Fractional representation approach to analysis and synthesis problems

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ANR MSDOS, Poitiers, 01/07/14

Introduction

- The fractional representation approach to analysis and synthesis problems developed by Desoer, Callier, Vidyasagar, Francis... was successful for finite-dimensional systems.

M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, MIT Press, 1985.

- However, it is still in progress for infinite-dimensional systems.

R. F. Curtain, H. J. Zwart, *An Introduction to ∞-Dimensional Linear Systems Theory*, TAM 21, Springer, 1991.

as well as for multidimensional systems:

Z. Lin, "Output feedback stabilizability and stabilization of linear nD systems", in *Multidimensional Signals, Circuits and Systems*, chapter 4, Taylor & Francis, 2001, 59 -76.

Transfer functions

- Ordinary differential equation:

$$\dot{z}(t) = z(t) + u(t), \;\; z(0) = 0 \quad \Rightarrow \; \widehat{z}(s) = \frac{1}{(s-1)}\,\widehat{u}(s).$$

- Differential time-delay equation:

$$\left\{ \begin{array}{l} \dot{z}(t) = z(t) + u(t), \;\; x(0) = 0, \\ y(t) = \left\{ \begin{array}{ll} 0, & 0 \leq t < h, \\ z(t-h), & t \geq h, \end{array} \right. \end{array} \right. \quad \Rightarrow \; \widehat{y}(s) = \frac{e^{-h\,s}}{(s-1)}\,\widehat{u}(s).$$

- Partial differential equation:

$$\left\{ \begin{array}{l} \dfrac{\partial^2 z}{\partial t^2}(x,t) - a^2\,\dfrac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \quad \dfrac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x},t), \end{array} \right. \quad \Rightarrow \; \widehat{y}(s) = \frac{\left(e^{-\frac{\overline{x}}{a}\,s} - e^{-\frac{(2\,l-\overline{x})\,s}{a}}\right)}{\left(1 - e^{-\frac{2\,a}{l}\,s}\right)}\,\widehat{u}(s).$$

Examples of transfer functions

- Heat equation:

$$\left\{\begin{array}{l} \frac{\partial z}{\partial t}(x,t) - \lambda^2 \frac{\partial^2 z}{\partial x^2}(x,t) = 0, \\ z(x,0) = 0, \\ z(0,t) = u(t), \quad z(l,t) = 0, \\ y(t) = z(\overline{x},t), \end{array}\right. \Rightarrow \widehat{y}(s) = \frac{\left(e^{\lambda\,(l-\overline{x})\,\sqrt{s}} - e^{-\lambda\,(l-\overline{x})\,\sqrt{s}}\right)}{\left(e^{\lambda\,l\,\sqrt{s}} - e^{-\lambda\,l\,\sqrt{s}}\right)}\,\widehat{u}(s).$$

- Telegraph equation:

$$\left\{\begin{array}{l} \frac{\partial^2 z}{\partial t^2}(x,t) - a^2 \frac{\partial^2 z}{\partial x^2}(x,t) - k\,z(x,t) = 0, \\ z(x,0) = 0, \quad \frac{\partial z}{\partial t}(x,0) = 0, \\ z(0,t) = u(t), \quad \lim_{x\to+\infty} z(x,t) = 0, \\ y(t) = z(\overline{x},t), \end{array}\right. \Rightarrow \widehat{y}(s) = e^{\frac{-\sqrt{s^2-k}}{a}\,\overline{x}}\,\widehat{u}(s).$$

