# Towards a definition of invariance in the behavioral approach (?)

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**The elements of** $\mathcal{B}$ **are called trajectories**

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- **There exists $g \in \mathbb{N}$ and a polynomial matrix $R(\underline{s}) \in \mathbb{R}^{g \times \mathtt{w}}[\underline{s}]$ such that**

$$\mathcal{B} = \{w \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) : R(\underline{\sigma})\, w = 0\}$$

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$R(\underline{s}) \rightarrow$ **kernel representation matrix of $\mathcal{B}$**

# nD Behaviors



**A behavior $\mathcal{B}$ is autonomous if for every $w \in \mathcal{B}$:**

$$[w(t) = 0,\ \forall t \in \mathcal{P}\ ] \Rightarrow [w \equiv 0];\ \forall \mathcal{P} \in \mathbb{P}$$

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$\mathcal{B} = \ker R$ **autonomous** $\Leftrightarrow$ $R$ **has full column rank**

# Invariance (1D State Space Systems)



$\dot{x} = Ax \;\longrightarrow$ **Autonomous state space system**

**State space** $\longrightarrow \;\; \mathcal{X} = \mathbb{R}^n$ **Associated behavior** $\longrightarrow \;\; \mathcal{B} = \ker(\sigma I - A)$

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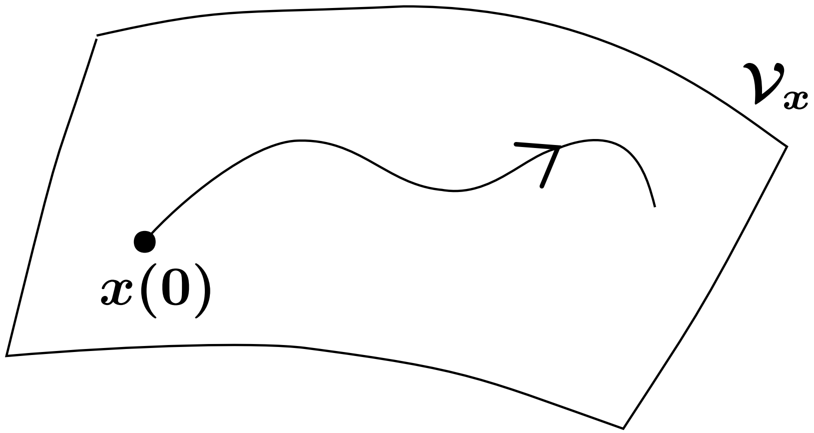

# Characterization

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**Proposition:** Consider a 1D state space system $\dot{x} = Ax$ with state space $\mathcal{X}$. Let $\mathcal{V}_{\mathcal{X}}$ be a subspace of $\mathcal{X}$. Then TFAE:

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**Elementary linear algebra**

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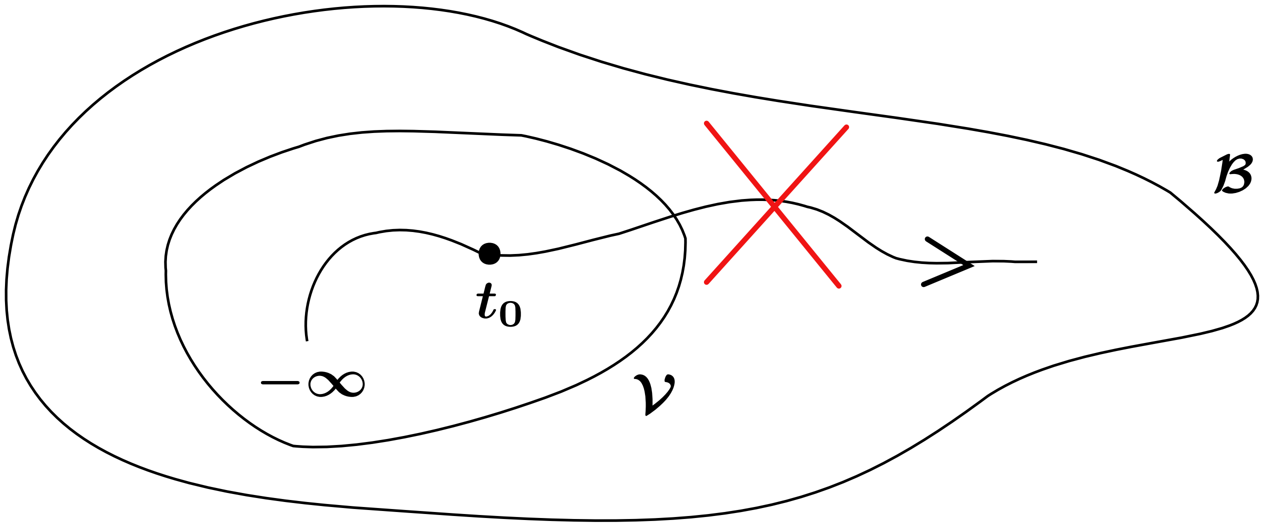


