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Stability Criterion for N-Dimensional Digital Filters

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***Abstract*—The stability requirement for one-dimensional recursive filters is well known. A stability theorem for n-dimensional recursive filters is proved wherein the denominator of the filter is an n-dimensional power series. A Tauberian theorem due to Wiener yields the desired result in the general case.**

I. Introduction

Linear digital filtering is a useful tool for processing discrete sequences of data [1],[2]. It is used in a variety of applications, including processing of seismic data, radar signals, cardiographic recordings, and many other "signals" which have been sampled and stored in digital form.

One of the more efficient types of digital filters is the "recursive filter" [3],[4]. For one-dimensional sequences, the recursive filter can be described by its z-transform

$$F(z) = \sum_{i=0}^{M} a_i z^i \Big/ \sum_{j=0}^{N} b_j z^j \tag{1}$$

where the a and b coefficients define the filter. In applying this filter to a data sequence, we use the recursive algorithm

$$y_n = \frac{1}{b_0}\left\{\sum_{i=0}^{M} a_i x_{n-i} - \sum_{j=1}^{N} b_j y_{n-j}\right\} \tag{2}$$

where the x_k, $k = 0,1,2,\cdots$, represent the input data sequence and the y_k, $k = 0,1,2,\cdots$, represent the output sequence. In using this algorithm, we assume that the x_k and y_k are zero for all $k < 0$.

This type of filter is used extensively in processing one-dimensional sampled data. It is also possible to extend this technique to n-dimensional data [5], [6], [10]. Such filters are useful in processing two-dimensional data, such as seismic data sections, digitized photographic data, and gravity and magnetic maps. In the case of a two-dimensional recursive filter, the filter can be described using two-dimensional polynomials or power series in (z_1,z_2), such as

$$F(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2) \tag{3}$$

where

$$A(z_1,z_2) = \sum_{i=0}^{M_1} \sum_{j=0}^{M_2} a_{ij} z_1^i z_2^j \tag{4}$$

and

$$B(z_1,z_2) = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} b_{kl} z_1^k z_2^l. \tag{5}$$

A filtering algorithm similar to (2) can be written for the two-dimensional and higher dimensional filters.

One of the problems in using recursive filters is stability. We require that the output of the filter not become unbounded if the input is bounded. The stability depends on the coefficients of the denominator of the recursive filters. In the case of the one-dimensional recursive filter, it has been shown in many places that the filter will be stable if the roots of the denominator polynomial $B(z)$ are all outside the z-plane unit circle [7]. However, these proofs all depend on our ability to factor the polynomial $B(z)$ into its distinct roots. In the case of n-dimensional polynomials or power series, no such factorization exists. Huang [11] has shown a proof for the two-dimensional case in which the $B(z_1,z_2)$ is a finite polynomial. Therefore, it is the purpose of this paper to state and prove the conditions on the denominator polynomial or power series of an n-dimensional recursive filter which will allow that filter to be stable.

II. Development

Let us begin by developing the rationale for the precise definition of stability which we shall use. It is well known that multiplication of two power series may be performed by convolving their sequences of coefficients; this is the process inherent in recursive digital filtering. We shall not distinguish between a power series and its sequence of coefficients but shall refer to a power series as a sequence, or vice versa, when convenient. The term stability of a filter is generally used to indicate that the result of convolving the filter with some bounded input sequence should have, in some sense, a bounded output. Since all of this is rather vague, let us be more precise. One of the simplest classes of power series which we might choose to work with is the class of power series in z and $1/z$ which has absolutely summable coefficients. That is, those series of the form

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

This class offers the advantage that a product (convolution) of two members of the class is again of this class. As a result, if a filter and an input sequence are chosen from this class, the output sequence must also be of this class, and so the filter is necessarily of the type we choose to think of as stable. Since the recursive filter is in general a quotient of two power series, we shall require that the two series be of this class and seek the conditions which will guarantee that the resulting quotient will again be represented by a power series in this class. Our procedure will be to use a Tauberian theorem proved by Wiener [9, p. 37] to derive the necessary criterion. Because this result does not rely on dimension, but only on the algebraic and topological structure of the class of absolutely summable sequences, we are able to derive the stability criterion for the large class of N-dimensional recursive filters. Our ultimate aim is to give the necessary and sufficient condition that, given an N-dimensional absolutely summable power series in the denominator of the filter, the filter will be stable no matter what N-dimensional absolutely summable numerator may be chosen for the recursive filter.