Example: An electric transmission line

$$\begin{cases} \dfrac{\partial V}{\partial x}(x,t) + L\,\dfrac{\partial I}{\partial t}(x,t) + R\,I(x,t) = 0, \\ \dfrac{\partial I}{\partial x}(x,t) + C\,\dfrac{\partial V}{\partial t}(x,t) + G\,V(x,t) = 0, \\ V(x,0) = 0, \quad I(x,0) = 0, \\ V(0,t) = u(t), \quad \lim_{x \to +\infty} V(x,t) = 0, \\ V(\overline{x},t) = y_1(t), \quad I(\overline{x},t) = y_2(t), \end{cases}$$

$$\Rightarrow \begin{pmatrix} \widehat{y}_1(s) \\ \widehat{y}_2(s) \end{pmatrix} = \begin{pmatrix} e^{-\sqrt{(L\,s+R)\,(C\,s+G)}\,\overline{x}} \\ \sqrt{\dfrac{C\,s+G}{L\,s+R}}\; e^{-\sqrt{(L\,s+R)\,(C\,s+G)}\,\overline{x}} \end{pmatrix} \widehat{u}(s).$$

Discrete systems & filters

- $\mathcal{Z}$-transform: $\forall \, (x_n)_{n\in\mathbb{Z}}$: $\quad \mathcal{Z}((x_n)_{n\in\mathbb{Z}})(z) = \sum_{n\in\mathbb{Z}} x_n \, z^{-n}$.

- Let us consider $(y_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ satisfying:

$$\begin{cases} y_{n+2} - 3\, y_{n+1} + 2\, y_n = 2\, u_{n+1} + 2\, u_n, \\ y_0 = 0, \\ y_1 = 0. \end{cases}$$

$$\Rightarrow \; \mathcal{Z}(y)(z) = \frac{2\,(z^{-1}+1)}{z^{-2}-3\,z^{-1}+2}\, \mathcal{Z}(u)(z) = \frac{2\,(z^2+z)}{2\,z^2-3\,z+1}\, \mathcal{Z}(u)(z).$$

- $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$,

$$\Rightarrow \; \mathbf{k} - \mathbf{j} = (k_1 - j_1, \ldots, k_n - j_n).$$

- n-D filters: $y(\mathbf{k}) = (h \star u)(\mathbf{k}) = \sum_{\mathbf{j}\in\mathbb{Z}^n} h(\mathbf{k}-\mathbf{j})\, u(\mathbf{j})$.

$$\mathcal{Z}((h(\mathbf{k}))_{\mathbf{k}\in\mathbb{Z}^n}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} h(\mathbf{k})\, z^{-\mathbf{k}}, \quad z^{-\mathbf{k}} := z_1^{-k_1} \ldots z_n^{-k_n}.$$

The fractional representation approach

• (Zames) The set of transfer functions has the structure of an algebra (parallel $+$, serie $\circ$, proportional feedback . by $\mathbb{R}$).

• (Vidyasagar) Let A be an algebra of stable transfer functions with a structure of an integral domain ($a\,b = 0 \;\Rightarrow\; a = 0 \vee b = 0$).

$Q(A) = \{p = n/d \mid 0 \neq d,\, n \in A\}$ represents the class of systems.

$$\boxed{p \in A \;\Leftrightarrow\; p \text{ is stable}, \quad p \in K \setminus A \;\Leftrightarrow\; p \text{ is unstable}}$$

• (Zames) The algebra A should be a normed algebra so that the errors in the modelization & approximation of the real plant by the mathematical model can be considered

(e.g., Banach algebra: $\| a\,b \|_A \leq \| a \|_A \, \| b \|_A, \quad \| 1 \|_A = 1$).

Example: Hardy algebra $H^\infty(\mathbb{C}_+)$

- Let us define the right half plane $\mathbb{C}_+ = \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 0\}$.

- The Hardy algebra $H^\infty(\mathbb{C}_+)$ (Banach algebra) is defined by:

$$H^\infty(\mathbb{C}_+) = \{\text{holomorphic fcts in } \mathbb{C}_+ \mid \; \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty\}.$$

- The Hardy algebra $H^\infty(\mathbb{C}_+)$ is the algebra of transfer functions of $L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+)$-stable shift-invariant ∞-dimensional systems.