**1D case**

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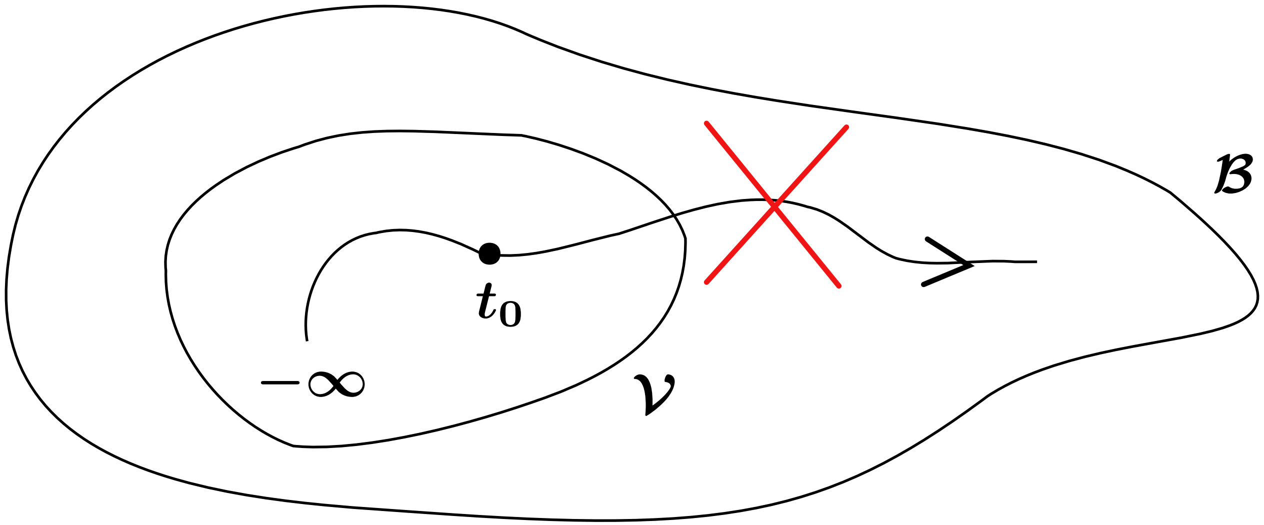


1D case

**Remark:** [Rocha & Wood 1997] $\mathcal{V}$ **hermetic sub-behavior** of $\mathcal{B}$.

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**Proposition:** Let $\mathcal{B}$ and $\mathcal{V} \subset \mathcal{B}$ be two nD behaviors. TFAE:

1. $\mathcal{V}$ is $\mathcal{B}$-invariant.
2. $\mathcal{B}/\mathcal{V}$ is autonomous.
3. If $\mathcal{B} = \ker \bar{R}V$, $\mathcal{V} = \ker V$, and $L$ is a MLA of $V$, then $\begin{bmatrix} \bar{R} \\ L \end{bmatrix}$ has full column rank.

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**Remark:** If $\mathcal{B}$ is autonomous then any sub-behavior $\mathcal{V}$ is $\mathcal{B}$-invariant.

**Given system** $\Sigma$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \sim \mathcal{B}_{(x,u)}$$

**Controller system** $\Sigma^{controller}$

$$u = -Kx \sim \mathcal{B}^{controller}_{(x,u)}$$

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**Closed-loop $x$-system** $\Sigma^{CL}$

$$\mathcal{B}^{CL}_{x} = \{x : \dot{x} = (A - BK)x\}$$

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**Aims**: Pole placement, stabilization, etc.

