11} & -A_{12} \\ -A_{21} & \lambda_2 \, I_{d_v} - A_{22} \end{pmatrix} \neq 0. \qquad (7)$$

where

$$\mathbb{S} := \left\{ (z_1, z_2) \in \overline{\mathbb{C}}^2 \mid \forall i = 1,\, 2,\, |z_i| \geq 1 \right\}.$$

Structural stability of an FM model

An FM model is said structurally stable if the associated autonomous model described by

$$x(i+1,j+1) = F_1\, x(i+1,j) + F_2\, x(i,j+1) + F_3\, x(i,j)$$

with $F_3 = 0$ is structurally stable *i.e.* if

$$\forall (\lambda_1, \lambda_2) \in \mathcal{D}, \quad \det(I_{d_x} - \lambda_1\, F_1 - \lambda_2\, F_2) \neq 0.$$

where

$$\mathcal{D} := \left\{ (z_1, z_2) \in \overline{\mathbb{C}}^2 \mid \forall i = 1,\, 2,\, |z_i| \leq 1 \right\}.$$

Meaning of structural stability

The meaning of structural stability is not so obvious. It is “highly suspected” to be a necessary and sufficient for asymptotic stability to hold (clearly proved in some cases)... See Nima’s show ! This is part of our questions.

It also seems to be a sufficient condition for bounded input-bounded output (BIBO) stability... See Oberst’s work.

Structural stability and algebraic approach

The fact that structural stabiliy is defined on the control-free system led us to specify the definition of linear systems $R\eta$ as follows :

$$R\,\eta = 0 \Longleftrightarrow (R_1 \quad R_2)\begin{pmatrix} x \\ u \end{pmatrix} = 0 \Longleftrightarrow R_1\,x + R_2\,u = 0.$$

In other words, we split η into two subvectors : the state vector x and the control vector u. Matrix R is splitted in accordance. R_1 and x correspond to the control-free part.

Definition of structural stability in the algebraic framework

A linear system $R_1\, x + R_2\, u = 0$ is said to be *structurally stable* if

$$\forall (\lambda_1, \lambda_2) \in \mathbb{S}, \quad \left(\overline{R_1}(\lambda_1, \lambda_2)\, y = 0 \Longrightarrow y = 0 \right).$$

where $\overline{R_1}(\lambda_1, \lambda_2)$ is the matrix obtained by replacing the shift operators σ_i and σ_j with complex variables λ_1 and λ_2.

This definition was proved to match those introduced for special cases of R model and FM model with $F_3 = 0$.

Structural stability and equivalence

Let two linear systems $R\eta = R_1 x + R_2 u = 0$ and $R'\eta' = R'_1 x' + R'_2 u' = 0$ be given. If the two control-free models $R_1 x = 0$ and $R'_1 x' = 0$ are equivalent in the sense of algebraic analysis then $R\eta = 0$ is structurally stable if and only if $R'\eta' = 0$ is structurally stable.

Remark : The equivalence of $R\eta = 0$ and $R'\eta' = 0$ does not necessarily imply the structural stability. Only the autonomous parts matter.

Control laws

Assume that a system is described by $R_1 x + R_2 u = 0$. A control law can be expressed by

$$T_1 x + T_2 u = 0$$

i.e. by another linear model which leads to a closed-loop model

$$R_s x_s = 0, \qquad R_s := \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \in D^{(q+d_u)\times(d_x+d_u)}$$

This model is considered as autonomous since u is no longer a vector of exogeneous signals.

(Note that a state feedback control law corresponds to $T_1 = -K$ and $T_2 = I_{d_u}$.)

Stabilizing control laws

We say that the system $R_1 x + R_2 u = 0$ is stabilized by the control law $T_1 x + T_2 u = 0$ if $R_s x_s = 0$ is structurally stable *i.e.*

$$\forall(\lambda_1, \lambda_2) \in \mathcal{S}, \quad \left(\overline{R_s}(\lambda_1, \lambda_2)\, y = 0 \Longrightarrow y = 0\right).$$

State feedback and equivalence

Let two models $R\eta = 0$ and $R'\eta' = 0$ be equivalent in the sense of algebraic equivalence (not only their autonomous parts). Let the matrix P involved in the 1-1 correspondence be splitted as follows :

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \in D^{(d_x+d_u)\times(d_{x'}+d_{u'})}.$$

Then applying the state feedback control law $u = Kx$ on $R\eta = 0$ amounts to applying the control law

$$(-K\,P_{11} + P_{21})\,x' + (-K\,P_{12} + P_{22})\,u' = 0$$

on $R'\eta' = 0$.

Stabilizing state feedback and equivalence

Since the closed-loop systems are autonomous and equivalent to each other, and since structural stability is preserved by an equivalent transformation of autonomous parts, then the consequence is as follows :

The state feedback control law $u = K\,x$ with $K \in \mathbb{Q}^{d_u \times d_x}$ stabilizes $R\eta = 0$ if and only if the control law $(-K\,P_{11} + P_{21})\,x' + (-K\,P_{12} + P_{22})\,u' = 0$ stabilizes $R'\eta' = 0$.
The state feedback control law $u' = K'\,x'$ with $K' \in \mathbb{Q}^{d_{u'} \times d_{x'}}$ stabilizes $R'\eta' = 0$ if and only if the control law $(-K'\,P'_{11} + P'_{21})\,x + (-K'\,P'_{12} + P'_{22})\,u = 0$ stabilizes $R\eta = 0$.

State feedback stabilization of a R model

Consider an open-loop R model $R'\eta' = 0$. We recently established a method to compute a stabilizing state feedback $u' = K'x$. This is based upon the solution of an LMI *(Linear Inequality Matrix)* system.

