Robust control, multidimensional systems and multivariable Nevanlinna-Pick interpolation

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Dedicated to Israel Gohberg on the occasion of his 80th birthday

Abstract. The connection between the standard H^∞-problem in control theory and Nevanlinna-Pick interpolation in operator theory was established in the 1980s, and has led to a fruitful cross-pollination between the two fields since. In the meantime, research in H^∞-control theory has moved on to the study of robust control for systems with structured uncertainties and to various types of multidimensional systems, while Nevanlinna-Pick interpolation theory has moved on independently to a variety of multivariable settings. Here we review these developments and indicate the precise connections which survive in the more general multidimensional/multivariable incarnations of the two theories.

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1. Introduction

Starting in the early 1980s with the seminal paper [139] of George Zames, there occurred an active interaction between operator theorists and control engineers in the development of the early stages of the emerging theory of H^∞-control. The cornerstone for this interaction was the early recognition by Francis-Helton-Zames [65] that the simplest case of the central problem of H^∞-control (the sensitivity minimization problem) is one and the same as a Nevanlinna-Pick interpolation problem which had already been solved in the early part of the twentieth century (see [110, 105]). For the standard problem of H^∞-control it was known early on that it could be brought to the so-called Model-Matching form (see [53, 64]). In the simplest cases, the Model-Matching problem converts easily to a Nevanlinna-Pick interpolation problem of classical type. Handling the more general problems of H^∞-control required extensions of the theory of Nevanlinna-Pick interpolation to tangential (or directional) interpolation conditions for matrix-valued functions; such extensions of the interpolation theory were pursued by both engineers and mathematicians (see e.g. [26, 58, 90, 86, 87]). Alternatively, the Model-Matching problem can be viewed as a Sarason problem which is suitable for application of Commutant Lifting theory (see [125, 62]). The approach of [64] used an additional conversion to a Nehari problem where existing results on the solution of the Nehari problem in state-space coordinates were applicable (see [69, 33]). The book of Francis [64] was the first book on H^∞-control and provides a good summary of the state of the subject in 1987.

While there was a lot of work emphasizing the connection of the H^∞-problem with interpolation and the related approach through J-spectral factorization ([26, 90, 91, 86, 87, 33, 24]), we should point out that the final form of the H^∞-theory parted ways with the connection with Nevanlinna-Pick interpolation. When calculations were carried out in state-space coordinates, the reduction to Model-Matching form via the Youla-Kučera parametrization of stabilizing controllers led to inflation of state-space dimension; elimination of non-minimal state-space nodes by finding pole-zero cancellations demanded tedious brute-force calculations (see [90, 91]). A direct solution in state-space coordinates (without reduction to Model-Matching form and any explicit connection with Nevanlinna-Pick interpolation) was finally obtained by Ball-Cohen [24] (via a J-spectral factorization approach) and in the more definitive coupled-Riccati-equation form of Doyle-Glover-Khargonekar-Francis [54]. This latter paper emphasizes the parallels with older control paradigms (e.g., the Linear-Quadratic-Gaussian and Linear-Quadratic-Regulator problems) and obtained parallel formulas for the related H^2-problem. The J-spectral factorization approach was further developed in the work of Kimura, Green, Glover, Limebeer, and Doyle [87, 70, 71]. A good review of the state of the theory to this point can be found in the books of Zhou-Doyle-Glover [141] and Green-Limebeer [72].

The coupled-Riccati-equation solution however has now been superseded by the Linear-Matrix-Inequality (LMI) solution which came shortly thereafter; we mention specifically the papers of Iwasaki-Skelton [78] and Gahinet-Apkarian [66]. This solution does not require any boundary rank conditions entailed in all the earlier approaches and generalizes in a straightforward way to more general settings (to be discussed in more detail below). The LMI form of the solution is particularly appealing from a computational point of view due to the recent advances in semidefinite programming (see [68]). The book of Dullerud-Paganini [57] gives an up-to-date account of these latest developments.

Research in H^∞-control has moved on in a number of different new directions, e.g., extensions of the H^∞-paradigm to sampled-data systems [47], nonlinear systems [126], hybrid systems [23], stochastic systems [76], quantum stochastic systems [79], linear repetitive processes [123], as well as behavioral frameworks [134]. Our focus here will be on the extensions to robust control for systems with structured uncertainties and related H^∞-control problems for multidimensional (N-D) systems—both frequency-domain and state-space settings. In the meantime, Nevanlinna-Pick interpolation theory has moved on to a variety of multivariable settings (polydisk, ball, noncommutative polydisk/ball); we mention in particular the papers [1, 49, 113, 3, 35, 19, 20, 21, 22, 30].

As the transfer function for a multidimensional system is a function of several variables, one would expect that the same connections familiar from the 1-D/single-variable case should also occur in these more general settings; however, while there had been some interaction between control theory and several-variable complex function theory in the older area of systems over rings (see [83, 85, 46]), to this point, with a few exceptions [73, 74, 32], there has not been such an interaction in connection with H^∞-control for N-D systems and related such topics. With this paper we wish to make precise the interconnections which do exist between the H^∞-theory and the interpolation theory in these more general settings. As we shall see, some aspects which are taken for granted in the 1-D/single-variable case become much more subtle in the N-D/multivariable case. Along the way we shall encounter a variety of topics that have gained attention recently, and sometimes less recently, in the engineering literature.

Besides the present Introduction, the paper consists of five sections which we now describe:

(1) In Section 2 we lay out four specific results for the classical 1-D case; these serve as models for the type of results which we wish to generalize to the N-D/multivariable settings.

(2) In Section 3 we survey the recent results of Quadrat [117, 118, 119, 120, 121, 122] on internal stabilization and parametrization of stabilizing controllers in an abstract ring setting. The main point here is that it is possible to parametrize the set of all stabilizing controllers in terms of a given stabilizing controller even in settings where the given plant may not have a double coprime factorization—resolving some issues left open in the book of Vidyasagar [136]. In the case where a double-coprime factorization is available, the parametrization formula is more efficient. Our modest new contribution here is to extend the ideas to the setting of the standard problem of H^∞-control (in the sense of the book of Francis [64]) where the given plant is assumed to have distinct disturbance and control inputs and distinct error and measurement outputs.

(3) In Section 4 we look at the internal-stabilization/H^∞-control problem for multidimensional systems. These problems have been studied in a purely frequency-domain framework (see [92, 93]) as well as in a state-space framework (see [81, 55, 56]). In Subsection 4.1, we give the frequency-domain formulation of the problem.

When one takes the stable plants to consist of the ring of structurally stable rational matrix functions, the general results of Quadrat apply. In particular, for this setting stabilizability of a given plant implies the existence of a double coprime factorization (see [119]). Application of the Youla-Kučera parametrization then leads to a Model-Matching form and, in the presence of some boundary rank conditions, the H^∞-problem converts to a polydisk version of the Nevanlinna-Pick interpolation problem. Unlike the situation in the classical single-variable case, this interpolation problem has no practical necessary-and-sufficient solution criterion and in practice one is satisfied with necessary and sufficient conditions for the existence of a solution in the more restrictive Schur-Agler class (see [1, 3, 35]).

In Subsection 4.2 we formulate the internal-stabilization/H^∞-control problem in Givone-Roesser state-space coordinates. We indicate the various subtleties involved in implementing the state-space version [104, 85] of the double-coprime factorization and associated Youla-Kučera parametrization of the set of stabilizing controllers. With regard to the H^∞-control problem, unlike the situation in the classical 1-D case, there is no useable necessary and sufficient analysis for solution of the problem; instead what is done (see e.g. [55, 56]) is the use of an LMI/Bounded-Real-Lemma analysis which provides a convenient set of sufficient conditions for solution of the problem. This sufficiency analysis in turn amounts to an N-D extension of the LMI solution [78, 66] of the 1-D H^∞-control problem and can be viewed as a necessary and sufficient analysis of a compromise problem (the "scaled" H^∞-problem).