# Behavioral Control



| | | | |
|---|---|---|---|
| **Given behavior** | $\mathcal{B}$ | $\sim$ | $R(\underline{\sigma})w = 0$ |
| **Controller behavior** | $\mathcal{B}^{controller}$ | $\sim$ | $K(\underline{\sigma})w = 0$ |

# Behavioral Control



**Given behavior** $\mathcal{B}$ $\sim$ $R(\underline{\sigma})w = 0$

**Controller behavior** $\mathcal{B}^{controller}$ $\sim$ $K(\underline{\sigma})w = 0$

**Controlled behavior**

$$\mathcal{B}^{controlled} = \underbrace{\mathcal{B} \cap \mathcal{B}^{controller}}_{\text{interconnection}} \sim \begin{bmatrix} R(\underline{\sigma}) \\ K(\underline{\sigma}) \end{bmatrix} w = 0$$

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**Aims: To obtain a specific controlled behavior or a controlled behavior with desired properties.**

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**Aims**: To obtain a specific controlled behavior or a controlled behavior with desired properties.

**Issues**: Solvability of the control problem; regular control; partial interconnections...

$\dot{x} = Ax + Bu$ $\longrightarrow$ **state space system**

**State space** $\longrightarrow$ $\mathcal{X} = \mathbb{R}^n$

**Associated behavior** $\longrightarrow$ $\mathcal{B}_{(x,u)} = \ker[(\sigma I - A) \quad -B]$

# Controlled invariance (1D State Space Systems- very simplified!)

$\dot{x} = Ax + Bu \;\longrightarrow\;$ state space system

State space $\;\longrightarrow\; \mathcal{X} = \mathbb{R}^n$

Associated behavior $\;\longrightarrow\; \mathcal{B}_{(x,u)} = \ker[(\sigma I - A) \quad -B]$

**Definition:** A subspace $\mathcal{V}_{\mathcal{X}}$ of $\mathcal{X}$ is said to be controlled-invariant if there exists a static feedback controller $u = -Kx$ such that $\mathcal{V}_{\mathcal{X}}$ is invariant for the closed-loop system, i.e., such that $\mathcal{V}_{\mathcal{X}}$ is $(A - BK)$-invariant.

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**Issues:** Computation of the largest controlled-invariant subspace, $\mathcal{V}^*(\mathcal{F})$, contained in a given subspace $\mathcal{F}$, etc.

# Behavioral controlled invariance

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**Definition:** Let $\mathcal{B}$ be a nD behavior. A sub-behavior $\mathcal{V}$ of $\mathcal{B}$ is **controlled-invariant** if **there exists** a (regular) controller behavior $\mathcal{B}^{controller}$ such that $\mathcal{V}$ is $(\mathcal{B} \cap \mathcal{B}^{controller})$**-invariant**.

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$\mathcal{B} = \ker \bar{R}V$ $\longrightarrow$ **behavior**

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$\mathcal{V}$ **controlled-invariant** $\Leftrightarrow$

**There exists** $\mathcal{B}^{controller} = \ker KV$ **such that**

$$\ker \left( \begin{bmatrix} \bar{R} \\ K \end{bmatrix} V \right) / \ker V \text{ is autonomous...}$$

# Characterization



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$$\ker\left(\begin{bmatrix}\bar{R}\\ K\end{bmatrix} V\right)/\ker V = \ker\begin{bmatrix}\bar{R}\\ K\\ L\end{bmatrix}$$

**with $L$ MLA of $V$**

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**Thus:**

$\mathcal{V}$ **controlled-invariant** $\Leftrightarrow$

$\exists K$, **nD polynomial matrix, such that** $\begin{bmatrix}\bar{R}\\ K\\ L\end{bmatrix}$ **has full column rank**

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**Problem:** this is always true!