Let us consider a nonnegative integer $\alpha \in \mathbb{N}$, and $\alpha + 1$ matrices $Q_i \in \mathbb{R}^{d_h \times d_h}$, $i = 0, \ldots \alpha$. We introduce the two positive integers

$$\nu = \left(\frac{\alpha\,(\alpha + 1)}{2} + 1 \right) d_h, \quad \delta = 2\,(\nu + d_h + d_v), \tag{8}$$

the set of matrices

$$\left(\begin{array}{c|c} \mathcal{A}_0 & \mathcal{B}_0 \\ \hline \mathcal{C}_0 & \mathcal{D}_0 \end{array} \right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_{d_h} \end{array} \right) \in \mathbb{R}^{2\,d_h \times 2\,d_h},$$

$$\left(\begin{array}{c|c} \mathcal{A}_1 & \mathcal{B}_1 \\ \hline \mathcal{C}_1 & \mathcal{D}_1 \end{array} \right) = \left(\begin{array}{c|c} 0 & I_{d_h} \\ \hline I_{d_h} & 0 \end{array} \right) \in \mathbb{R}^{2\,d_h \times 2\,d_h},$$

State feedback stabilization of a R model

$$\forall i \in \{2, \ldots, \alpha\},$$

$$\left(\begin{array}{c|c} \mathcal{A}_i & \mathcal{B}_i \\ \hline \mathcal{C}_i & \mathcal{D}_i \end{array}\right) = \left(\begin{array}{cc|c} 0 & I_{(i-1)\,d_h} & 0 \\ 0 & 0 & I_{d_h} \\ \hline I_{d_h} & 0 & 0 \end{array}\right) \in \mathbb{R}^{(i+1)\,d_h \times (i+1)\,d_h},$$

and then $A_{\mathbf{Q}} \in \mathbb{R}^{\nu \times \nu}$, $B_{\mathbf{Q}} \in \mathbb{R}^{\nu \times d_h}$, $C_{\mathbf{Q}} \in \mathbb{R}^{d_h \times \nu}$, and $D_{\mathbf{Q}} \in \mathbb{R}^{d_h \times d_h}$ such that

$$\left(\begin{array}{c|c} A_{\mathbf{Q}} & B_{\mathbf{Q}} \\ \hline C_{\mathbf{Q}} & D_{\mathbf{Q}} \end{array}\right) = \left(\begin{array}{c|c} \operatorname{diag}(\mathcal{A}_0, \ldots, \mathcal{A}_\alpha) & \begin{matrix} \mathcal{B}_0 \\ \mathcal{B}_1 \\ \vdots \\ \mathcal{B}_\alpha \end{matrix} \\ \hline \mathbf{Q}\,\operatorname{diag}(\mathcal{C}_0, \ldots, \mathcal{C}_\alpha) & \mathbf{Q} \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \vdots \\ \mathcal{D}_\alpha \end{pmatrix} \end{array}\right)$$

where

$$\mathbf{Q} = (Q_0 \quad Q_1 \quad \cdots \quad Q_\alpha)$$

From this, we define $J_1 \in \mathbb{R}^{(4\,\nu+2\,d_v)\times\delta}$ and $J_3 \in \mathbb{R}^{2\,d_h\times\delta}$ as follows :

$$J_1 = \begin{pmatrix} I_\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & I_\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d_v} & 0 & 0 & 0 \\ A_{\mathbf{Q}} & 0 & 0 & 0 & 0 & B_{\mathbf{Q}} \\ 0 & A_{\mathbf{Q}} & 0 & B_{\mathbf{Q}} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{d_v} & 0 \end{pmatrix},$$

$$J_3 = \begin{pmatrix} C_{\mathbf{Q}} & 0 & 0 & 0 & 0 & D_{\mathbf{Q}} \\ 0 & C_{\mathbf{Q}} & 0 & D_{\mathbf{Q}} & 0 & 0 \end{pmatrix}.$$

State feedback stabilization of a R model

From the Roesser model (1), we also define $\mathbf{A} \in \mathbb{R}^{\delta \times (d_v + d_h)}$ and $\mathbf{B} \in \mathbb{R}^{\delta \times d_u}$ given by :

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ A_{22} & A_{21} \\ A_{12} & A_{11} \\ -I_{d_v} & 0 \\ 0 & -I_{d_h} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B_2 \\ B_1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, we consider $X_1, X_2 \in \mathbb{R}^{2 \times 2}$ given by

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

as well as $J_2 \in \mathbb{R}^{2\,d_h \times \delta}$ and $L \in \mathbb{R}^{(d_v+d_h)\times\delta}$ given by

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & I_{d_h} \\ 0 & 0 & 0 & I_{d_h} & 0 & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 0 & \beta_2\, I_{d_v} & 0 & -I_{d_v} & 0 \\ 0 & 0 & 0 & \beta_1\, I_{d_h} & 0 & -I_{d_h} \end{pmatrix},$$

where $\beta_1,\, \beta_2$ are free parameters in $\mathbb{C}$.