To simplify our work in N-dimensions, let us use the following notation.

Notation: We shall represent the integers by Z, the set of nonnegative integers by P, and the set of nonpositive integers by N.

The sequences (coefficients of power series) which we use must be indexed. We shall consider index sets which belong to the set $Z^\alpha \times P^\beta \times N^\gamma$ where α, β, γ are nonnegative integers. A zero exponent on

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a set indicates that no index takes its value in that set. We denote indices by lower case letters and will always indicate the range of an index (the set in which it takes its values) to avoid confusion.

If $S = Z^\alpha \times P^\beta \times N^\gamma$ and $n \in S$, then, by the symbol z^n, we shall mean $z_1^{n_1}, z_2^{n_2}, \cdots, z_p^{n_p}$ where each z_j is a complex number and $n = (n_1, n_2, \cdots, n_p)$ with n_j in Z, P, or N, and $p = \alpha + \beta + \gamma$. We denote by R_j the range of n_j.

We shall consider complex valued functions defined on an index set $S = Z^\alpha \times P^\beta \times N^\gamma$ and shall represent these by lower case letters subscripted by the index, or by the lower case letter with the index appearing as an argument; that is, d_n and $d(n)$ are to have the same meaning. We do this to follow familiar usage of indices for various purposes.

Let the index set $S = Z^\alpha \times P^\beta \times N^\gamma$ be chosen, where $\alpha + \beta + \gamma > 0$.

Lemma: Let the complex function $d(n)$, $n \in S$, satisfy the functional equation $d(n + m) = d(n)d(m)$. Then

$$d(n) = z^n = z_1^{n_1}, z_2^{n_2}, \cdots, z_p^{n_p}$$

for some collection of complex numbers $z_1, \cdots, z_p$.

Proof: First consider the case in which n is an integer belonging to Z, P, or N. If $d \equiv 0$, we take $z = z_1 = 0$. If $d \not\equiv 0$, then it is easy to verify that $d(n) = d(1)^n$ for all n. Setting $z_1 = d(1)$, the result follows.

We proceed to the general case and let

$$n = (n_1, n_2, \cdots, n_p) \in S$$

$$m = (m_1, m_2, \cdots, m_p) \in S$$

and suppose

$$d(n + m) = d(n)d(m) \tag{6}$$

for all choices of $n, m \in S$. Setting

$$n_2 = \cdots = n_p = m_2 = \cdots = m_p = 0,$$

we get

$$d(n_1 + m_1, 0, \cdots, 0) = d(n_1, 0, \cdots, 0)d(m_1, 0, \cdots, 0).$$

If we define

$$d(n_1, 0, \cdots, 0) = r(n_1),$$

we get

$$r(n_1 + m_1) = r(n_1)r(m_1)$$

where n_1, m_1 are integers, and so $r(n_1) = r(1)^{n_1} = z^{n_1}$ from above, where

$$z_1 = r(1) = d(1, 0, \cdots, 0).$$

Continuing in this manner, we conclude that

$$d(0, \cdots 0, n_k, 0, \cdots, 0) = z_k^{n_k}$$

where $z_k = d(0, \cdots, 0, 1, 0, \cdots, 0)$.

We now choose $m_1 = m_3 = \cdots = 0$ and $n_2 = n_3 = \cdots = 0$ in (6). We then obtain the relationship

$$d(n_1, m_2, 0, \cdots, 0) = d(n_1, 0, \cdots, 0)d(0, m_2, 0, \cdots, 0)$$

so that

$$d(n_1, m_2, 0, \cdots, 0) = z_1^{n_1} z_2^{m_2}.$$

Choosing

$$n_3 = \cdots = n_p = 0$$

$$m_1 = m_2 = m_4 = \cdots = m_p = 0,$$

we get

$$d(n_1, n_2, m_3, 0, \cdots, 0) = (z_1^{n_1} z_2^{n_2})(z_3^{m_3}) = z_1^{n_1} z_2^{n_2} z_3^{m_3}.$$

Continuing this process, we conclude

$$d(n) \equiv d(n_1, n_2, \cdots, n_p) = z_1^{n_1} z_2^{n_2}, \cdots, z_p^{n_p} = z^n$$

where the z_n are defined above.