# 1D Observers (State Space Systems - very simplified!)

[Trentelman-Stoorvogel-Hautus 2001]

**Given system** $\Sigma$ (with state space $\mathcal{X}$) $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases} \sim \mathcal{B}_{(y,x)}$

**Observer system** $\Omega$ (with state space $\mathcal{X}$) $\left\{ \dot{\widehat{x}} = P\widehat{x} + Ry \right. \sim \widehat{\mathcal{B}}_{(y,\widehat{x})}$

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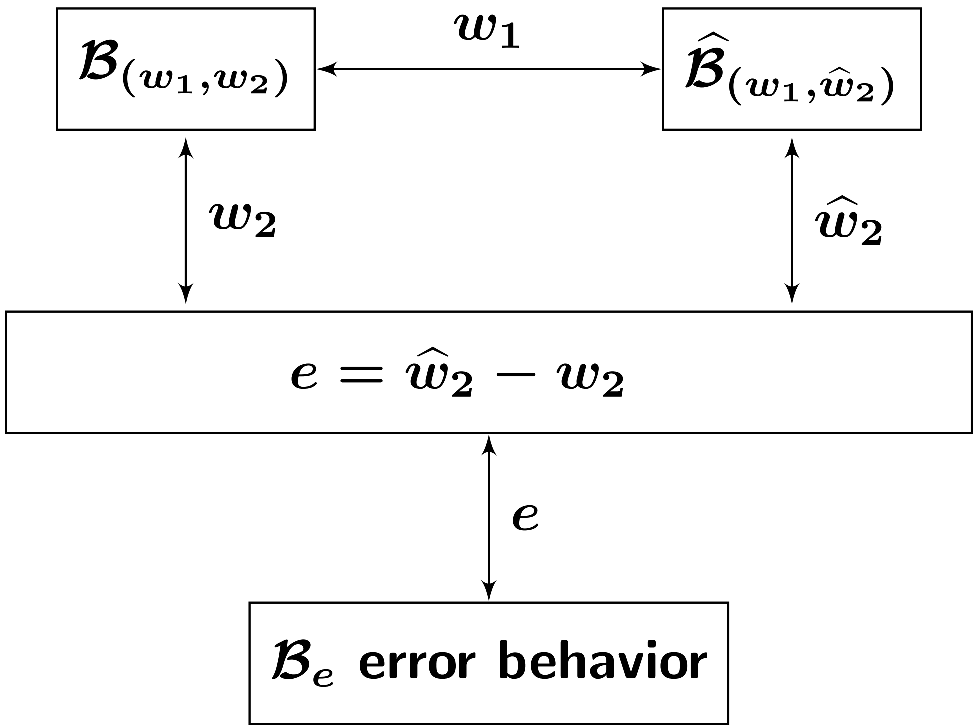
(State space) autonomy of the error system

# 1D Behavioral observers

[Valcher&Willems 1999, Trumpf-Trentelman-Willems 2011]