While stabilization and H^∞-control problems have been studied in the state-space setting [81, 55, 56] and in the frequency-domain setting [92, 93] separately, there does not seem to have been much work on the precise connections between these two settings. The main point of Subsection 4.3 is to study this relationship; while solving the state-space problem implies a solution of the frequency-domain problem, the reverse direction is more subtle and it seems that only partial results are known. Here we introduce a notion of *modal stabilizability* and *modal detectability* (a modification of the notions of *modal controllability* and *modal observability* introduced by Kung-Levy-Morf-Kailath [88]) to obtain a partial result on relating a solution of the frequency-domain problem to a solution of the associated state-space problem. This result suffers from the same weakness as a corresponding result in [88]: just as the authors in [88] were unable to prove that *minimal* (i.e., simultaneously modally controllable and modally observable) realizations for a given transfer matrix exist, so also we are unable to prove that a simultaneously modally stabilizable and modally detectable realization exists. A basic difficulty in translating from frequency-domain to state-space coordinates is the failure of the State-Space-Similarity theorem and related Kalman state-space reduction for N-D systems. Nevertheless, the result is a natural analogue of the corresponding 1-D result.

There is a parallel between the control-theory side and the interpolation-theory side in that in both cases one is forced to be satisfied with a compromise solution: the scaled-H^∞ problem on the control-theory side, and the Schur-Agler class (rather than the Schur class) on the interpolation-theory side. We include some discussion on the extent to which these compromises are equivalent.

(4) In Section 5 we discuss several 1-D variations on the internal-stabilization and H^∞-control problem which lead to versions of the N-D/multivariable problems discussed in Section 4. It was observed early on that an H^∞-controller has good robustness properties, i.e., an H^∞-controller not only provides stability of the closed-loop system associated with the given (or *nominal*) plant for which the control was designed, but also for a whole neighborhood of plants around the nominal plant. This idea was refined in a number of directions, e.g., robustness with respect to additive or multiplicative plant uncertainty, or with respect to uncertainty in a normalized coprime factorization of the plant (see [100]). Another model for an uncertainty structure is the Linear-Fractional-Transformation (LFT) model used by Doyle and coworkers (see [97, 98]). Here a key concept is the notion of structured singular value $\mu(A)$ for a finite square matrix A introduced by Doyle and Safonov [52, 124] which simultaneously generalizes the norm and the spectral radius depending on the choice of uncertainty structure (a C^*-algebra of matrices with a prescribed block-diagonal structure); we refer to [107] for a comprehensive survey. If one assumes that the controller has on-line access to the uncertainty parameters one is led to a gain-scheduling problem which can be identified as the type of multidimensional control problem discussed in Section 4.2—see [106, 18]; we survey this material in Subsection 5.1. In Subsection 5.2 we review the purely frequency-domain approach of Helton [73, 74] toward gain-scheduling which leads to the frequency-domain internal-stabilization/H^∞-control problem discussed in Section 4.1. Finally, in Section 5.3 we discuss a hybrid frequency-domain/state-space model for structured uncertainty which leads to a generalization of Nevanlinna-Pick interpolation for single-variable functions where the constraint that the norm be uniformly bounded by 1 is replaced by the constraint that the μ-singular value be uniformly bounded by 1; this approach has only been analyzed for very special cases of the control problem but does lead to interesting new results for operator theory and complex geometry in the work of Bercovici-Foias-Tannenbaum [38, 39, 40, 41], Agler-Young [5, 6, 7, 8, 9, 10, 11, 12, 13], Huang-Marcantognini-Young [77], and Popescu [114].

(5) The final Section 6 discusses an enhancement of the LFT-model for structured uncertainty to allow dynamic time-varying uncertainties. If the controller is allowed to have on-line access to these more general uncertainties, then the solution of the internal-stabilization/H^∞-control problem has a form completely analogous to the classical 1-D case. Roughly, this result corresponds to the fact that, with this noncommutative enhanced uncertainty structure, the a priori upper bound $\widehat{\mu}(\mathbf{A})$ for the structured singular value $\mu(\mathbf{A})$ is actually equal to $\mu(\mathbf{A})$, despite the fact that for non-enhanced structures, the gap between μ and $\widehat{\mu}$ can be arbitrarily large (see [133]). In this precise form, the result appears for the first time in the thesis of Paganini [108] but various versions of this type of result have also appeared elsewhere (see [37, 42, 60, 99, 129]). We discuss this enhanced noncommutative LFT-model in Subsection 6.1. In Subsection 6.2 we introduce a noncommutative frequency-domain control problem in the spirit of Chapter 4 of the thesis of Lu [96], where the underlying polydisk occurring in Section 4.1 is now replaced by the noncommutative polydisk consisting of all d-tuples of contraction operators on a fixed separable infinite-dimensional Hilbert space $\mathcal{K}$ and the space of H^∞-functions is replaced by the space of scalar multiples of the noncommutative Schur-Agler class introduced in [28]. Via an adaptation of the Youla-Kučera parametrization of stabilizing controllers, the internal-stabilization/H^∞-control problem can be reduced to a Model-Matching form which has the interpretation as a noncommutative Sarason interpolation problem. In the final Subsection 6.3, we show how the noncommutative state-space problem is exactly equivalent to the noncommutative frequency-domain problem and thereby obtain an analogue of the classical case which is much more complete than for the commutative-variable case given in Section 4.3. In particular, if the problem data are given in terms of state-space coordinates, the noncommutative Sarason problem can be solved as an application of the LMI solution of the H^∞-problem. While there has been quite a bit of recent activity on this kind of noncommutative function theory (see e.g. [14, 22, 75, 82, 115, 116]), the noncommutative Sarason problem has to this point escaped attention; in particular, it is not clear how the noncommutative Nevanlinna-Pick interpolation problem studied in [22] is connected with the noncommutative Sarason problem.

Finally we mention that each section ends with a "Notes" subsection which discusses more specialized points and makes some additional connections with existing literature.

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2. The 1-D systems/single-variable case

Let $\mathbb{C}[z]$ be the space of polynomials with complex coefficients and $\mathbb{C}(z)$ the quotient field consisting of rational functions in the variable z. Let $\mathcal{R}H^\infty$ be the subring of *stable* elements of $\mathbb{C}(z)$ consisting of those rational functions which are analytic and bounded on the unit disk $\mathbb{D}$, i.e., with no poles in the closed unit disk $\overline{\mathbb{D}}$. We assume to be given a plant $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right] : \mathcal{W} \oplus \mathcal{U} \to \mathcal{Z} \oplus \mathcal{Y}$ which is given as a block matrix of appropriate size with entries from $\mathbb{C}(z)$. Here the spaces $\mathcal{U}$, $\mathcal{W}$, $\mathcal{Z}$ and $\mathcal{Y}$ have the interpretation of *control-signal space*, *disturbance-signal space*, *error-signal space* and *measurement-signal space*, respectively, and consist of column vectors of given sizes $n_\mathcal{U}$, $n_\mathcal{W}$, $n_\mathcal{Z}$ and $n_\mathcal{Y}$, respectively, with entries from $\mathbb{C}(z)$. For this plant G we seek to design a controller $K \colon \mathcal{Y} \to \mathcal{U}$, also given as a matrix over $\mathbb{C}(z)$, that stabilizes the feedback system $\Sigma(G, K)$ obtained from the signal-flow diagram in Figure 1 in a sense to be defined precisely below.

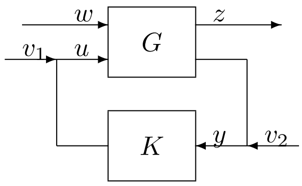

Figure 1. Feedback with tap signals

Note that the various matrix entries G_{ij} of G are themselves matrices with entries from $\mathbb{C}(z)$ of compatible sizes (e.g., G_{11} has size $n_{\mathcal{Z}} \times n_{\mathcal{W}}$) and K is a matrix over $\mathbb{C}(z)$ of size $n_{\mathcal{U}} \times n_{\mathcal{Y}}$.