III. Main Result

In Section II we gave the rationale for a reasonable definition of stability, and at this point we would like to make this definition precise. Our notation is that introduced in Section II. Let an index set S be chosen as $S = Z^\alpha \times P^\beta \times N^\gamma$.

Definition: $L_1(S)$ is defined to be the space of functions $b(n)$ defined on S which satisfies $\sum_{n \in S} |b(n)| < \infty$ with convolution as multiplication.

We define the z-transform $b(z)$ of a function $b(n)$ in $L_1(S)$ by

$$b(z) = \sum_{n \in S} b_n z^n = \sum_{n \in S} b_{n_1, \ldots, n_p} z_1^{n_1}, \cdots, z_p^{n_p}.$$

If $b(n) \in L_1(S)$, we shall call $1/b(z)$ stable if and only if there exists $a(n) \in L_1(S)$ such that $a(z) = 1/b(z)$ [or $a(n) * b(n) = \delta$ (convolution identity)]; that is, $b(n)$ has a convolution inverse in $L_1(S)$.

It is easy to see that this definition agrees with our previous indication of the meaning of stability since, if $1/b(z)$ has a representation as an absolutely summable power series, then it follows (previous discussion) that the filter $a(z)/b(z)$ is represented by an absolutely summable power series whenever $a(z)$ is represented by an absolutely summable power series. Hence, multiplying this filter by any absolutely convergent power series (input) results in an absolutely convergent output, and so we would agree to call the filter stable.

It is easy to verify that the space $L_1(S)$ is a commutative Banach algebra with identity (with convolution as multiplication), and it is well known that its dual is $L_\infty(S)$ [8, p. 239].

Let us state the Tauberian theorem due to Wiener in the form which we shall use (see [9, p. 37]).

Theorem: Let A be a commutative Banach algebra with identity, and let A' be its dual. An element $a \in A$ is invertible if and only if the equation $\Phi(a) = 0$ is not satisfied by any homomorphism Φ in the (topological) dual of A.

In our analysis, we have equated stability with invertibility in $L_1(S)$. We shall show that the homomorphisms in the dual of $L_1(S)$ correspond to evaluation of the z-transform of an element of $L_1(S)$ at a point in a suitable subset of a cross product of complex planes. The Tauberian theorem then says that an element $b(n)$ in $L_1(S)$ is invertible if and only if its z-transform does not vanish (equal zero) on that subset, which in turn yields our desired result.

We may now state our result in the following desired form.

Theorem (Desired Form): Let $S = Z^\alpha \times P^\beta \times N^\gamma$ where $\alpha + \beta + \gamma = p > 0$, and let $b = \{b_n\} \in L_1(S)$. Then $1/b(z)$ is stable if and only if $b(z) \neq 0$ for

$$|z_k| \begin{cases} = 1, & \text{if } R_k = Z \\ \leq 1, & \text{if } R_k = P \\ \geq 1, & \text{if } R_k = N \end{cases}$$

and $1 \leq k \leq p$, where R_k = range of the kth index.

Proof: If ψ is a continuous linear functional on $L_1(S)$, then there is a function $c(n)$ on S satisfying

$$\sup_{n \in S} |c(n)| < \infty$$

and

$$\psi(a) = \sum_{n \in S} c_n a_n$$

for all a in $L_1(S)$. This follows from the fact that the dual of $L_1(S)$ is $L_\infty(S)$. We wish to determine those continuous linear functionals ψ which are nonzero homomorphisms from $L_1(S)$ to the scalar field. Any such functional ψ must satisfy the equation

$$\psi(a*b) = \psi(a)\,\psi(b) \quad \text{(the homomorphism condition)}$$

for all functions a,b in $L_1(S)$.

We recall that

$$\psi(a*b) = \sum_{m\in S} c_m \sum_{n\in S} a_n b_{m-n} \quad \text{(from above)}$$

where $\sup_{m\in S} |c_m| < \infty$; but then,

$$\psi(a*b) = \sum_{n\in S} a_n \sum_{m\in S} c_m b_{m-n} = \sum_{n\in S} a_n \sum_{m\in S} c_{n+m} b_m \qquad (7)$$

for all functions a,b in $L_1(S)$. We now let n_0, m_0 be fixed elements of S and define

$$a(n) = \delta_n{}^{n_0}, \qquad b(m) = \delta_m{}^{m_0}$$

where δ is the Kronecker delta. Clearly, $a,b \in L_1(S)$ and

$$\psi(a*b) = c_{n_0+m_0}$$

from (7). On the other hand,

$$\psi(a)\,\psi(b) = \sum_{n\in S} a_n c_n \sum_{n\in S} b_m c_m = c_{n_0} c_{m_0};$$

so that, if ψ is a homomorphism and a,b are chosen as indicated, we obtain the following necessary condition on ψ:

$$c_{n_0+m_0} = c_{n_0} c_{m_0}$$

for all n_0, $m_0 \in S$.