State feedback stabilization of a R model

Let $(\beta_1, \beta_2) \in \mathbb{C}^2 : |\beta_i| < 1$, $i = 1, 2$. There exists $u' = K'x'$ which stabilizes $R'\eta' = 0$ if and only if there exists a non-negative integer $\alpha \leq \frac{d_v\,(d_h^2+d_h-2)}{2}$ such that there exist matrices $Q_i \in \mathrm{I\!R}^{d_h \times d_h}$, $i = 0, \ldots, \alpha$ as well as $S_1 \in \mathrm{I\!R}^{(d_h+d_v)\times(d_h+d_v)}$, $S_2 \in \mathrm{I\!R}^{d_u\times(d_h+d_v)}$, $P_1 \in \mathrm{I\!R}^{(2\,\nu+d_v)\times(2\,\nu+d_v)}$, and $P_2 \in \mathrm{I\!R}^{\nu\times\nu}$ such that $P_i = P_i^T > 0$, $i = 1, 2$, and which satisfy the following two LMIs :

$$J_1^* \,(X_2 \otimes P_1)\, J_1 + ((J_2^* \,(X_1 \otimes I_{d_h})\, J_3)^H + ((\mathbf{A}\, S_1 + \mathbf{B}\, S_2)\, L)^H < 0,$$

$$\begin{pmatrix} C_{\mathbf{Q}} & D_{\mathbf{Q}} \\ 0 & I_{d_h} \end{pmatrix}^* \begin{pmatrix} 0 & -I_{d_h} \\ -I_{d_h} & 0 \end{pmatrix} \begin{pmatrix} C_{\mathbf{Q}} & D_{\mathbf{Q}} \\ 0 & I_{d_h} \end{pmatrix} + \begin{pmatrix} I_\nu & 0 \\ A_{\mathbf{Q}} & B_{\mathbf{Q}} \end{pmatrix}^* (X_2 \otimes P_2) \begin{pmatrix} I_\nu & 0 \\ A_{\mathbf{Q}} & B_{\mathbf{Q}} \end{pmatrix} < 0.$$

State feedback stabilization of a R model

In this event, a structurally stabilizing gain is given by

$$K' = \begin{pmatrix} K'_1 & K'_2 \end{pmatrix} \in \mathbb{R}^{d_u \times (d_h + d_v)}, \quad \begin{pmatrix} K'_2 & K'_1 \end{pmatrix} = S_2 \, S_1^{-1}.$$

Now the question is to know if it can be used when the model if FM

Stabilization of an FM model

Let us a FM model $R\eta = 0$ and the equivalent Roesser $R'\eta' = 0$. From a previously introduced result, we deduce that the state feedback control law

$$u' = K' \begin{pmatrix} x^h \\ x^v \end{pmatrix}, \quad K' = (K'_1 \quad K'_2),$$

stabilizes the R model if and only if the control law

$$\left(-K'_1\,(I_{d_x}\,\sigma_j - F_1) - K'_2 \begin{pmatrix} I_{d_x} \\ 0 \end{pmatrix}\right) x +$$

$$\left(K'_1\,G_1 - K'_2 \begin{pmatrix} 0 \\ I_{d_u} \end{pmatrix} + I_{d_u}\,\sigma_j\right) u = 0,$$

stabilizes the FM model.

Stabilization of an FM model

Another way to express such a result is as follows :

$$u'(i,j) = K' \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix},$$

$$K' = (K'_1 \quad K'_{21} \quad K'_{22}) \in \mathbb{R}^{d_u \times (2\,d_x + d_u)},$$

$$K'_1 \in \mathbb{R}^{d_u \times d_x},\; K'_{21} \in \mathbb{R}^{d_u \times d_x},\; K'_{22} \in \mathbb{R}^{d_u \times d_u}$$

stabilizes the R model if and only if

$$u(i,j+1) = K'_1\, x(i,j+1) + (K'_{21} - K'_1\, F_1)\, x(i,j) + (K'_{22} - K'_1\, G_1)\, u(i,j),$$

stabilizes the FM model.

Note that the obtained control law is dynamic and causal.

Stabilization of an FM model

Consider the FM model given by

$$(F_1 \mid F_2 \mid F_3) =$$

$$\left(\begin{array}{ccc|ccc|ccc} 0.7815 & 0.8189 & 0.1054 & 0.3652 & 0.3623 & 0.0537 & 0.5008 & 0.0225 & 1.21 \\ 0.3428 & 0.2393 & 0.2070 & 0.0596 & 0.6569 & 0.1596 & 0.6066 & 1.3271 & 0.47 \\ 0.0480 & 0.0369 & 0.3624 & 0.3674 & 0.3029 & 0.8198 & 0.4145 & 0.5905 & 0.46 \end{array} \right.$$

$$(G_1 \mid G_2 \mid G_3) = \left(\begin{array}{c|c|c} 0.3922 & 0.7060 & 0.0462 \\ 0.6555 & 0.0318 & 0.0971 \\ 0.1712 & 0.2769 & 0.8235 \end{array} \right).$$

This model is not structurally stable

Stabilization of an FM model

This model is equivalently transformed into a R model which is also (fortunately) not stable.

We solve the LMI system for $\beta_1 = \beta_2 = 0$ and $\alpha = 2$:

$$(K_1' \mid K_{21}' \mid K_{22}') =$$

$$(-0.9469 \; -1.1764 \; -0.8413 \mid -1.3196 \; -1.2745 \; -0.8113 \mid -1.4124)\,.$$

This leads to a control law for the original FM model and the obtained closed-loop FM model is simulated for a given set of boundary conditions.

Stabilization of an FM model

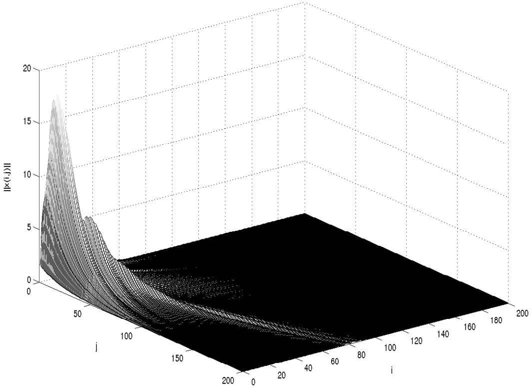

FIGURE: Evolution of the norm $||x(i,j)||$

What can be done

What can be done now

Extension to the case of dynamic control law
(First discussions in February).