The system equations associated with the signal-flow diagram of Figure 1 can be written as

$$\begin{bmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{bmatrix} \begin{bmatrix} z \\ u \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ G_{21} & 0 & I \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}. \tag{2.1}$$

Here v_1 and v_2 are tap signals used to detect stability properties of the internal signals u and y. We say that the system $\Sigma(G, K)$ is *well-posed* if there is a well-defined map from $\left[\begin{smallmatrix} w \\ v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ u \\ y \end{smallmatrix}\right]$. It follows from a standard Schur complement computation that the system is well-posed if and only if $\det(I - G_{22}K) \neq 0$, and that in that case the map from $\left[\begin{smallmatrix} w \\ v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ u \\ y \end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = \Theta(G, K) \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

where

$$\Theta(G,K) := \begin{bmatrix} I & -G_{12} & 0 \\ 0 & I & -K \\ 0 & -G_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ G_{21} & 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} & G_{12}[I + K(I - G_{22}K)^{-1}G_{22}] & G_{12}K(I - G_{22}K)^{-1} \\ K(I - G_{22}K)^{-1}G_{21} & I + K(I - G_{22}K)^{-1}G_{22} & K(I - G_{22}K)^{-1} \\ (I - G_{22}K)^{-1}G_{21} & (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21} & G_{12}(I - KG_{22})^{-1} & G_{12}(I - KG_{22})^{-1}K \\ (I - KG_{22})^{-1}KG_{21} & (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ [I + G_{22}(I - KG_{22})^{-1}K]G_{21} & G_{22}(I - KG_{22})^{-1} & I + G_{22}(I - KG_{22})^{-1}K \end{bmatrix}. \tag{2.2}$$

We say that the system $\Sigma(G, K)$ is *internally stable* if $\Sigma(G, K)$ is well-posed and, in addition, if the map $\Theta(G, K)$ maps $\mathcal{R}H^\infty_{\mathcal{W}} \oplus \mathcal{R}H^\infty_{\mathcal{U}} \oplus \mathcal{R}H^\infty_{\mathcal{Y}}$ into $\mathcal{R}H^\infty_{\mathcal{Z}} \oplus \mathcal{R}H^\infty_{\mathcal{U}} \oplus$

$\mathcal{R}H^{\infty}_{\mathcal{Y}}$, i.e., stable inputs w, v_1, v_2 are mapped to stable outputs z, u, y. Note that this is the same as the condition that the entries of $\Sigma(G, K)$ be in $\mathcal{R}H^{\infty}$.

We say that the system $\Sigma(G, K)$ *has performance* if $\Sigma(G, K)$ is internally stable and in addition the transfer function T_{zw} from w to z has supremum-norm over the unit disk bounded by some tolerance which we normalize to be equal to 1:

$$\|T_{zw}\|_{\infty} := \sup\{\|T_{zw}(\lambda)\| \colon \lambda \in \mathbb{D}\} \leq 1.$$

Here $\|T_{zw}(\lambda)\|$ refers to the induced operator norm, i.e., the largest singular value for the matrix $T_{zw}(\lambda)$. We say that the system $\Sigma(G, K)$ has *strict performance* if in addition $\|T_{zw}\|_{\infty} < 1$. The *stabilization problem* then is to describe all (if any exist) internally stabilizing controllers K for the given plant G, i.e., all $K \in \mathbb{C}(z)^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ so that the associated closed-loop system $\Sigma(G, K)$ is internally stable. The *standard H^{∞}-problem* is to find all internally stabilizing controllers which in addition achieve performance $\|T_{zw}\|_{\infty} \leq 1$. The strictly suboptimal H^{∞}-problem is to describe all internally stabilizing controllers which also achieve strict performance $\|T_{zw}\|_{\infty} < 1$.

2.1. The model-matching problem

Let us now consider the special case where $G_{22} = 0$, so that G has the form $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix}\right]$. In this case well-posedness is automatic and $\Theta(G, K)$ simplifies to

$$\Theta(G, K) = \begin{bmatrix} G_{11} + G_{12}KG_{21} & G_{12} & G_{12}K \\ KG_{21} & I & K \\ G_{21} & 0 & I \end{bmatrix}.$$

Thus internal stability for the closed-loop system $\Sigma(G, K)$ is equivalent to stability of the four transfer matrices G_{11}, G_{12}, G_{21} and K. Hence internal stabilizability of G is equivalent to stability of G_{11}, G_{12} and G_{21}; when the latter holds a given K internally stabilizes G if and only if K itself is stable.

Now assume that G_{11}, G_{12} and G_{21} are stable. Then the H^{∞}-performance problem for G consists of finding stable K so that $\|G_{11} + G_{12}KG_{21}\|_{\infty} \leq 1$. Following the terminology of [64], the problem is called the Model-Matching Problem. Due to the influence of the paper [125], this problem is usually referred to as the Sarason problem in the operator theory community; in [125] it is shown explicitly how the problem can be reduced to an interpolation problem.

In general control problems the assumption that $G_{22} = 0$ is an unnatural assumption. However, after making a change of coordinates using the Youla-Kučera parametrization or the Quadrat parametrization, discussed below, it turns out that the general H^{∞}-problem can be reduced to a model-matching problem.

2.2. The frequency-domain stabilization and H^{∞} problem

The following result on characterization of stabilizing controllers is well known (see e.g. [64] or [136, 137] for a more general setting).

Theorem 2.1. *Suppose that we are given a rational matrix function $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ of size $(n_{\mathcal{Z}} + n_{\mathcal{Y}}) \times (n_{\mathcal{W}} + n_{\mathcal{U}})$ with entries in $\mathbb{C}(z)$ as above. Assume that G is stabilizable, i.e., there exists a rational matrix function K of size $n_{\mathcal{U}} \times n_{\mathcal{Y}}$ so that* *the nine transfer functions in* (2.2) *are all stable. Then a given rational matrix function K stabilizes G if and only if K stabilizes G_{22}, i.e., $\Theta(G,K)$ in* (2.2) *is stable if and only if*

$$\Theta(G_{22},K) := \begin{bmatrix} I + K(I-G_{22}K)^{-1} & K(I-G_2K)^{-1} \\ (I-G_{22}K)^{-1} & (I-G_{22}K)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} (I-KG_{22})^{-1} & (I-KG_{22})^{-1}K \\ G_{22}(I-KG_{22})^{-1} & I+G_{22}(I-KG_{22})^{-1}K \end{bmatrix}$$

is stable. Moreover, if we are given a double coprime factorization for G_{22}, i.e., stable transfer matrices D, N, X, Y, $\widetilde{D}$, $\widetilde{N}$, $\widetilde{X}$ and $\widetilde{Y}$ so that the determinants of D, $\widetilde{D}$, X and $\widetilde{X}$ are all nonzero (in $\mathcal{R}H^\infty$) and

$$G_{22} = D^{-1}N = \widetilde{N}\widetilde{D}^{-1}, \quad \begin{bmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{bmatrix} \begin{bmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{bmatrix} = \begin{bmatrix} I_{n_\mathcal{Y}} & 0 \\ 0 & I_{n_\mathcal{U}} \end{bmatrix} \tag{2.3}$$

(such double coprime factorizations always exists since $\mathcal{R}H^\infty$ is a Principal Ideal Domain), then the set of all stabilizing controllers K is given by either of the formulas

$$K = (Y + \widetilde{D}\Lambda)(X + \widetilde{N}\Lambda)^{-1} = (\widetilde{X} + \Lambda N)^{-1}(\widetilde{Y} + \Lambda D),$$

where Λ is a free stable parameter from $\mathcal{R}H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}$ such that $\det(X + \widetilde{N}\Lambda) \neq 0$ *or equivalently* $\det(\widetilde{X} + \Lambda N) \neq 0$.

Through the characterization of the stabilizing controllers, those controllers that, in addition, achieve performance can be obtained from the solutions of a Model-Matching/Sarason interpolation problem.