By the lemma we conclude that

$$c_n = z^n = z_1{}^{n_1} z_2{}^{n_2}, \cdots, z_p{}^{n_p}$$

for some sequence of complex numbers $z_1,\cdots,z_p \neq 0$. Since we must require

$$\sup_{n\in S} |z^n| < \infty \quad (\text{since } \sup |c_n| < \infty),$$

then, in particular,

$$\sup_{n_k\in R_k} |z_k{}^{n_k}| < \infty, \qquad 1 \le k \le p,$$

from which it follows that

$$\begin{aligned} |z_k| &= 1 && \text{if } R_n = Z \\ |z_k| &\le 1 && \text{if } R_k = P \\ |z_k| &\le 1 && \text{if } R_k = N. \end{aligned} \qquad (8)$$

It follows that ψ is a continuous homomorphism from $L_1(S)$ to the scalar field if and only if $\psi = \{z^n\}$ where $|z_k|$ satisfies the conditions prescribed in (8). (It is easy to check that these necessary conditions are also sufficient.)

It follows that ψ is a continuous homomorphism on $L_1(a)$ if and only if $\psi(b) = b(z)$ for all $b \in L_1(S)$ and some z satisfying (8).

By the Tauberian theorem due to Wiener [9, p. 37], we conclude that $1/b(z)$ is stable if and only if $b(z)$ satisfies the hypotheses of the theorem.

IV. Conclusion

In the case of two-dimensional causal recursive filters first considered by Shanks *et al.* [3], [5], [6], we see that the above theorem yields the result that the filter $1/B(z_1,z_2)$ is stable if and only if $B(z_1,z_2)$ has no zeros for $|z_1| \le 1$, $|z_2| \le 1$ simultaneously (where $B(z_1,z_2)$ has absolutely summable coefficients). However, we go significantly beyond this level in allowing the filter to be defined in any number of variables and in dropping the requirement that it should be (though it might be) causal in any variable. We have further dropped the requirement that the number of nonzero coefficients of the filter denominator should be finite.

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A General Dynamic Programming Solution of Discrete-Time Linear Optimal Control Problems

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Abstract—**An optimal control problem with linear dynamics and quadratic criterion is imbedded in a family of problems characterized by both initial and terminal points. The optimal value function is jointly quadratic in initial and terminal points, and the optimal control is jointly linear. Recursive formulas for the coefficients of these functions are developed. This generalized procedure can be used to solve several versions of the problem not solvable by the standard one-ended imbedding technique. In particular, a procedure doubling the number of stages at each iteration is given for problems with time-invariant coefficients.**

I. Introduction

We consider the following problem. Choose y_i, $i = 1,\cdots,N-1$, that minimize J where

$$J = \tfrac{1}{2} \sum_{i=1}^{N-1} (x_i{}^T A_i x_i + y_i{}^T C_i y_i) \qquad (1)$$

subject to

$$x_{i+1} = G_i x_i + H_i y_i, \qquad x_1 = b_1, \qquad x_N = b_N \qquad (2)$$

where the subscript denotes the stage and superscript T denotes transpose, A_i and G_i are $n \times n$ matrices, C_i is $m \times m$, H_i is $n \times m$, x_i, b_1, and b_N are $n \times 1$ vectors, and y_i is $m \times 1$ with $m \le n$. We assume, with no loss of generality, that A_i and C_i are symmetric.

The usual dynamic programming computational algorithm for the solution of this problem ([1], [4], [5]) proceeds backwards from the terminal time and solves for the optimal control as a function of the initial stage and state. This is the case whether the terminal point is free to be chosen optimally as a function of the initial point or is specified (independently of the initial point) or is to be chosen subject to linear constraints. The optimal control is a linear function of the initial state with coefficients depending on the stage. The optimal value is a quadratic function of the state with coefficients depending on the stage. The coefficients are determined by backwards solution of Riccati and linear recurrence relations with terminal con-

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