Observed state feedback

I tried to find an LMI approach to the derivation of stabilizing dynamic controllers but could not reach a result as satisfactory as for state feedback. Hence the idea to focus on a particular structure of dynamic control law.

Roesser model with output equation

We add an output equation to Roesser model :

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix}}_{x(i,j)} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} u(i,j),$$

$$y(i,j) = \underbrace{\begin{pmatrix} C_1 & C_2 \end{pmatrix}}_{C} \begin{pmatrix} x^h(i,j) \\ x^v(i,j) \end{pmatrix} + D\, u(i,j). \tag{9}$$

Only y can be measured, not x.

2D Kalman-Luenberger observer

The idea is to extend the classic Kalman-Luenberger observer to the 2D case.

Let the following observer be given :

$$\begin{aligned}
\begin{pmatrix} \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{pmatrix} &= \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{pmatrix}}_{\hat{x}(i,j)} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} u(i,j) + \\
&\quad Z(\hat{y}(i,j) - y(i,j)), \\
\hat{y}(i,j) &= \underbrace{\begin{pmatrix} C_1 & C_2 \end{pmatrix}}_{C} \hat{x}(i,j) + D\, u(i,j).
\end{aligned} \tag{10}$$

2D Kalman-Luenberger observer

The observation error is defined by

$$\epsilon(i,j) = \hat{x}(i,j) - x(i,j) = \begin{pmatrix} \epsilon^h(i,j) \\ \epsilon^v(i,j) \end{pmatrix}, \tag{11}$$

which satisfies

$$\begin{pmatrix} \epsilon^h(i+1,j) \\ \epsilon^v(i,j+1) \end{pmatrix} = (A + ZC)\epsilon(i,j). \tag{12}$$

Process and observer in open loop

The original system model together with its observer comply with

$$\begin{aligned}
\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \\ \epsilon^h(i+1,j) \\ \epsilon^v(i,j+1) \end{pmatrix} &= \underbrace{\begin{pmatrix} A & 0 \\ 0 & (A+ZC) \end{pmatrix}}_{A_{bo}} \underbrace{\begin{pmatrix} x^h(i,j) \\ x^v(i,j) \\ \epsilon^h(i,j) \\ \epsilon^v(i,j) \end{pmatrix}}_{\kappa(i,j)} + \underbrace{\begin{pmatrix} B \\ 0 \end{pmatrix}}_{B_{bo}} u(i,j), \\
y(i,j) &= \underbrace{\begin{pmatrix} C & 0 \end{pmatrix}}_{C_{bo}} \kappa(i,j) + \underbrace{D}_{D_{bo}} u(i,j).
\end{aligned} \tag{13}$$

This is not exactly a Roesser model but...

Process and observer in open loop

... with the next permutation matrix,

$$M = \begin{pmatrix} I_{d_h} & 0 & 0 & 0 \\ 0 & 0 & I_{d_h} & 0 \\ 0 & I_{d_v} & 0 & 0 \\ 0 & 0 & 0 & I_{d_v} \end{pmatrix} \quad (14)$$

and the change of basis $\mu(i,j) = M\kappa(i,j)$, it comes

Process and observer in open loop

$$\begin{aligned} \begin{pmatrix} x^h(i+1,j) \\ \epsilon^h(i+1,j) \\ x^v(i,j+1) \\ \epsilon^v(i,j+1) \end{pmatrix} &= MA_{bo}M\mu(i,j) + MB_{bo}u(i,j), \\ y(i,j) &= C_{bo}M\mu(i,j) + D_{bo}\,u(i,j), \end{aligned} \qquad (15)$$

which is a R model.

This change of basis is clearly an isomorphism !

Process and observer in closed loop

It is assumed that only y can be measured, not x (a classic and practically reasonable assumption). The idea is to use $\hat{x}$ rather than x and thus to apply the control law

$$u(i,j) = \underbrace{\begin{pmatrix} K_1 & K_2 \end{pmatrix}}_{K} \hat{x}(i,j) = K(x(i,j) + \epsilon(i,j)). \tag{16}$$

It leads to

$$\begin{aligned} \begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \\ \epsilon^h(i+1,j) \\ \epsilon^v(i,j+1) \end{pmatrix} &= \underbrace{\begin{pmatrix} A+BK & BK \\ 0 & (A+ZC) \end{pmatrix}}_{A_{bf}} \underbrace{\begin{pmatrix} x^h(i,j) \\ x^v(i,j) \\ \epsilon^h(i,j) \\ \epsilon^v(i,j) \end{pmatrix}}_{\kappa(i,j)}, \\ y(i,j) &= \underbrace{\begin{pmatrix} C+DK & DK \end{pmatrix}}_{C_{bf}} \kappa(i,j), \end{aligned} \tag{17}$$

which is not a R model. But using $\mu(i,j) = M\kappa(i,j)$ again, it comes

Process and observer in closed loop

$$\begin{aligned}\begin{pmatrix} x^h(i+1,j) \\ \epsilon^h(i+1,j) \\ x^v(i,j+1) \\ \epsilon^v(i,j+1) \end{pmatrix} &= MA_{bf}M\mu(i,j) \\ y(i,j) &= C_{bf}M\mu(i,j),\end{aligned} \tag{18}$$

which is an autonomous R model.

Is it (structurally) stable ?

Closed-loop stability

We know that one can compute K such that $(A + BK)$ is "stable".

By duality, one can compute Z such that $(A + ZC)$ is "stable".

(In the 1D-case, this is a classic issue in a control course.)

The structural stability is completely determined by $\mathbf{A} = MA_{bf}M$.