Theorem 2.2. *Assume that $G \in \mathbb{C}(z)^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$ is stabilizable and that G_{22} admits a double coprime factorization* (3.9)*. Let $K \in \mathbb{C}(z)^{n_\mathcal{U}\times n_\mathcal{Y}}$. Then K is a solution to the standard H^∞ problem for G if and only if*

$$K = (Y + \widetilde{D}\Lambda)(X + \widetilde{N}\Lambda)^{-1} = (\widetilde{X} + \Lambda N)^{-1}(\widetilde{Y} + \Lambda D),$$

where $\Lambda \in \mathcal{R}H^\infty_{\mathcal{L}(\mathcal{U},\mathcal{Y})}$ so that $\det(X + \widetilde{N}\Lambda) \neq 0$, *or equivalently* $\det(\widetilde{X} + \Lambda N) \neq 0$, *is any solution to the Model-Matching/Sarason interpolation problem for $\widetilde{G}_{11}$, $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ defined by*

$$\widetilde{G}_{11} := G_{11} + G_{12}YDG_{21}, \quad \widetilde{G}_{12} := G_{12}\widetilde{D}, \quad \widetilde{G}_{21} := DG_{21},$$

i.e., so that

$$\|\widetilde{G}_{11} + \widetilde{G}_{12}\Lambda\widetilde{G}_{21}\|_\infty \leq 1.$$

We note that in case $\widetilde{G}_{12}$ is injective and $\widetilde{G}_{21}$ is surjective on the unit circle, by absorbing outer factors into the free parameter Λ we may assume without loss of generality that $\widetilde{G}_{12}$ is inner (i.e., $\widetilde{G}_{12}(z)$ is isometric for z on unit circle) and $\widetilde{G}_{21}$ is co-inner (i.e., $\widetilde{G}_{21}(z)$ is coisometric for z on the unit circle). Let $\Gamma\colon L^2_\mathcal{W} \ominus \widetilde{G}^*_{21}H^{2\perp}_\mathcal{U} \to L^2_\mathcal{Z} \ominus \widetilde{G}_{12}H^2_\mathcal{U}$ be the compression of multiplication by $\widetilde{G}_{11}$ to the spaces

$L^2_{\mathcal{W}} \ominus \widetilde{G}_{21}^* H^{2\perp}_{\mathcal{U}}$ and $L^2_{\mathcal{Z}} \ominus \widetilde{G}_{12} H^2_{\mathcal{U}}$, i.e., $\Gamma = P_{L^2_{\mathcal{Z}} \ominus \widetilde{G}_{12} H^2_{\mathcal{U}}} \widetilde{G}_{11}|_{L^2_{\mathcal{W}} \ominus \widetilde{G}_{21}^* H^{2\perp}_{\mathcal{Y}}}$. Then, as a consequence of the Commutant Lifting theorem (see [63, Corollary 10.2 pages 40–41]), one can see that the strict Model-Matching/Sarason interpolation problem posed in Theorem 2.2 has a solution if and only if $\|\Gamma\|_{op} < 1$. Alternatively, in case $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ are square and invertible on the unit circle, one can convert this Model-Matching/Commutant-Lifting problem to a bitangential Nevanlinna-Pick interpolation problem (see [26, Theorem 16.9.3]), a direct generalization of the connection between a model-matching/Sarason interpolation problem with Nevanlinna-Pick interpolation as given in [125, 65] for the scalar case, but we will not go into the details of this here.

2.3. The state-space approach

We now restrict the classes of admissible plants and controllers to the transfer matrices whose entries are in $\mathbb{C}(z)_0$, the space of rational functions without a pole at 0 (i.e., analytic in a neighborhood of 0). In that case, a transfer matrix $F : \mathcal{U} \to \mathcal{Y}$ with entries in $\mathbb{C}(z)_0$ admits a *state-space realization*: There exists a quadruple $\{A, B, C, D\}$ consisting of matrices whose sizes are given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}, \tag{2.4}$$

where the *state-space* $\mathcal{X}$ is finite dimensional, so that

$$F(z) = D + zC(I - zA)^{-1}B$$

for z in a neighborhood of 0. Sometimes we consider quadruples $\{A, B, C, D\}$ of operators, of compatible size as above, without any explicit connection to a transfer matrix, in which case we just speak of a realization.

Associated with the realization $\{A, B, C, D\}$ is the linear discrete-time system of equations

$$\Sigma := \left\{ \begin{array}{rcl} x(n+1) & = & Ax(n) + Bu(n), \\ y(n) & = & Cx(n) + Du(n). \end{array} \right. \qquad (n \in \mathbb{Z}_+)$$

The system Σ and function F are related through the fact that F is the *transfer-function* of Σ. The two-by-two matrix (2.4) is called the *system matrix* of the system Σ.

For the rest of this section we shall say that an operator A on a finite-dimensional state space $\mathcal{X}$ is *stable* if all its eigenvalues are in the open unit disk, or, equivalently, $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$. The following result deals with two key notions for the stabilizability problem on the state-space level.

Theorem 2.3. *(I) Suppose that $\{A, B\}$ is an* input pair*, i.e., A, B are operators with $A \colon \mathcal{X} \to \mathcal{X}$ and $B \colon \mathcal{U} \to \mathcal{X}$ for a finite-dimensional state space $\mathcal{X}$ and a finite-dimensional input space $\mathcal{U}$. Then the following are equivalent:*

1. $\{A, B\}$ *is* operator-stabilizable*, i.e., there exists a state-feedback operator $F \colon \mathcal{X} \to \mathcal{U}$ so that the operator $A + BF$ is stable.*

2. $\{A, B\}$ *is* Hautus-stabilizable*, i.e., the matrix pencil* $\begin{bmatrix} I - zA & B \end{bmatrix}$ *is surjective for each* z *in the closed unit disk* $\overline{\mathbb{D}}$*.*
3. *The Stein inequality*

$$AXA^* - X - BB^* < 0$$

has a positive-definite solution X*. Here* $\Gamma < 0$ *for a square matrix* Γ *means that* $-\Gamma$ *is positive definite.*

(II) Dually, if $\{C, A\}$ *is an output pair, i.e.,* C, A *are operators with* $A\colon \mathcal{X} \to \mathcal{X}$ *and* $C\colon \mathcal{X} \to \mathcal{Y}$ *for a finite-dimensional state space* $\mathcal{X}$ *and a finite-dimensional output space* $\mathcal{Y}$*, then the following are equivalent:*

1. $\{C, A\}$ *is* operator-detectable*, i.e., there exists an output-injection operator* $L\colon \mathcal{Y} \to \mathcal{X}$ *so that* $A + LC$ *is stable.*
2. $\{C, A\}$ *is* Hautus-detectable*, i.e., the matrix pencil* $\left[\begin{smallmatrix} I - zA \\ C \end{smallmatrix}\right]$ *is injective for all* z *in the closed disk* $\overline{\mathbb{D}}$*.*
3. *The Stein inequality*

$$A^*YA - Y - C^*C < 0$$

has a positive definite solution Y*.*

When the input pair $\{A, B\}$ satisfies any one (and hence all) of the three equivalent conditions in part (I) of Theorem 2.3, we shall say simply that $\{A, B\}$ is *stabilizable.* Similarly, if (C, A) satisfies any one of the three equivalent conditions in part (II), we shall say simply that $\{C, A\}$ is detectable. Given a realization $\{A, B, C, D\}$, we shall say that $\{A, B, C, D\}$ is stabilizable and detectable if $\{A, B\}$ is stabilizable and $\{C, A\}$ is detectable.

In the state-space formulation of the internal stabilization/H^∞-control problem, one assumes to be given a state-space realization for the plant G:

$$G(z) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + z \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - zA)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \tag{2.5}$$

where the system matrix has the form

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \\ \mathcal{Y} \end{bmatrix}. \tag{2.6}$$

One then seeks a controller K which is also given in terms of a state-space realization

$$K(z) = D_K + zC_K(I - zA_K)^{-1}B_K$$

which provides internal stability (in the state-space sense to de defined below) and/or H^∞-performance for the closed-loop system. Well-posedness of the closed-loop system is equivalent to invertibility of $I - D_{22}D_K$. To keep various formulas affine in the design parameters A_K, B_K, C_K, D_K, it is natural to assume that $D_{22} = 0$; this is considered not unduly restrictive since under the assumption of well-posedness this can always be arranged via a change of variables (see [78]). Then the closed loop system $\Theta(G,K)$ admits a state space realization $\{A_{cl}, B_{cl}, C_{cl}, D_{cl}\}$ given by its system matrix

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array}\right] \tag{2.7}$$

and *internal stability (in the state-space sense)* is taken to mean that $A_{cl} = \left[\begin{smallmatrix} A+B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{smallmatrix}\right]$ should be stable, i.e., all eigenvalues are in the open unit disk.