- $RH_\infty = \mathbb{R}(s) \cap H^\infty(\mathbb{C}_+)$

$$= \left\{ \frac{n}{d} \;\middle|\; 0 \neq d,\, n \in \mathbb{R}[s],\ \deg n \leq \deg d,\ d(s_\star) = 0 \Rightarrow \mathrm{Re}\, s_\star < 0 \right\}$$

is the algebra of exponentially-stable finite-dimensional plants.

Example: Wiener algebra

- $L^1(\mathbb{R}_+) = \{f : [0, +\infty[\to \mathbb{R} \mid \; \| f \|_1 = \int_0^{+\infty} |f(t)| \, dt < +\infty\}$,

$$l^1(\mathbb{Z}_+) = \{a : \mathbb{Z}_+ = \{0, 1, \ldots\} \to \mathbb{R} \mid \; \| (a_i)_{i \in \mathbb{Z}_+} \|_1 = \sum_{i=0}^{+\infty} |a_i| < +\infty\}.$$

- The Wiener algebra $\mathcal{A}$ is defined by:

$$\mathcal{A} = \{f = g + \textstyle\sum_{i=0}^{+\infty} a_i \, \delta_{(t-h_i)} \mid \; g \in L^1(\mathbb{R}_+), \; (a_i)_{i \in \mathbb{Z}_+} \in l^1(\mathbb{Z}_+), \; 0 = h_0 \leq h_1 \leq h_2 \leq \cdots\}.$$

- $\mathcal{A}$ is a Banach algebra w.r.t. $\| f \|_{\mathcal{A}} = \| g \|_1 + \| (a_i)_{i \in \mathbb{Z}_+} \|_1$.
- $\hat{A} = \{\mathcal{L}(f) \mid f \in \mathcal{A}\}$, $\quad \| \widehat{f} \|_{\widehat{\mathcal{A}}} = \| f \|_{\mathcal{A}}$.
- $\mathcal{A}$ is the algebra of $L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$-stable shift-invariant ∞-dimensional systems.

Example: structural stabilizable n-D systems

- $\overline{\mathbb{D}}^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \forall\, i = 1, \ldots, n, \; |z_i| \leq 1\}$ the closed unit polydisc of $\mathbb{C}^n$.

- The ring of structural stabilizable n-D systems:

$$\mathbb{R}(z)_S := \left\{ \frac{r}{s} \mid 0 \neq s, \; r \in \mathbb{R}[z_1, \ldots, z_n], \; s(\underline{z}) = 0 \Rightarrow \underline{z} \notin \overline{\mathbb{D}}^n \right\}.$$

- $\mathcal{Z}(h(\mathbf{k})_{\mathbf{k} \in \mathbb{Z}^n}) \in A$ implies the BIBO stability of the filter

$$u \in l^\infty(\mathbb{Z}^n) \longmapsto y = h \star u \in l^\infty(\mathbb{Z}^n),$$

i.e., we have $(h(\mathbf{k})_{\mathbf{k} \in \mathbb{Z}^n}) \in l^1(\mathbb{Z}^n)$, i.e.:

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |h(\mathbf{k})| < +\infty.$$

- M. Benidir, M. Barret, *Stabilité des filtres et des systèmes linéaires*, Dunod, 1999.

Examples

- RH_∞ (algebra of exponentially-stable finite-dimensional plants):

$$p = \frac{1}{s-1} = \frac{\left(\frac{1}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{1}{s+1}, \quad \frac{s-1}{s+1} \in RH_\infty \;\Rightarrow\; p \in Q(RH_\infty).$$

- $\hat{\mathcal{A}}$ (algebra of BIBO-stable ∞-dimensional plants):

$$p = \frac{e^{-h\,s}}{s-1} = \frac{\left(\frac{e^{-h\,s}}{s+1}\right)}{\left(\frac{s-1}{s+1}\right)}, \quad \frac{e^{-h\,s}}{s+1}, \quad \frac{s-1}{s+1} \in \hat{\mathcal{A}} \;\Rightarrow\; p \in Q(\hat{\mathcal{A}}).$$