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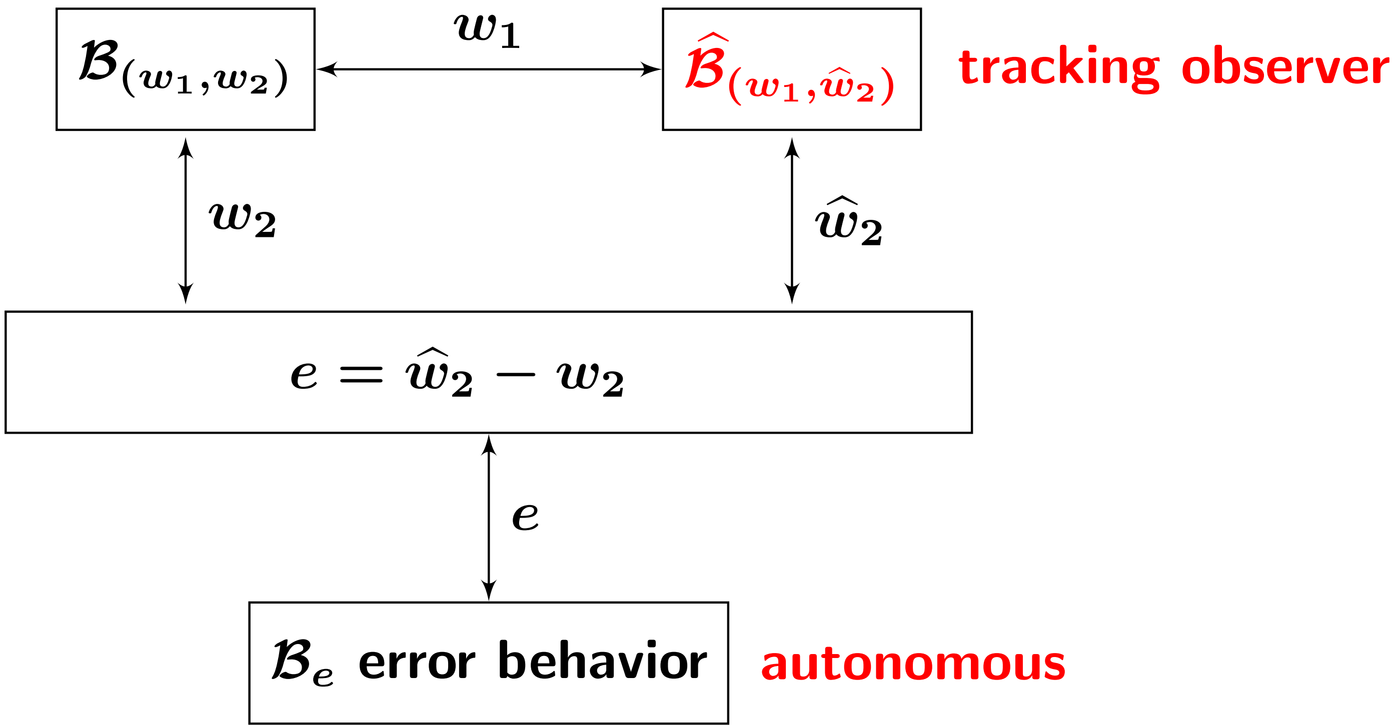

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**Remark:** $\ker R_2$ = hidden behavior of $w_2$

$$\mathcal{N}_{w_2} = \{w_2 \mid (0, w_2) \in \mathcal{B}_{(w_1,w_2)}\}$$

Hence

Trackability $\Leftrightarrow$ Autonomy of the hidden behavior

# Example: Behavioral tracking observer $\neq$ Tracking state observer



$$\mathcal{B}_{(y,x)} \;\longrightarrow\; \begin{cases} \dot{x} = x \\ y = x \end{cases} \;\longrightarrow x(t) = e^{t}x(0)$$

$$\widehat{\mathcal{B}}_{(y,\widehat{x})} \;\longrightarrow\; \dot{\widehat{x}} = 0\widehat{x} + 0y \longrightarrow \widehat{x}(t) \equiv \widehat{x}(0)$$

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**Therefore** $\widehat{x}(t) = x(t)$ **for** $t \in (-\infty, 0] \Rightarrow \widehat{x} \equiv x \equiv 0$

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**But, for** $x(t) = e^{t}$ **and** $\widehat{x}(t) \equiv 1$

$[x(0) = \widehat{x}(0)]$, **but** $[x(t) = e^{t} \neq 1 = \widehat{x}(t)$ **for** $t > 0]$

$\widehat{\mathcal{B}}_{(y,\widehat{x})}$ **is not a tracking state observer for** $\mathcal{B}_{(y,x)}$.

# Observers and trackability - nD case

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**Same definitions and results as for the 1D case.**

# Conditioned Invariance (1D State Space Systems)

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[Basile-Marro 1969]



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**Observer system** $\Omega$ (with state space $\mathcal{X}$) $\left\{ \dot{\widehat{x}} = P\widehat{x} + Ry \right. \sim \widehat{\mathcal{B}}_{(y,\widehat{x})}$

**Error system** $\Sigma_e$ with error-trajectory set $\mathcal{B}_e^{ss}$, as before

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**Definition:** $\Omega$ **observer for $x/\mathcal{V}_{\mathcal{X}}$ ($\mathcal{V}_{\mathcal{X}}$ subspace of $\mathcal{X}$) if**

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$\mathcal{V}_\mathcal{X}$ invariant subspace of $\mathcal{X}$ under the error dynamics

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# Conditioned Invariance (1D state space systems)