The following result characterizes when a given G is internally stabilizable in the state-space sense.

Theorem 2.4. *(See Proposition 5.2 in* [57]*.) Suppose that we are given a system matrix as in* (2.6) *with* $D_{22} = 0$ *with associated transfer matrix* G *as in* (2.5)*. Then there exists a* $K(z) = D_K + zC_K(I - zA_K)^{-1}B_K$ *which internally stabilizes* G *(in the state-spaces sense) if and only if* $\{A, B_2\}$ *is stabilizable and* $\{C_2, A\}$ *is detectable. In this case one such controller is given by the realization* $\{A_K, B_K, C_K, D_K\}$ *with system matrix*

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} A + B_2 F + L C_2 & -L \\ F & 0 \end{bmatrix}$$

where F *and* L *are state-feedback and output-injection operators chosen so that* $A + B_2 F$ *and* $A + LC_2$ *are stable.*

In addition to the state-space version of the stabilizability problem we also consider a (strict) state-space H^∞ problem, namely to find a controller K given by a state-space realization $\{A_K, B_K, C_K, D_K\}$ of compatible size so that the transfer-function T_{zw} of the closed loop system, given by the system matrix (2.7), is stable (in the state-space sense) and has a supremum norm $\|T_{zw}\|_\infty$ of at most 1 (less than 1).

The definitive solution of the H^∞-control problem in state-space coordinates for a time was the coupled-Riccati-equation solution due to Doyle-Glover-Khargonekar-Francis [54]. This solution has now been superseded by the LMI solution of Gahinet-Apkarian [66] which can be stated as follows. Note that the problem can be solved directly without first processing the data to the Model-Matching form.

Theorem 2.5. *Let* $\{A, B, C, D\} = \left\{A, \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} C_1 & C_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{smallmatrix}\right]\right\}$ *be a given realization. Then there exists a solution for the strict state-space* H^∞*-control problem associated with* $\{A, B, C, D\}$ *if and only if there exist positive-definite matrices*

X, Y satisfying the LMIs

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AYA^* - Y & AYC_1^* & B_1 \\ C_1YA^* & C_1YC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0, \quad Y > 0, \tag{2.8}$$

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*XA - X & A^*XB_1 & C_1^* \\ B_1^*XA & B_1^*XB_1 - I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0, \quad X > 0, \tag{2.9}$$

and the coupling condition

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{2.10}$$

Here N_c and N_o are matrices chosen so that

$$N_c \text{ is injective and } \operatorname{Im} N_c = \operatorname{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \text{ and}$$
$$N_o \text{ is injective and } \operatorname{Im} N_o = \operatorname{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}.$$

We shall discuss the proof of Theorem 2.5 in Section 4.2 below in the context of a more general multidimensional-system H^∞-control problem.

The next result is the key to transferring from the frequency-domain version of the internal-stabilization/H^∞-control problem to the state-space version.

Theorem 2.6. *(See Lemma 5.5 in* [57]*.) Suppose that the realization $\{A, B_2, C_2, 0\}$ for the plant G_{22} and the realization $\{A_K, B_K, C_K, D_K\}$ for the controller K are both stabilizable and detectable. Then K internally stabilizes G_{22} in the state-space sense if and only if K stabilizes G_{22} in the frequency-domain sense, i.e., the closed-loop matrix $A_{cl} = \left[\begin{smallmatrix} A+B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{smallmatrix}\right]$ is stable if and only if the associated transfer matrix*

$$\Theta(G_{22}, K) = \begin{bmatrix} I & D_K \\ 0 & I \end{bmatrix} + z \begin{bmatrix} D_KC_2 & C_K \\ C_2 & 0 \end{bmatrix} (I - zA_{cl})^{-1} \begin{bmatrix} B_2 & B_2D_K \\ 0 & B_K \end{bmatrix}$$

has all matrix entries in $\mathcal{R}H^\infty$.

2.4. Notes

In the context of the discussion immediately after the statement of Theorem 2.2, in case $\widetilde{G}_{12}$ and/or $\widetilde{G}_{21}$ drop rank at points on the unit circle, the Model-Matching problem in Theorem 2.2 may convert to a boundary Nevanlinna-Pick interpolation problem for which there is an elaborate specialized theory (see e.g. Chapter 21 of [26] and the more recent [43]). However, if one sticks with the strictly suboptimal version of the problem, one can solve the problem with the boundary interpolation conditions if and only if one can solve the problem without the boundary interpolation conditions, i.e., boundary interpolation conditions are irrelevant as far as existence criteria are concerned. This is the route taken in the LMI solution of the H^∞-problem and provides one explanation for the disappearance of any rank conditions in the formulation of the solution of the problem. For a complete analysis of the relation between the coupled-Riccati-equation of [54] versus the LMI solution of [66], we refer to [127].

3. The fractional representation approach to stabilizability and performance

In this section we work in the general framework of the fractional representation approach to stabilization of linear systems as introduced originally by Desoer, Vidyasagar and coauthors [50, 137] in the 1980s and refined only recently in the work of Quadrat [118, 121, 122]. For an overview of the more recent developments we recommend the survey article [117] and for a completely elementary account of the generalized Youla-Kučera parametrization with all the algebro-geometric interpretations stripped out we recommend [120].

The set of *stable* single-input single-output (SISO) transfer functions is assumed to be given by a general ring $\mathbb{A}$ in place of the ring $\mathcal{R}H^\infty$ used for the classical case as discussed in Section 2; the only assumption which we shall impose on $\mathbb{A}$ is that it be a commutative integral domain. It therefore has a quotient field $\mathbb{K} := Q(\mathbb{A}) = \{n/d\colon d, n \in \mathbb{A}, d \neq 0\}$ which shall be considered as the set of all possible SISO transfer functions (or plants). Examples of $\mathbb{A}$ which come up include the ring $\mathbb{R}_s(z)$ of real rational functions of the complex variable z with no poles in the closed right half plane, the Banach algebra $RH^\infty(\mathbb{C}_+)$ of all bounded analytic functions on the right half plane $\mathbb{C}_+$ which are real on the positive real axis, and their discrete-time analogues: (1) real rational functions with no poles in the closed unit disk (or closed exterior of the unit disk depending on how one sets conventions), and (2) the Banach algebra $RH^\infty(\mathbb{D})$ of all bounded holomorphic functions on the unit disk $\mathbb{D}$ with real values on the real interval $(-1, 1)$. There are also Banach subalgebras of $RH^\infty(\mathbb{C}_+)$ or $RH^\infty(\mathbb{D})$ (e.g., the Wiener algebra and its relatives such as the Callier-Desoer class—see [48]) which are of interest. In addition to these examples there are multivariable analogues, some of which we shall discuss in the next section.

We now introduce some notation. We assume that the *control-signal space* $\mathcal{U}$, the *disturbance-signal space* $\mathcal{W}$, the *error-signal space* $\mathcal{Z}$ and the *measurement signal space* $\mathcal{Y}$ consist of column vectors of given sizes $n_\mathcal{U}$, $n_\mathcal{W}$, $n_\mathcal{Z}$ and $n_\mathcal{Y}$, respectively, with entries from the quotient field $\mathbb{K}$ of $\mathbb{A}$:

$$\mathcal{U} = \mathbb{K}^{n_\mathcal{U}}, \quad \mathcal{W} = \mathbb{K}^{n_\mathcal{W}}, \quad \mathcal{Z} = \mathbb{K}^{n_\mathcal{Z}}, \quad \mathcal{Y} = \mathbb{K}^{n_\mathcal{Y}}.$$

We are given a plant $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right] : \mathcal{W} \oplus \mathcal{U} \to \mathcal{Z} \oplus \mathcal{Y}$ and seek to design a controller $K\colon \mathcal{Y} \to \mathcal{U}$ that stabilizes the system $\Sigma(G, K)$ of Figure 1 as given in Section 2. The various matrix entries G_{ij} of G are now matrices with entries from $\mathbb{K}$ (rather than $\mathcal{R}H^\infty$ as in the classical case) of compatible sizes (e.g., G_{11} has size $n_\mathcal{W} \times n_\mathcal{U}$) and K is a matrix over $\mathbb{K}$ of size $n_\mathcal{U} \times n_\mathcal{Y}$. Again v_1 and v_2 are tap signals used to detect stability properties of the internal signals u and y.