- $H^\infty(\mathbb{C}_+)$ (algebra of $L^2(\mathbb{R}_+)$-stable ∞-dimensional plants):

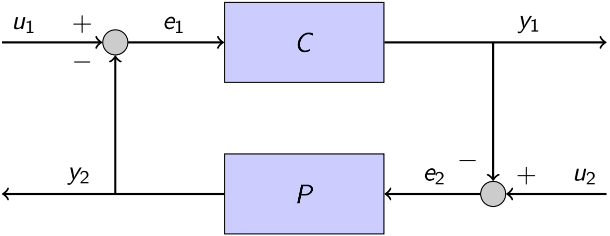
$$p = \frac{(1+e^{-2\,s})}{(1-e^{-2\,s})} \in Q(H^\infty(\mathbb{C}_+)) : \; 1+e^{-2\,s},\; 1-e^{-2\,s} \in H^\infty(\mathbb{C}_+).$$

- $R(z)_S$ (algebra of structural stable filters):

$$p = \frac{2\,z\,(z+1)}{2\,z^2-3\,z+1} = \frac{r}{s}, \quad r = \frac{2\,z\,(z+1)}{(z+2)^2}, \quad s = \frac{2\,z^2-3\,z+1}{(z+2)^2}.$$

Internal stabilization

- Let A be an algebra of stable transfer function, $K := Q(A)$.
- Let $P \in K^{q \times r}$ be a plant and $C \in K^{r \times q}$ a controller.



$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P, C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \; H(P, C) := \begin{pmatrix} I_q & P \\ C & I_r \end{pmatrix}^{-1}.$$

- Definition: $P \in K^{q \times r}$ is internally stabilizable iff there exists a stabilizing controller $C \in K^{r \times q}$, i.e., such that:

$$H(P, C) = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ -C\,(I_q - P\,C)^{-1} & I_r + C\,(I_q - P\,C)^{-1}\,P \end{pmatrix} \in A^{(q+r) \times (q+r)}.$$

Idempotents of $A^{(q+r)\times(q+r)}$

- Lemma: $P \in K^{q\times r}$ is stabilized by $C \in K^{r\times q}$ iff $1 \Leftrightarrow 2$ holds:

1. The matrix

$$\Pi_C = \begin{pmatrix} (I_q - P\,C)^{-1} & -(I_q - P\,C)^{-1}\,P \\ C\,(I_q - P\,C)^{-1} & -C\,(I_q - P\,C)^{-1}\,P \end{pmatrix}$$

satisfies $\Pi_C^2 = \Pi_C \in A^{(q+r)\times(q+r)}$.

2. The matrix

$$\Pi_P = \begin{pmatrix} -P\,(I_r - C\,P)^{-1}\,C & P\,(I_r - C\,P)^{-1} \\ -(I_r - C\,P)^{-1}\,C & (I_r - C\,P)^{-1} \end{pmatrix}$$

satisfies $\Pi_P^2 = \Pi_P \in A^{(q+r)\times(q+r)}$.

- We have $\Pi_C + \Pi_P = I_{q+r}$: complementary idempotents.

$$\begin{pmatrix} e_1 \\ y_1 \end{pmatrix} = \Pi_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} y_2 \\ e_2 \end{pmatrix} = \Pi_P \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Characterization of stabilizability

$$\Pi_C^2 = \Pi_C \in A^{(q+r)\times(q+r)} \Rightarrow \Pi_C\, A^{(q+r)\times 1} \oplus \ker_A(\Pi_C .) = A^{q+r}.$$

$$\Pi_P^2 = \Pi_P \in A^{(q+r)\times(q+r)} \Rightarrow A^{1\times(q+r)}\, \Pi_P \oplus \ker_A(.\Pi_P) = A^{1\times(q+r)}.$$

- Definition: A finitely generated (f.g.) A-module M is projective if there exist $r \in \mathbb{Z}_{\geq 0}$ and an A-module P such that $M \oplus P \cong A^r$.