Closed-loop stability

Stability of **A**

$$\Leftrightarrow \forall(\lambda_1,\lambda_2) \in \mathbb{S}, \quad \det\left(\begin{pmatrix} \lambda_1 I_{2d_h} & 0 \\ 0 & \lambda_2 I_{2d_v} \end{pmatrix} - \mathbf{A}\right) \neq 0$$

(where $\mathbb{S} = \{(z1, z_2) \in \mathbb{C} \cup \{\infty\};\ |z_i| \geq 1,\ i = 1, 2\}$).

$$\Leftrightarrow \forall(\lambda_1,\lambda_2) \in \mathbb{S}, \quad \det\left(\underbrace{\begin{pmatrix} \lambda_1 I_{2d_h} & 0 \\ 0 & \lambda_2 I_{2d_v} \end{pmatrix}}_{\tilde{H}(\lambda)} - MA_{bf}M\right) \neq 0$$

Closed-loop stability

$$\Leftrightarrow \forall(\lambda_1, \lambda_2) \in \mathbb{S}, \quad \det\left(M\tilde{H}(\lambda)M - A_{bf}\right) \neq 0$$

$$\Leftrightarrow \forall(\lambda_1, \lambda_2) \in \mathbb{S}, \quad \det\left(\left(\begin{array}{c|c} \underbrace{\begin{pmatrix} \lambda_1 I_{d_h} & 0 \\ 0 & \lambda_2 I_{d_v} \end{pmatrix}}_{H(\lambda)} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \lambda_1 I_{d_h} & 0 \\ 0 & \lambda_2 I_{d_v} \end{pmatrix} \end{array}\right) - A_{bf}\right) \neq 0$$

By recalling that

$$A_{bf} = \begin{pmatrix} A + BK & BK \\ 0 & A + ZC \end{pmatrix}$$

the condition for stability is re-expressed as follows :

Closed-loop stability

$$\forall(\lambda_1, \lambda_2) \in \mathbb{S},$$
$$\det\left(H(\lambda) - (A + BK)\right) \det\left(H(\lambda) - (A + ZC)\right) \neq 0.$$

As a result, the closed-loop model is structurally stable if and only if $(A + BK)$ and $(A + ZC)$ are both “stable”... what we can obtain.

Coming back to the observer itself

The observer can be written

$$\begin{pmatrix} \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{pmatrix} = (A+ZC)\hat{x}(i,j) + (B+ZD)u(i,j) - Zy(i,j). \tag{19}$$

This is the form (a priori) used for implementation. It has two inputs : u et y. The actually relevant output is $\hat{x}$.

“practical closed loop”

Consider the state vector

$$\xi(i,j) = \begin{pmatrix} x(i,j) \\ \hat{x}(i,j) \end{pmatrix}. \tag{20}$$

With such a vector, the closed-loop model becomes

$$\begin{pmatrix} x^h(i+1,j) \\ x^v(i,j+1) \\ \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{pmatrix} = \underbrace{\begin{pmatrix} A & BK \\ -ZC & (A+ZC+BK) \end{pmatrix}}_{\mathcal{A}} \xi(i,j). \tag{21}$$

Of course, with the change of basis $\gamma(i,j) = M\xi(i,j)$, the next model is obtained

“practical closed loop”

$$\begin{pmatrix} x^h(i+1,j) \\ \hat{x}^h(i+1,j) \\ x^v(i,j+1) \\ \hat{x}^v(i,j+1) \end{pmatrix} = \underbrace{M\mathcal{A}M}_{\mathfrak{A}}\gamma(i,j), \tag{22}$$

which is a R model.

The change of basis is isomorphic so the above model is stable.

Algebraic point of view

... as far as I can understand.

For the present purpose, we need to further specify the definition of linear models... At least I think so.

Linear system

There are three kinds of signals : state $x \in \mathbb{Q}^{d_x}$, control $u \in \mathbb{Q}^{d_u}$ u, and measured output $y \in \mathbb{Q}^{d_y}$.
Therefore, the global signal vector is defined as

$$\eta = \begin{pmatrix} x^T & u^T & y^T \end{pmatrix}^T = \begin{pmatrix} x^T & z^T \end{pmatrix}^T . \quad (23)$$

Then a linear system can be defined

$$R\eta = 0, \quad (24)$$

where

$$R = \begin{pmatrix} R_D \\ R_{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} R_{D_x} & R_{D_u} & R_{D_y} \\ R_{\mathbb{Q}_x} & R_{\mathbb{Q}_u} & R_{\mathbb{Q}_y} \end{pmatrix} = \begin{pmatrix} \Pi_x & \Pi_z \end{pmatrix} \quad (25)$$

with $R_{D_x} \in \mathbb{Q} < \sigma_i, \sigma_j >^{q_D \times d_x}$, $R_{D_u} \in \mathbb{Q} < \sigma_i, \sigma_j >^{q_D \times d_u}$, $R_{D_y} \in \mathbb{Q} < \sigma_i, \sigma_j >^{q_D \times d_y}$, $R_{\mathbb{Q}x} \in \mathbb{Q}^{q_\mathbb{Q} \times d_x}$, $R_{\mathbb{Q}u} \in \mathbb{Q}^{q_\mathbb{Q} \times d_u}$, $R_{\mathbb{Q}_y} \in \mathbb{Q}^{q_\mathbb{Q} \times d_y}$, $\Pi_x \mathbb{Q} x \in \mathbb{Q} < \sigma_i, \sigma_j >^{(q_D + q_\mathbb{Q}) \times d_x}$.

Linear system

$R_D \eta = 0$ corresponds to the q_D dynamic equations of the model involving the shift operators σ_i and σ_j.

$R_{\mathbb{Q}} \eta = 0$ corresponds to $q_{\mathbb{Q}}$ static equations independent from the shift operators σ_i and σ_j (usually called “output equations").

A reasonable pratical assumption is $R_{D_y} = 0$ (the output is not involved in the dynamic subsystem). We adopt it.