Just as was explained in Section 2 for the classical case, the system $\Sigma(G, K)$ is *well-posed* if there is a well-defined map from $\left[\begin{smallmatrix} w \\ v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ u \\ y \end{smallmatrix}\right]$ and this happens exactly when $\det(I - G_{22}K) \neq 0$ (where the determinant now is an element of $\mathbb{A}$);

when this is the case, the map from $\left[\begin{smallmatrix} w \\ v_1 \\ v_2 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} z \\ u \\ y \end{smallmatrix}\right]$ is given by

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = \Theta(G, K) \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}$$

where $\Theta(G, K)$ is given by (2.2). We say that the system $\Sigma(G, K)$ is *internally stable* if $\Sigma(G, K)$ is well-posed and, in addition, if the map $\Theta(G, K)$ maps $\mathbb{A}^{n_{\mathcal{W}}} \oplus \mathbb{A}^{n_{\mathcal{U}}} \oplus \mathbb{A}^{n_{\mathcal{Y}}}$ into $\mathbb{A}^{n_{\mathcal{Z}}} \oplus \mathbb{A}^{n_{\mathcal{U}}} \oplus \mathbb{A}^{n_{\mathcal{Y}}}$, i.e., stable inputs w, v_1, v_2 are mapped to stable outputs z, u, y. Note that this is the same as the entries of $\Sigma(G, K)$ being in $\mathbb{A}$.

To formulate the standard problem of H^∞-control, we assume that $\mathbb{A}$ is equipped with a positive-definite inner product making $\mathbb{A}$ at least a pre-Hilbert space with norm $\|\cdot\|_{\mathbb{A}}$; in the classical case, one takes this norm to be the L^2-norm over the unit circle. Then we say that the system $\Sigma(G, K)$ *has performance* if $\Sigma(G, K)$ is internally stable and in addition the transfer function T_{zw} from w to z has induced operator norm bounded by some tolerance which we normalize to be equal to 1:

$$\|T_{zw}\|_{op} := \sup\{\|z\|_{\mathbb{A}^{n_{\mathcal{Z}}}} : \|w\|_{\mathbb{A}^{n_{\mathcal{W}}}} \leq 1, v_1 = 0,\ v_2 = 0\} \leq 1.$$

We say that the system $\Sigma(G, K)$ has *strict performance* if in fact $\|T_{zw}\|_{op} < 1$. The *stabilization problem* then is to describe all (if any exist) internally stabilizing controllers K for the given plant G, i.e., all $K \in \mathbb{K}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ so that the associated closed-loop system $\Sigma(G, K)$ is internally stable. The *standard H^∞-problem* is to find all internally stabilizing controllers which in addition achieve performance $\|T_{zw}\|_{op} \leq 1$. The strictly suboptimal H^∞-problem is to describe all internally stabilizing controllers which achieve strict performance $\|T_{zw}\|_{op} < 1$.

The H^∞-control problem for the special case where $G_{22} = 0$ is the Model-Matching problem for this setup. With the same arguments as in Subsection 2.1 it follows that stabilizability forces G_{11}, G_{12} and G_{21} all to be stable (i.e., to have all matrix entries in $\mathbb{A}$) and then K stabilizes exactly when also K is stable.

3.1. Parametrization of stabilizing controllers in terms of a given stabilizing controller

We return to the general case i.e., $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right] : \mathcal{W} \oplus \mathcal{U} \to \mathcal{Z} \oplus \mathcal{Y}$. Now suppose we have a stabilizing controller $K \in \mathbb{K}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$. Set

$$U = (I - G_{22}K)^{-1} \quad \text{and} \quad V = K(I - G_{22}K)^{-1}. \tag{3.1}$$

Then $U \in \mathbb{A}^{n_{\mathcal{Y}} \times n_{\mathcal{Y}}}$, $V \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$, $\det U \neq 0 \in \mathbb{A}$, $K = VU^{-1}$ and $U - G_{22}V = I$. Furthermore, $\Theta(G, K)$ can then be written as

$$\Theta(G, K) = \Theta(G; U, V) := \begin{bmatrix} G_{11} + G_{12}VG_{21} & G_{12} + G_{12}VG_{22} & G_{12}V \\ VG_{21} & I + VG_{22} & V \\ UG_{21} & UG_{22} & U \end{bmatrix}. \tag{3.2}$$

It is not hard to see that if $U \in \mathbb{A}^{n_{\mathcal{Y}} \times n_{\mathcal{Y}}}$ and $V \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ are such that $\det U \neq 0$, $U - G_{22}V = I$ and (3.2) is stable, i.e., in $\mathbb{A}^{(n_{\mathcal{Z}} + n_{\mathcal{U}} + n_{\mathcal{Y}}) \times (n_{\mathcal{W}} + n_{\mathcal{U}} + n_{\mathcal{Y}})}$, then $K =$

VU^{-1} is a stabilizing controller. A dual result holds if we set

$$\widetilde{U} = (I - KG_{22})^{-1} \quad \text{and} \quad \widetilde{V} = (I - KG_{22})^{-1}K. \tag{3.3}$$

In that case $\widetilde{U} \in \mathbb{A}^{n_\mathcal{U} \times n_\mathcal{U}}$, $\widetilde{V} \in \mathbb{A}^{n_\mathcal{U} \times n_\mathcal{Y}}$, $\det \widetilde{U} \neq 0 \in \mathbb{A}$, $K = \widetilde{U}^{-1}\widetilde{V}$, $\widetilde{U} - \widetilde{V}G_{22} = I$ and we can write $\Theta(G, K)$ as

$$\Theta(G, K) = \Theta(G; \widetilde{U}, \widetilde{V}) = \begin{bmatrix} G_{11} + G_{12}\widetilde{V}G_{21} & G_{12}\widetilde{U} & G_{12}\widetilde{V} \\ \widetilde{V}G_{21} & \widetilde{U} & \widetilde{V} \\ (I + G_{22}\widetilde{V})G_{21} & G_{22}\widetilde{U} & I + G_{22}\widetilde{V} \end{bmatrix}, \tag{3.4}$$

while conversely, for any $\widetilde{U} \in \mathbb{A}^{n_\mathcal{U} \times n_\mathcal{U}}$ and $\widetilde{V} \in \mathbb{A}^{n_\mathcal{U} \times n_\mathcal{Y}}$ with $\det \widetilde{U} \neq 0$ and $\widetilde{U} - \widetilde{V}G_{22} = I$ and such that (3.4) is stable, we have that $K = \widetilde{U}^{-1}\widetilde{V}$ is a stabilizing controller.

This leads to the following first-step more linear reformulation of the definition of internal stabilization.

Theorem 3.1. *A plant G defined by a transfer matrix $G \in \mathbb{K}^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$ is internally stabilizable if and only if one of the following equivalent assertions holds:*

1. *There exists $L = \left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \in \mathbb{A}^{(n_\mathcal{U}+n_\mathcal{Y})+n_\mathcal{Y}}$ with $\det U \neq 0$ such that:*
   (a) *The block matrix (3.2) is stable (i.e., has all matrix entries in $\mathbb{A}$), and*
   (b) $\begin{bmatrix} -G_{22} & I \end{bmatrix} L = I$.

   Then the controller $K = VU^{-1}$ internally stabilizes the plant G and we have:

   $$U = (I - G_{22}K)^{-1}, \quad V = K(I - G_{22}K)^{-1}.$$

2. *There exists $\widetilde{L} = \left[\begin{smallmatrix} \widetilde{U} & -\widetilde{V} \end{smallmatrix}\right] \in \mathbb{A}^{n_\mathcal{U}\times(n_\mathcal{U}+n_\mathcal{Y})}$ with $\det \widetilde{U} \neq 0$ such that:*
   (a) *The block matrix (3.4) is stable (i.e., has all matrix entries in $\mathbb{A}$), and*
   (b) $\widetilde{L}\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] := \left[\begin{smallmatrix} \widetilde{U} & -\widetilde{V} \end{smallmatrix}\right]\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] = I$.