- The f.g. A-module $M_C := \Pi_C\, A^{(q+r)\times 1}$ is projective of rank q.
- The f.g. A-module $M_P := A^{1\times(q+r)}\, \Pi_P$ is projective of rank r.

- Theorem (A.Q. 06): P is internally stabilizable iff 1 ⇔ 2 holds:
  1. $\mathcal{L} := (I_q \quad -P)\, A^{(q+r)\times 1}$ is a projective lattice of K^q isomorphic to $M_C := \Pi_C\, A^{(q+r)\times 1}$.
  2. $\mathcal{M} := A^{1\times(q+r)}\, (P^T \quad I_r^T)^T$ is a projective lattice of $K^{1\times r}$ isomorphic to $M_P := A^{1\times(q+r)}\, \Pi_P$.

Characterization of stabilizability

- Let $C \in K^{r\times q}$ be a stabilizing controller of $P \in K^{q\times r}$ and:
  - $S_o = (I_q - P\,C)^{-1}$ is the output sensitivity transfer matrix.
  - $U = C\,(I_q - P\,C)^{-1} = (I_r - C\,P)^{-1}\,C$.
  - $S_i = (I_r - C\,P)^{-1}$ is the input sensitivity transfer matrix.

$$\begin{array}{rcl} \mathcal{L} = (I_q \quad -P)\,A^{(q+r)\times 1} & \xrightarrow{\iota_1} & M_C = \Pi_C\,A^{(q+r)\times 1} \\ \eta_1 = (I_q \quad -P)\,\xi & \longmapsto & (S_o^T \quad U^T)^T\,\eta_1 = \Pi_C\,\xi, \end{array}$$

$$\begin{array}{rcl} \mathcal{M} = A^{1\times(q+r)}\,(P^T \quad I_r^T)^T & \xrightarrow{\iota_2} & M_P = A^{1\times(q+r)}\,\Pi_P \\ \eta_2 = \mu\,(P^T \quad I_r^T)^T & \longmapsto & \eta_2\,(-S_i \quad U) = \mu\,\Pi_P. \end{array}$$

- ι_1 and ι_2 are two injective A-linear maps since:

$$S_o - P\,U = I_q \quad \Rightarrow \quad \eta_1 = (S_o - P\,U)\,\eta_1 = (I_q \quad -P)\,\iota_1(\eta_1),$$

$$S_i - U\,P = I_r \quad \Rightarrow \quad \eta_2 = \eta_2\,(S_i - U\,P) = -\iota_2(\eta_2)\,(I_r^T \quad P^T)^T.$$

$$\Rightarrow \mathcal{L} \cong \iota_1(\mathcal{L}) = M_C, \quad \mathcal{M} \cong \iota_2(\mathcal{M}) = M_P.$$

Coprime factorization

• Definition: A transfer matrix $P \in K^{q \times r}$ admits a left-coprime factorization if there exist $D \in A^{q \times q}$, $\det D \neq 0$, $N \in A^{q \times r}$, $X \in A^{q \times q}$ and $Y \in A^{r \times q}$ such that

$$P = D^{-1} N, \quad D X - N Y = I_q.$$

• Definition: A transfer matrix $P \in K^{q \times r}$ admits a right-coprime factorization if there exist $\widetilde{D} \in A^{r \times r}$, $\det \widetilde{D} \neq 0$, $\widetilde{N} \in A^{q \times r}$, $\widetilde{X} \in A^{r \times r}$, $\widetilde{Y} \in A^{r \times q}$, such that:

$$P = \widetilde{N} \widetilde{D}^{-1}, \quad -\widetilde{Y} \widetilde{N} + \widetilde{X} \widetilde{D} = I_r.$$

• Definition: A transfer matrix $P \in K^{q \times r}$ admits a doubly-coprime factorization if it admits a left and a right coprime factorization $P = D^{-1} N = \widetilde{N} \widetilde{D}^{-1}$ such that:

$$\begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r}.$$

Coprime factorization

- Definition: A f.g. A-module M is free if there exists $r \in \mathbb{Z}_{\geq 0}$ such that $M \cong A^r$, i.e., amits a basis.