Autonomous linear system

The linear system $R\eta = 0$ is said to be autonomous if $d_u = 0$ and, if not, the autonomous system associated to $R\eta = 0$ is given by

$$\begin{pmatrix} R_{D_x} & 0 \\ R_{\mathbb{Q}_x} & R_{\mathbb{Q}_y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \quad (26)$$

Structural (spectral) stability

The linear system $R\eta = 0$ is said structurally stable (stable) if the associated autonomous system is stable *i.e.* if

$$\forall(\lambda_1, \lambda_2) \in \mathcal{S},\ \bar{R}_{D_x}(\lambda)x = 0 \Rightarrow x = 0. \qquad (27)$$

$\bar{R}_{D_x}(\lambda) \in \mathbb{C}^{q\times d_x}$ is obtained from $R_{D_x} \in \mathbb{Q} < \sigma_i, \sigma_j >^{q_D\times d_x}$ by replacing the shift operator σ_i (resp. σ_j) with the complex variable λ_1 (resp. λ_2).

Note that the static equation (*i.e.* $R_{\mathbb{Q}_x}x + R_{\mathbb{Q}_y}y = 0$) is not involved in this definition.

Dynamic control law

The dynamic control law is itself a linear model interacting with the original model (called “plant”).

Let the vecteur μ be defined by

$$\mu = \begin{pmatrix} w^T & u^T & y^T \end{pmatrix}^T = \begin{pmatrix} w^T & z^T \end{pmatrix}^T, \quad \text{avec} \quad w \in \mathbb{Q}^{d_w}. \tag{28}$$

(It involves u and y but also a state vector of the control law denoted by w.)

Dynamic control law

The here-considered control law, called « controller », can be written :

$$T\mu = 0 \tag{29}$$

où

$$T = \begin{pmatrix} T_D \\ T_{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} T_{D_w} & T_{D_u} & T_{D_y} \\ T_{\mathbb{Q}_w} & T_{\mathbb{Q}_u} & T_{\mathbb{Q}_y} \end{pmatrix} = \begin{pmatrix} \Theta_x & \Theta_z \end{pmatrix} \tag{30}$$

with $T_{D_w} \in \mathbb{Q} < \sigma_i, \sigma_j >^{l_D \times d_w}$, $T_{D_u} \in \mathbb{Q} < \sigma_i, \sigma_j >^{l_D \times d_u}$, $T_{D_y} \in \mathbb{Q} < \sigma_i, \sigma_j >^{l_D \times d_y}$, $T_{\mathbb{Q}w} \in \mathbb{Q}^{l_{\mathbb{Q}} \times d_w}$, $T_{\mathbb{Q}u} \in \mathbb{Q}^{l_{\mathbb{Q}} \times d_u}$, $T_{\mathbb{Q}_y} \in \mathbb{Q}^{l_{\mathbb{Q}} \times d_y}$, $\Theta_x \in \mathbb{Q} < \sigma_i, \sigma_j >^{(l_D + l_{\mathbb{Q}}) \times d_w}$

Closed-loop system

By applying the controller to the plant, one gets the autonomous closed-loop model

$$\mathbf{R}\nu = 0, \tag{31}$$

where $\nu = \begin{pmatrix} x^T & w^T & z^T \end{pmatrix}^T = \begin{pmatrix} x^T & \nu^T \end{pmatrix}^T$ and

$$\Leftrightarrow \mathbf{R} = \begin{pmatrix} \Pi_{D_x} & 0 & \Pi_z \\ 0 & \Theta_w & \Theta_z \end{pmatrix}. \tag{32}$$

This system is autonomous since u and y are now inner signals and no longer exogenous signals. More precisely, u is now computed through the controller and y, if still the output, is also involved in the controller, thus involved in the dynamics of the system.

Algebraic equivalence

Let two linear $R\eta = 0$ and $R'\eta' = 0$. They are algebraically equivalent (if I well understood) if and only if matrices (polynomial w.r.t. σ_i and σ_j) P, P', Q, Q', Z et Z' exist such that

$$RP = QR', \quad R'P' = Q'R,$$

$$PP' + ZR = I, \quad P'P + Z'R' = I.$$

This corresponds to the change of variables

$$\eta = P\eta', \quad \eta' = P'\eta.$$

Equivalence and control laws

Let two linear systems $R\eta = 0$ and $R'\eta' = 0$ be algebraically equivalent and let the next control law be applied to the second system :

$$T'\mu' = 0 = \begin{pmatrix} \Theta'_w & \Theta'_z \end{pmatrix} \begin{pmatrix} w' \\ u' \\ y' \end{pmatrix} = \begin{pmatrix} \Theta'_w & \Theta'_z \end{pmatrix} \begin{pmatrix} w' \\ z' \end{pmatrix} = 0.$$

If matrix P' is splitted as follows, $P' = \begin{pmatrix} P'_{xx} & P'_{xz} \\ P'_{zx} & P'_{zz} \end{pmatrix}$, then the control law is equivalent (to be proved ?) to the next law, applied to $R\eta = 0$,

$$\begin{pmatrix} \Theta_x & \Theta_w & \Theta_z \end{pmatrix} \begin{pmatrix} x \\ w \\ z \end{pmatrix} = \begin{pmatrix} \Theta'_z P'_{zx} & \Theta'_w & \Theta'_z P'_{zz} \end{pmatrix} \begin{pmatrix} x \\ w \\ z \end{pmatrix}$$

with $w = w'$.

Stability ?

A natural question is raised : Is stability preserved ?

This question is related to the previous one : Algebraic equivalence of $\mathbf{R}\nu = 0$ et $\mathbf{R}'\nu' = 0$ must be proved.

Stability ?