   If this is the case, then the controller $K = \widetilde{U}^{-1}\widetilde{V}$ internally stabilizes the plant G and we have:

   $$\widetilde{U} = (I - KG_{22})^{-1}, \quad \widetilde{V} = (I - KG_{22})^{-1}K.$$

With this result in hand, we are able to get a parametrization for the set of all stabilizing controllers in terms of an assumed particular stabilizing controller.

Theorem 3.2.

1. *Let $K_* \in \mathbb{K}^{n_\mathcal{U}\times n_\mathcal{Y}}$ be a stabilizing controller for $G \in \mathbb{K}^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$. Define $U_* = (I - G_{22}K_*)^{-1}$ and $V_* = K(I - G_{22}K_*)^{-1}$. Then the set of all stabilizing controllers is given by*

   $$K = (V_* + Q)(U_* + G_{22}Q)^{-1}, \tag{3.5}$$

   where $Q \in \mathbb{K}^{n_\mathcal{U}\times n_\mathcal{Y}}$ is an element of the set

$$\Omega := \left\{ Q \in \mathbb{K}^{n_\mathcal{U}\times n_\mathcal{Y}} : \begin{bmatrix} G_{12} \\ I \\ G_{22} \end{bmatrix} Q \begin{bmatrix} G_{21} & G_{22} & I \end{bmatrix} \in \mathbb{A}^{(n_\mathcal{Z}+n_\mathcal{U}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U}+n_\mathcal{Y})} \right\} \tag{3.6}$$

such that in addition $\det(U_* + G_{22}Q) \neq 0$.

2. *Let* $K_* \in \mathbf{K}^{n_\mathcal{U} \times n_\mathcal{Y}}$ *be a stabilizing controller for* $G \in \mathbb{K}^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$. *Define* $\widetilde{U}_* = (I - K_*G_{22})^{-1}$ *and* $\widetilde{V}_* = (I - K_*G_{22})^{-1}K_*$. *Then the set of all controllers is given by*
$$K = (\widetilde{U}_* + QG_{22})^{-1}(\widetilde{V}_* + Q), \tag{3.7}$$
where $Q \in \mathbb{K}^{n_\mathcal{U} \times n_\mathcal{Y}}$ *is an element of the set* Ω (3.6) *such that in addition* $\det(\widetilde{U}_* + QG_{22}) \neq 0$.

Moreover, if $Q \in \Omega$*, that* $\det(U_* + G_{22}Q) \neq 0$ *if and only if* $\det(\widetilde{U}_* + QG_{22}) \neq 0$*, and the formulas* (3.5) *and* (3.7) *give rise to the same controller* K.

Proof. By Theorem 3.1, if K is a stabilizing controller for G, then K has the form $K = VU^{-1}$ with $L = \left[\begin{smallmatrix} U \\ V \end{smallmatrix}\right]$ as in part (1) of Theorem 3.1 and then $\Theta(G, K)$ is as in (3.2). Similarly $\Theta(G, K_*)$ is given as $\Theta(G; U_*, V_*)$ in (3.2) with U_*, V_* in place of U, V. As by assumption $\Theta(G; U_*, V_*)$ is stable, it follows that $\Theta(G; U, V)$ is stable if and only if $\Theta(G; U, V) - \Theta(G; U_*, V_*)$ is stable. Let $Q = V - V_*$; as $U = I + G_{22}V$ and $U_* = I + G_{22}V_*$, it follows that $U - U_* = G_{22}Q$. From (3.2) we then see that the stable quantity $\Theta(G; U, V) - \Theta(G; U_*, V_*)$ is given by
$$\Theta(G; U, V) - \Theta(G; U_*, V_*) = \begin{bmatrix} G_{12} \\ I \\ G_{22} \end{bmatrix} Q \begin{bmatrix} G_{21} & G_{22} & I \end{bmatrix}.$$
Thus
$$K = VU^{-1} = (V_* + (V - V_*))(U_* + (U - U_*))^{-1} = (V_* + Q)(U_* + G_{22}Q)^{-1},$$
where Q is an element of Ω such that $\det(U_* + G_{22}Q) \neq 0$.

Conversely, suppose K has the form $K = (V_* + Q)(U_* + G_{22}Q)^{-1}$ where $Q \in \Omega$ and $\det(U_* + G_{22}Q) \neq 0$. Define $V = V_* + Q$, $U = U_* + G_{22}Q$. Then one easily checks that
$$\Theta(G; U, V) = \Theta(G; U_*, V_*) + \begin{bmatrix} G_{12} \\ I \\ G_{22} \end{bmatrix} Q \begin{bmatrix} G_{21} & G_{22} & I \end{bmatrix}$$
is stable and
$$\begin{bmatrix} -G_{22} & I \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} -G_{22} & I \end{bmatrix} \begin{bmatrix} V_* \\ U_* \end{bmatrix} + \begin{bmatrix} -G_{22} & I \end{bmatrix} \begin{bmatrix} Q \\ G_{22}Q \end{bmatrix} = I + 0 = I.$$
So $K = VU^{-1}$ stabilizes G by part (1) of Theorem 3.1. This completes the proof of the first statement of the theorem. The second part follows in a similar way by using the second statement in Theorem 3.1 and $Q = \widetilde{V} - \widetilde{V}_*$. Finally, since $V = \widetilde{V}$ and $V_* = \widetilde{V}_*$, we find that indeed $\det(U_* + G_{22}Q) \neq 0$ if and only if $\det(\widetilde{U}_* + QG_{22}) \neq 0$, and the formulas (3.5) and (3.7) give rise to the same controller K. □

The drawback of the parametrization of the stabilizing controllers in Theorem 3.2 is that the set Ω is not really a free-parameter set. By definition, $Q \in \Omega$ if Q

itself is stable (from the (1,3) entry in the defining matrix for the Ω in (3.6)), but, in addition, the eight additional transfer matrices

$$\begin{gathered} G_{12}QG_{21}, \quad G_{12}QG_{22}, \quad G_{12}Q, \quad QG_{21}, \\ QG_{22}, \quad G_{22}QG_{21}, \quad G_{22}QG_{22}, \quad G_{22}Q \end{gathered}$$

should all be stable as well. The next lemma shows how the parameter set Ω can in turn be parametrized by a free stable parameter Λ of size $(n_{\mathcal{U}}+n_{\mathcal{Y}})\times(n_{\mathcal{U}}+n_{\mathcal{Y}})$.

Lemma 3.3. *Assume that G is stabilizable and that K_* is a particular stabilizing controller for G. Let $Q \in \mathbb{K}^{n_{\mathcal{U}}\times n_{\mathcal{Y}}}$. Then the following are equivalent:*

(i) *Q is an element of the set Ω in* (3.6)*,*

(ii) $\begin{bmatrix} I \\ G_{22} \end{bmatrix} Q \begin{bmatrix} G_{22} & I \end{bmatrix}$ *is stable,*

(iii) *Q has the form $Q = \widetilde{L}\Lambda L$ for a stable free-parameter $\Lambda \in \mathbb{A}^{(n_{\mathcal{U}}+n_{\mathcal{Y}})\times(n_{\mathcal{U}}+n_{\mathcal{Y}})}$, where $\widetilde{L} \in \mathbb{A}^{n_{\mathcal{U}}\times(n_{\mathcal{U}}+n_{\mathcal{Y}})}$ and $L \in \mathbb{A}^{(n_{\mathcal{U}}+n_{\mathcal{Y}})\times n_{\mathcal{Y}}}$ are given by*

$$\widetilde{L} = \begin{bmatrix}(I-K_*G_{22})^{-1} & -(I-K_*G_{22})^{-1}K_*\end{bmatrix}, \quad L = \begin{bmatrix} -K_*(I-G_{22}K_*)^{-1} \\ (I-G_{22}K_*)^{-1} \end{bmatrix}. \tag{3.8}$$