- If $P = D^{-1} N$ is a left coprime factorization of $P \in K^{q \times r}$ with

$$(D \quad -N) \begin{pmatrix} X \\ Y \end{pmatrix} = I_q \quad \Rightarrow \; (D \quad -N) \, A^{(q+r)\times 1} = A^q.$$

$$\Rightarrow \quad \mathcal{L} = (I_q \quad -P) \, A^{(q+r)\times 1} = (I_q \quad -D^{-1} N) \, A^{(q+r)\times 1}$$

$$= D^{-1} ((D \quad -N) \, A^{(q+r)\times 1}) = D^{-1} \, A^{q \times 1},$$

$\Rightarrow$ the columns of D^{-1} form a basis of $\mathcal{L}$, i.e., $\mathcal{L}$ is a free lattice.

- If $\mathcal{L}$ is lattice, then there exists $U \in K^{q \times q}$, $\det U \neq 0$, such that $\mathcal{L} = (I_q \quad -P) \, A^{(q+r)\times 1} = U \, A^q$, i.e.,

$$\exists \; D \in A^{q \times q}, \quad N \in A^{q \times r}, \qquad X \in A^{q \times q}, \qquad Y \in A^{r \times q} :$$

$$I_q = U \, D, \qquad P = U \, N, \qquad U = I_q \, X - P \, Y$$

$$\Rightarrow U^{-1} = D, \quad P = D^{-1} \, N, \quad D \, X - N \, Y = I_q.$$

Coprime factorization

- Proposition (A.Q. 06): $P \in K^{q \times r}$ admits a left-coprime factorization iff there exists $D \in A^{q \times q}$ such that $\det D \neq 0$ and

$$\mathcal{L} := (I_q \quad -P)\, A^{q+r} = D^{-1}\, A^q,$$

i.e., iff $\mathcal{L}$ is a free lattice of K^q, namely, $\mathcal{L} \cong A^q$.

- Proposition (A.Q. 06): $P \in K^{q \times r}$ admits a right-coprime factorization if there exists $\widetilde{D} \in A^{r \times r}$ such that $\det \widetilde{D} \neq 0$ and

$$\mathcal{M} := A^{1 \times (q+r)} \begin{pmatrix} P \\ I_r \end{pmatrix} = A^{1 \times r}\, \widetilde{D}^{-1},$$

i.e., iff $\mathcal{M}$ is a free lattice of $K^{1 \times r}$, namely, $\mathcal{M} \cong A^{1 \times r}$.

- Remark: A f.g. free A-module M is projective ($M \oplus 0 \cong A^r$).
- Question: When does a f.g. projective A-module free?

Parametrization of all the stabilizing controllers

• Theorem (A.Q. 06): Let $P \in K^{q \times r}$ be an internally stabilizable plant. Then, all stabilizing controllers of P have the form

$$C(Q) = (U + \Lambda)\,(S_o + P\,\Lambda)^{-1} = (S_i + \Lambda\,P)^{-1}\,(U + \Lambda), \quad (\star)$$

where C_* is any stabilizing controller of P,

$$\begin{cases} S_o = (I_q - P\,C_*)^{-1} \in A^{q \times q}, \\ U = C_*\,(I_q - P\,C_*)^{-1} = (I_r - C_*\,P)^{-1}\,C_* \in A^{r \times q}, \\ S_i = (I_r - C_*\,P)^{-1} \in A^{r \times r}, \end{cases}$$