Consider $R\eta \simeq R'\eta'$ *i.e.*

$$\exists \left(P, P', Q, Q', Z = \begin{pmatrix} Z_x \\ Z_z \end{pmatrix}, Z' = \begin{pmatrix} Z'_x \\ Z'_z \end{pmatrix} \right) :$$

$$RP = QR', \quad R'P' = Q'R,$$
$$PP' + ZR = I, \quad P'P + Z'R' = I.$$

One needs to know if $\mathbf{R}\nu = 0 \simeq \mathbf{R}'\nu' = 0$ *i.e.* if there exist $(\mathbf{P}, \mathbf{P}', \mathbf{Q}, \mathbf{Q}', \mathbf{Z}, \mathbf{Z}')$ such that

$$\mathbf{RP} = \mathbf{QR}', \quad \mathbf{R}'\mathbf{P}' = \mathbf{Q}'\mathbf{R},$$
$$\mathbf{PP}' + \mathbf{ZR} = I, \quad \mathbf{P}'\mathbf{P} + \mathbf{Z}'\mathbf{R}' = I.$$

with

$$\mathbf{R} = \begin{pmatrix} R_{D_x} & 0 & (R_{D_u} & 0) \\ \Theta'_z P'_{zx} & \Theta'_w & \Theta'_z P'_{zz} & \end{pmatrix}, \quad \mathbf{R}' = \begin{pmatrix} R'_{D_x} & 0 & (R'_{D_u} & 0) \\ 0 & \Theta'_w & \Theta'_z & \end{pmatrix}.$$

Stability ?

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The answer is *yes !*

$$\mathbf{P} = \begin{pmatrix} P_{xx} & 0 & P_{xz} \\ 0 & I & 0 \\ P_{zx} & 0 & P_{zz} \end{pmatrix}, \quad \mathbf{P}' = \begin{pmatrix} P'_{xx} & 0 & P'_{xz} \\ 0 & I & 0 \\ P'_{zx} & 0 & P'_{zz} \end{pmatrix},$$

$$\mathbf{Z} = \begin{pmatrix} Z_x & 0 \\ 0 & 0 \\ Z_z & 0 \end{pmatrix}, \quad \mathbf{Z}' = \begin{pmatrix} Z'_x & 0 \\ 0 & 0 \\ Z'_z & 0 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} Q & 0 \\ -T' \begin{pmatrix} Z'_x \\ 0 \\ Z'_z \end{pmatrix} & I \end{pmatrix}, \quad \mathbf{Q}' = \begin{pmatrix} Q' & 0 \\ 0 & I \end{pmatrix}.$$

And since the equivalence in the sense of algebraic approach preserves stability... $\Omega E \Delta$

Fornasini-Marchesini Model

The FM model with an output equation is given by

$$\begin{cases} x(i+1,j+1) = F_1\, x(i+1,j) + F_2\, x(i,j+1) + F_3\, x(i,j) + G_1\, u(i+1,j) + G_2\, u(i,j+1) + G_3\, u(i,j), \\ y(i,j) = Hx(i,j) + Ju(i,j). \end{cases} \tag{33}$$

The Roesser model is reminded :

$$\begin{aligned} \begin{pmatrix} x'^h(i+1,j) \\ x'^v(i,j+1) \end{pmatrix} &= \underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x'^h(i,j) \\ x'^v(i,j) \end{pmatrix}}_{x'(i,j)} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} u'(i,j), \\ y'(i,j) &= \underbrace{\begin{pmatrix} C_1 & C_2 \end{pmatrix}}_{C} \begin{pmatrix} x'^h(i,j) \\ x'^v(i,j) \end{pmatrix} + D\, u'(i,j). \end{aligned} \tag{34}$$

Equivalence between FM and R ?

Thomas proved that the FM model could be equivalently transformed into a R model without the output equation in y (resp. y') :

$$R\eta = 0, \quad \text{avec}$$

$$R = \begin{pmatrix} I_{d_x}\,\sigma_i\,\sigma_j - F_1\,\sigma_i - F_2\,\sigma_j - F_3 & -G_1\,\sigma_i - G_2\,\sigma_j - G_3 \end{pmatrix}$$

is equivalent to

$$R'\eta' = 0 \quad \text{avec}$$

$$R' = \begin{pmatrix} I_{d_x}\,\sigma_i - F_2 & -(F_2\,F_1 + F_3) & -(F_2\,G_1 + G_3) & -G_2 \\ -I_{d_x} & I_{d_x}\,\sigma_j - F_1 & -G_1 & 0 \\ 0 & 0 & I_{d_u}\,\sigma_j & -I_{d_u} \end{pmatrix}$$

Equivalence between FM and R ?

To take the output equation into account, it suffices to consider $y = y'$ in addition to $\eta' = P'\eta$, which amounts to

$$\eta \leftarrow \begin{pmatrix} \eta \\ y \end{pmatrix}, \quad \eta' \leftarrow \begin{pmatrix} \eta' \\ y' = y \end{pmatrix}$$

and leads to

$$R = \begin{pmatrix} I_{d_x}\,\sigma_i\,\sigma_j - F_1\,\sigma_i - F_2\,\sigma_j - F_3 & -G_1\,\sigma_i - G_2\,\sigma_j - G_3 & 0 \\ H & J & -I_{d_y} \end{pmatrix}$$

$$R' = \begin{pmatrix} I_{d_x}\,\sigma_i - F_2 & -(F_2\,F_1 + F_3) & -(F_2\,G_1 + G_3) & -G_2 & 0 \\ -I_{d_x} & I_{d_x}\,\sigma_j - F_1 & -G_1 & 0 & 0 \\ 0 & 0 & I_{d_u}\,\sigma_j & -I_{d_u} & 0 \\ 0 & H & J & 0 & -I_{d_y} \end{pmatrix}$$

Equivalence between FM and R ?