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| | | |
|---|---|---|
| Conditioned invariance of $\mathcal{V}_{\mathcal{X}}$ | $\Leftrightarrow$ | Existence of a state space observer s.t. $\mathcal{V}_{\mathcal{X}}$ is an invariant subspace of $\mathcal{X}$ under the corresponding error dynamics |

# Behavioral conditioned invariance

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$\mathcal{B}_{(w_1,w_2)}$ → **given nD behavior**

$\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ → **observer behavior**

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$\mathcal{V}_e$ → **sub-behavior of** $\mathcal{B}_e$

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**Definition:**

$\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ **observer for** $w_2$ **from** $w_1$ **for** $\mathcal{B}_{(w_1,w_2)}$ **modulo** $\mathcal{V}_e$ **if**

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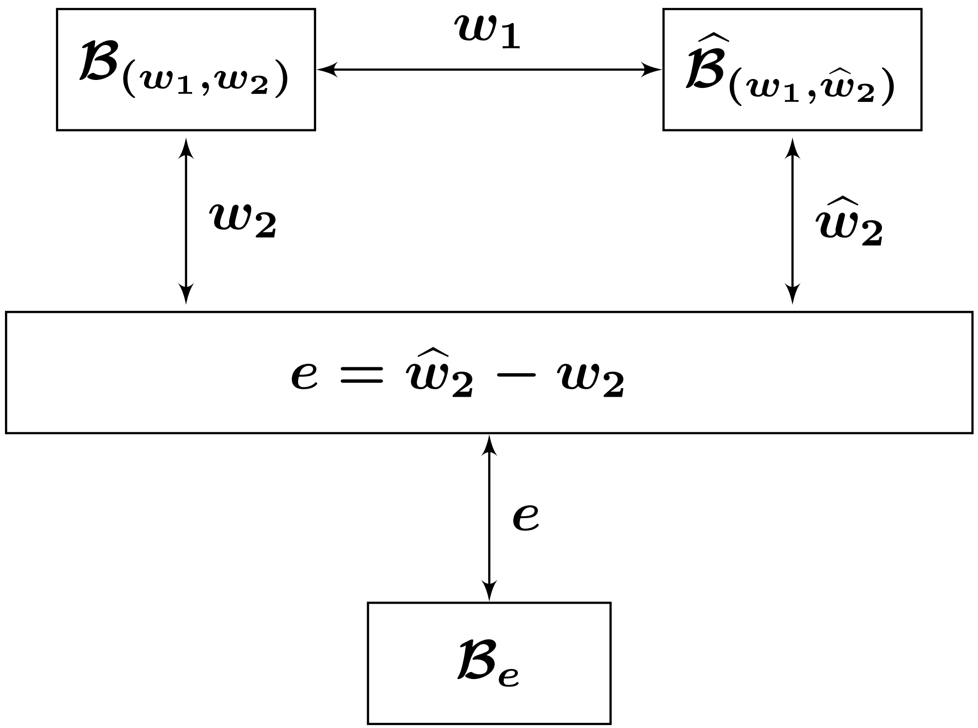
$\mathcal{V}_e$ is $\mathcal{B}_e$-invariant