and Q is any matrix which belongs to the projective A-module

$$\begin{aligned} \Omega &= \{\Lambda \in A^{r \times q} \mid \Lambda\,P \in A^{r \times r},\; P\,\Lambda \in A^{q \times q}, P\,\Lambda\,P \in A^{q \times r}\}, \\ &= (-U \quad S_i)\,A^{(q+r)\times(q+r)}\,(S_o^T \quad U^T)^T. \end{aligned}$$

and satisfies $\det(S_o + P\,\Lambda) \neq 0$ and $\det(S_i + \Lambda\,P) \neq 0$.

Youla-Kučera parametrization

- Corollary: Let $P \in K^{q \times r}$ be a transfer matrix which admits a doubly coprime factorization:

$$\begin{cases} P = D^{-1}\, N = \widetilde{N}\, \widetilde{D}^{-1}, \\ \begin{pmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{pmatrix} \begin{pmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{pmatrix} = I_{q+r}. \end{cases}$$

Then, we have:

$$S_o = X\, D, \quad U = Y\, D = \widetilde{D}\, \widetilde{Y}, \quad S_i = \widetilde{D}\, \widetilde{X}.$$

Moreover, the set Ω of free parameters of $(\star)$ satisfies:

$$\Omega = \widetilde{D}\, A^{r \times q}\, D, \text{ i.e., } \Lambda = \widetilde{D}\, Q\, D, \; Q \in A^{r \times q}.$$

Then all stabilizing controllers of P have the form

$$C(Q) = (Y + \widetilde{D}\, Q)\,(X + \widetilde{N}\, Q)^{-1} = (\widetilde{X} + Q\, N)^{-1}\,(\widetilde{Y} + Q\, D),$$

for all $Q \in A^{r \times q}$ s.t. $\det(X + \widetilde{N}\, Q) \neq 0$ and $\det(\widetilde{X} + Q\, N) \neq 0$.

Z. Lin's conjecture

- Z. Lin's problems (1998):

1. Determine whether or not an internally stabilizable n-D linear system defined by $P \in \mathbb{R}(z_1, \ldots, z_n)^{q \times r}$ admits a doubly coprime factorization over A.

2. Parametrize all stabilizing controllers for an internally stabilizable n-D system $P \in \mathbb{R}(z_1, \ldots, z_n)^{q \times r}$.

- Z. Lin's conjecture: If P is internally stabilizable, then P admits a doubly coprime factorization.

Z. Lin's conjecture is true

- Theorem: (Deligne; Kamen-Khargonekar-Tannenbaum (ACAP 84), Byrnes-Spong-Tarn (MST 84)):

The ring of structural stabilizable n-D systems

$$\mathbb{R}(z)_S = \left\{ \frac{r}{s} \mid 0 \neq s,\ r \in \mathbb{R}[z_1, \ldots, z_n],\ s(\underline{z}) = 0 \Rightarrow \underline{z} \notin \overline{\mathbb{D}}^n \right\}.$$

is projective-free, i.e., finitely generated projective A-modules are free.

- Corollary (Q. 06): Z. Lin's conjecture is true.

- Remark: Deligne's proof is not constructive!

Neutral time-delay systems

- $0 < h_1 < \ldots < h_n$ rationally independent set of reals.
- $\partial\, x(t) = \dot{x}(t), \quad \delta_i\, x(t) = x(t - h_i),\ i = 1, \ldots, n.$
- Differential time-delay linear systems:

$$D(\delta_1, \ldots, \delta_n)\, \frac{dx(t)}{dt} = A(\delta_1, \ldots, \delta_n)\, x(t) + B(\delta_1, \ldots, \delta_n)\, u(t).$$

$$\Rightarrow\ \frac{dx(t)}{dt} = (D(\delta)^{-1}\, A(\delta))\, x(t) + (D(\delta)^{-1}\, B(\delta))\, u(t).$$