This amounts to update the matrices defining the isomorphism as follows :

$$P \leftarrow P \oplus I_{d_y}, \quad P' \leftarrow P' \oplus I_{d_y},$$

$$Q \leftarrow \begin{pmatrix} Q & 0 \\ -\begin{pmatrix} 0 & H & J & 0 \end{pmatrix} \begin{pmatrix} Z' \\ 0 \end{pmatrix} & I_{d_y} \end{pmatrix}, \quad Q' \leftarrow Q' \oplus I_{d_y},$$

$$Z \leftarrow Z \oplus 0_{d_y}, \quad Z' \leftarrow Z' \oplus 0_{d_y}.$$

Hence, once again, $R\eta = 0 \simeq R'\eta' = 0$.

Coming back to the observer of R’

I has been seen that the observer can be written

$$\begin{pmatrix} \hat{x}'^h(i+1,j) \\ \hat{x}'^v(i,j+1) \end{pmatrix} = (A+ZC)\hat{x}'(i,j) + (B+ZD)u'(i,j) - Zy'(i,j). \tag{35}$$

to which one must add the control law $u'(i,j) = K\hat{x}(i,j)$. By considering $w' = w = \hat{x}'$, it comes

$$\Theta'_w = \begin{pmatrix} \begin{pmatrix} \sigma_i Id_h & 0 \\ 0 & \sigma_j I_{d_v} \end{pmatrix} - A - ZC \\ K \end{pmatrix},$$

$$\Theta'_z = \begin{pmatrix} (B+ZD) & -Z \\ -I_{d_u} & 0 \end{pmatrix}$$

Coming back to the observer of R'

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From Thomas's work, if the output equation is taken into account, P' becomes

$$P' = \begin{pmatrix} I_{d_x}\,\sigma_j - F_1 & -G_1 & 0 \\ I_{d_x} & 0 & 0 \\ 0 & I_{d_u} & 0 \\ 0 & I_{d_u}\,\sigma_j & 0 \\ 0 & 0 & I_{d_y} \end{pmatrix}$$

Coming back to the observer

With taking the observer into account *i.e.* considering the closed-loop, one gets

$$\mathbf{P}' = \begin{pmatrix} P'_{xx} & 0 & P_{xz} \\ 0 & I_{d_x} & 0 \\ P'_{zx} & 0 & P_{zz} \end{pmatrix} = \left(\begin{array}{c|c|cc} I_{d_x}\,\sigma_j - F_1 & 0 & -G_1 & 0 \\ I_{d_x} & 0 & 0 & 0 \\ 0 & 0 & I_{d_u} & 0 \\ \hline 0 & I_{d_x} & 0 & 0 \\ \hline 0 & 0 & I_{d_u}\,\sigma_j & 0 \\ 0 & 0 & 0 & I_{d_y} \end{array}\right)$$

Recalling that

$$\begin{pmatrix} \Theta_x & \Theta_w & \Theta_{uy} \end{pmatrix} \begin{pmatrix} x \\ w \\ z \end{pmatrix} = \begin{pmatrix} \Theta'_{uy} P'_{zx} & \Theta'_w & \Theta'_{uy} P'_{zz} \end{pmatrix} \begin{pmatrix} x \\ w \\ z \end{pmatrix}$$

it comes...

Coming back to the observer

$$\Theta_x = 0$$

$$\Theta_w = \left(\begin{pmatrix} \sigma_i Id_h & 0 \\ 0 & \sigma_j I_{d_v} \end{pmatrix} - A - ZC \atop K \right),$$

$$\Theta_z = \begin{pmatrix} \sigma_j(B+ZD) & -Z \\ -\sigma_j I_{d_u} & 0 \end{pmatrix}.$$

This corresponds (with no surprise) to the controller

$$\begin{aligned} \begin{pmatrix} \hat{x}^h(i+1,j) \\ \hat{x}^v(i,j+1) \end{pmatrix} &= (A+ZC)\hat{x}(i,j) + (B+ZD)u(i,j+1) - Zy(i,j), \\ u(i,j+1) &= K\hat{x}(i,j). \end{aligned} \tag{36}$$

Without surprise, indeed, but now, one is sure that it stabilizes the FM model.

The final controller

If the controller is expressed from the matrices of the FM model, one gets

$$\begin{aligned}\begin{pmatrix}\hat{x}^h(i+1,j)\\ \hat{x}^v(i,j+1)\end{pmatrix} &= \left(\begin{pmatrix}F_2 & F_2\,F_1+F_3 & F_2\,G_1+G_3\\ I_{d_x} & F_1 & G_1\\ 0 & 0 & 0\end{pmatrix} + Z\begin{pmatrix}0 & H & J\end{pmatrix}\right)\hat{x}(i,j)\\ &+ \begin{pmatrix}G_2\\ 0\\ I_{d_u}\end{pmatrix} u(i,j+1) - Zy(i,j),\\ u(i,j+1) &= K\hat{x}(i,j),\end{aligned} \tag{37}$$

or, under the classic form of a strictly proper controller,

$$\begin{aligned}\begin{pmatrix}\hat{x}^h(i+1,j)\\ \hat{x}^v(i,j+1)\end{pmatrix} &= \left(\begin{pmatrix}F_2 & F_2\,F_1+F_3 & F_2\,G_1+G_3\\ I_{d_x} & F_1 & G_1\\ 0 & 0 & 0\end{pmatrix} + Z\begin{pmatrix}0 & H & J\end{pmatrix} + \begin{pmatrix}G_2\\ 0\\ I_{d_u}\end{pmatrix} K\right)\hat{x}(i,j)\\ &- Zy(i,j),\\ u(i,j+1) &= K\hat{x}(i,j).\end{aligned} \tag{38}$$

The end !