# Behavioral conditioned invariance - Diagram





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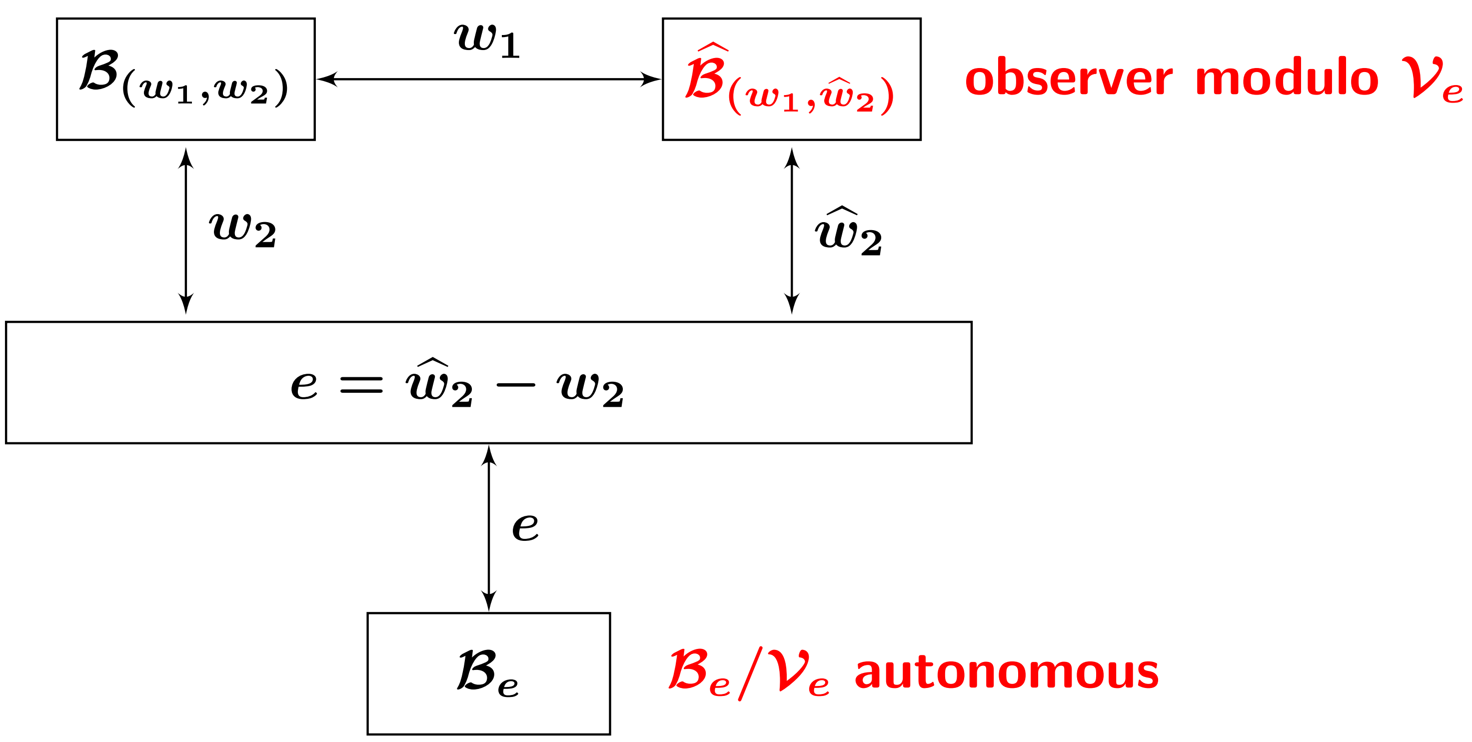

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A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is conditioned invariant if a (behavioral) observer for $w_2$ from $w_1$ for $\mathcal{B}_{(w_1,w_2)}$ modulo $\mathcal{V}$ exists.

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| Conditioned invariance of $\mathcal{V}$ $\Leftrightarrow$ | Existence of a behavioral observer with error behavior $\mathcal{B}_e$ s.t. $\mathcal{V} \subset \mathcal{B}_e$ is $\mathcal{B}_e$-invariant |

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**Example (1D case)** $\mathcal{B}_{(w_1,w_2)} \sim (\sigma+1)[\sigma+2 \quad \sigma+3]w_2 = w_1$

**Hidden behavior:** $\mathcal{N}_{w_2} \sim (\sigma+1)[\sigma+2 \quad \sigma+3]w_2 = 0$

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**Proposition:**

$\mathcal{B}_{(w_1,w_2)} \longrightarrow$ nD behavior

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**Remark:** The sufficient conditions are not necessary.

# Conditioned invariance - Summary



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| Observer (tracking) | Error system state space autonomous | Error behavior autonomous |
| Existence of observers (trackability) | Always<br>Just take $\Omega = \Sigma$ | Hidden behavior autonomous |
| Observer modulo $\mathcal{V}$ | $\mathcal{V}$ subspace of state space<br>$\mathcal{V}$ is an invariant subspace w.r.t. the error dynamics | $\mathcal{V}$ sub-behavior of $\mathcal{B}_e$<br>$\mathcal{V}$ is behaviorally invariant under the error dynamics ($\mathcal{B}_e$-invariant) |
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**Thank you!**