- $\mathcal{L}(D(\delta_1, \ldots, \delta_n) = D(e^{-h_1\, s}, \ldots, e^{-h_n\, s})$, where:

$$\forall\ i = 1, \ldots, n, \quad |e^{-h_i\, s}| \leq 1.$$

- Uniform asymptotical stability independent of delay

$\Rightarrow\ \forall\ h_i \geq 0,\ i = 1, \ldots, n,\ \forall\ s \in \mathbb{C}_+,\ \det(D(e^{-h_1\, s}, \ldots, e^{-h_n\, s})) \neq 0$

$\Rightarrow$ systems over the ring $\mathbb{R}(\delta)_S$ ($D = \mathbb{R}(\delta)_S[\frac{d}{dt}]$):

$$\dot{x}(t) = A'(\delta)\, x(t) + B'(\delta)\, u(t),\ A'(\delta) \in \mathbb{R}(z)_S^{n \times n},\ B \in \mathbb{R}(z)_S^{n \times m}.$$

Scientific program

- Participants: Yacine, Thomas, Hugues, Alban.
- Based on computer algebra techniques

  (Gröbner bases, CAD, computational algebraic geometry, ...),

constructive module theory and constructive homological algebra, we want to study the following problems:

1. For $d \in \mathbb{R}[z_1, \ldots, z_n]$, study the semi-algebraic set:

$$E(d) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid d(z_1, \ldots, z_n) = 0, \ |z_i| \leq 1, \ i = 1, \ldots, n\}.$$

If $z_k = a_k + i\, b_k, \quad a_k, \, b_k \in \mathbb{R}$, the problem is equivalent to:

$$\begin{cases} \Re(d(a_1 + i\, b_1, \ldots, a_n + i\, b_n)) = \alpha(a_1, b_1, \ldots, a_n, b_n) = 0, \\ \Im(d(a_1 + i\, b_1, \ldots, a_n + i\, b_n)) = \beta(a_1, b_1, \ldots, a_n, b_n) = 0, \\ a_k^2 + b_k^2 - 1 \leq 0, \quad k = 1, \ldots, n. \end{cases}$$

In particular, effectively check whether or not $E(d) = \emptyset$.

Scientific program

2. Algorithmically study the algebra $A := \mathbb{R}(z)_S$:

- Membership problem:

  Let $P_1, \ldots, P_{m+1} \in A$, check whether or not:

  $$P_{m+1} \in I := \sum_{i=1}^{m} A\, P_i.$$

  Given $R \in A^{q \times p}$ and $\lambda \in A^{1 \times p}$, check whether or not:

  $$\exists\, \mu \in A^{1 \times q} : \; \mu\, R = \lambda.$$

- Study the set $\mathrm{Max}(A)$ of the maximal ideals of the ring A:

  $$\Rightarrow \; \exists\, Q_1, \ldots, Q_m \in A : \; \sum_{i=1}^{m} Q_i\, P_i = 1.$$

- Given $R \in A^{q \times p}$, compute the left kernel of matrices:

  $$\ker_A(.R) := \{\lambda \in A^{1 \times q} \mid \lambda\, R = 0\}.$$

- Compute invariants of A: Krull and homological dimensions.

Scientific program

3. Develop a constructive approach of module theory over A:
   - free resolutions,
   - functors $\mathrm{ext}^i_A(\,\cdot\,,\, M)$ and $\mathrm{tor}^A_i(\,\cdot\,,\, N)$,
   - projective dimensions,
   - properties: torsion, torsion-free, reflexive, projective, ...

4. Develop a constructive version of Deligne's proof:

$$\text{projective} = \text{free}.$$

5. Implement the different algorithms in dedicated Maple package.

6. Develop a H^∞ control theory for multidimensional systems following Ball's, ..., works.

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