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. Suppose that $\Lambda = \left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] Q \left[\begin{smallmatrix} G_{22} & I \end{smallmatrix}\right]$ is stable. Note that

$$\begin{aligned} \widetilde{L}\Lambda L &= \begin{bmatrix}(I-K_*G_{22})^{-1} & -(I-K_*G_{22})^{-1}K_*\end{bmatrix} \begin{bmatrix} I \\ G_{22} \end{bmatrix} Q \times \\ &\qquad\qquad \times \begin{bmatrix} G_{22} & I \end{bmatrix} \begin{bmatrix} -K_*(I-G_{22}K_*)^{-1} \\ (I-G_{22}K_*)^{-1} \end{bmatrix} \\ &= Q. \end{aligned}$$

Hence (ii) implies (iii). Finally assume $Q = \widetilde{L}\Lambda L$ for a stable Λ. To show that $Q \in \Omega$, as Λ is stable, it suffices to show that

$$L_1 := \begin{bmatrix} G_{12} \\ I \\ G_{22} \end{bmatrix} \widetilde{L} \text{ is stable, and } L_2 := L \begin{bmatrix} G_{21} & G_{22} & I \end{bmatrix} \text{ is stable.}$$

Spelling out L_1, using the definition of $\widetilde{L}$ from (3.8), gives

$$L_1 = \begin{bmatrix} G_{12} \\ I \\ G_{22} \end{bmatrix} \begin{bmatrix}(I-K_*G_{22})^{-1} & -(I-K_*G_{22})^{-1}K_*\end{bmatrix}.$$

We note that each of the six matrix entries of L_1 are stable, since they all occur among the matrix entries of $\Theta(G, K_*)$ (see (2.2)) and K_* stabilizes G by assumption. Similarly, each of the six matrix entries of L_2 given by

$$L_2 = \begin{bmatrix} -K_*(I-G_{22}K_*)^{-1} \\ (I-G_{22}K_*)^{-1} \end{bmatrix} \begin{bmatrix} G_{21} & G_{22} & I \end{bmatrix}$$

is stable since K_* stabilizes G. It therefore follows that $Q \in \Omega$ as wanted. $\square$

We say that K stabilizes G_{22} if the map $\left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in Figure 1 is stable, i.e., the usual stability holds with $w = 0$ and z ignored. This amounts to the stability of the lower right 2×2 block in $\Theta(G, K)$:

$$\begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ G_{22}(I - KG_{22})^{-1} & I + G_{22}(I - KG_{22})^{-1}K \end{bmatrix}.$$

The equivalence of (i) and (ii) in Lemma 3.3 implies the following result.

Corollary 3.4. *Assume that G is stabilizable. Then K stabilizes G if and only if K stabilizes G_{22}.*

Proof. Assume $K_* \in \mathbb{K}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ stabilizes G. Then in particular the lower left 2×2 block in $\Theta(G, K_*)$ is stable. Thus K_* stabilizes G_{22}. Moreover, K stabilizes G_{22} if and only if K stabilizes G when we impose $G_{11} = 0$, $G_{12} = 0$ and $G_{21} = 0$, that is, K is of the form (3.5) with U_* and V_* as in Theorem 3.2 and $Q \in \mathbb{K}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ is such that $\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] Q \left[\begin{smallmatrix} G_{22} & I \end{smallmatrix}\right]$ is stable. But then it follows from the implication (ii) $\Longrightarrow$ (i) in Lemma 3.3 that Q is in Ω, and thus, by Theorem 3.2, K stabilizes G (without $G_{11} = 0$, $G_{12} = 0$, $G_{21} = 0$). ☐

Combining Lemma 3.3 with Theorem 3.2 leads to the following generalization of Theorem 2.1 giving a parametrization of stabilizing controllers without the assumption of any coprime factorization.

Theorem 3.5. *Assume that $G \in \mathbb{K}^{(n_{\mathcal{Z}}+n_{\mathcal{Y}})\times(n_{\mathcal{W}}+n_{\mathcal{U}})}$ is stabilizable and that K_* is one stabilizing controller for G. Define $U_* = (I - G_{22}K_*)^{-1}$, $V_* = K_*(I - G_{22}K_*)^{-1}$, $\widetilde{U}_* = (I - K_*G_{22})^{-1}$ and $\widetilde{V}_* = (I - K_*G_{22})^{-1}K_*$. Then the set of all stabilizing controllers for G are given by*

$$K = (V_* + Q)(U_* + G_{22}Q)^{-1} = (\widetilde{U}_* + QG_{22})^{-1}(\widetilde{V}_* + Q),$$

where $Q = \widetilde{L}\Lambda L$ where $\widetilde{L}$ and L are given by (3.8) and Λ is a free stable parameter of size $(n_{\mathcal{U}}+n_{\mathcal{Y}}) \times (n_{\mathcal{U}}+n_{\mathcal{Y}})$ so that $\det(U_* + G_{22}Q) \neq 0$ or equivalently $\det(\widetilde{U}_* + QG_{22}) \neq 0$.

3.2. The Youla-Kučera parametrization

There are two drawbacks to the parametrization of the stabilizing controllers obtained in Theorem 3.5, namely, to find all stabilizing controllers one first has to find a particular stabilizing controller, and secondly, the map $\Lambda \mapsto Q$ given in Part (iii) of Lemma 3.3 is in general not one-to-one. We now show that, under the additional hypothesis that G_{22} admits a double coprime factorization, both issues can be remedied, and we are thereby led to the well known Youla-Kučera parametrization for the stabilizing controllers.

Recall that G_{22} has a double coprime factorization in case there exist stable transfer matrices D, N, X, Y, $\widetilde{D}$, $\widetilde{N}$, $\widetilde{X}$ and $\widetilde{Y}$ so that the determinants of D, $\widetilde{D}$, X and $\widetilde{X}$ are all nonzero (in $\mathbb{A}$) and

$$G_{22} = D^{-1}N = \widetilde{N}\widetilde{D}^{-1}, \quad \begin{bmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{bmatrix}\begin{bmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{bmatrix} = \begin{bmatrix} I_{n_{\mathcal{Y}}} & 0 \\ 0 & I_{n_{\mathcal{U}}} \end{bmatrix}. \qquad (3.9)$$

According to Corollary 3.4 it suffices to focus on describing the stabilizing controllers of G_{22}. Note that K stabilizes G_{22} means that

$$\begin{bmatrix} (I-KG_{22})^{-1} & (I-KG_{22})^{-1}K \\ G_{22}(I-KG_{22})^{-1} & I+G_{22}(I-KG_{22})^{-1}K \end{bmatrix}$$

is stable, or, by Theorem 3.2, that K is given by (3.5) or (3.7) for some $Q \in \mathbb{K}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ so that $\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] Q \left[\begin{smallmatrix} G_{22} & I \end{smallmatrix}\right]$ is stable.

In case G_{22} has a double coprime factorization Quadrat shows in [120, Proposition 4] that the equivalence of (ii) and (iii) in Lemma 3.3 has the following refinement. We provide a proof for completeness.

Lemma 3.6. *Suppose that G_{22} has a double coprime factorization* (3.9)*. Let $Q \in \mathbb{K}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$. Then $\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] Q \left[\begin{smallmatrix} G_{22} & I \end{smallmatrix}\right]$ is stable if and only if $Q = \widetilde{D}\Lambda D$ for some $\Lambda \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$.*

Proof. Let $Q = \widetilde{D}\Lambda D$ for some $\Lambda \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$. Then

$$\begin{bmatrix} I \\ G_{22} \end{bmatrix} Q \begin{bmatrix} G_{22} & I \end{bmatrix} = \begin{bmatrix} QG_{22} & Q \\ G_{22}QG_{22} & G_{22}Q \end{bmatrix} = \begin{bmatrix} \widetilde{D}\Lambda N & \widetilde{D}\Lambda D \\ \widetilde{N}\Lambda N & \widetilde{N}\Lambda D \end{bmatrix}.$$

Hence $\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] Q \left[\begin{smallmatrix} G_{22} & I \end{smallmatrix}\right]$ is stable.

Conversely, assume that $\left[\begin{smallmatrix} I \\ G_{22} \end{smallmatrix}\right] Q \left[\begin{smallmatrix} G_{22} & I \end{smallmatrix}\right]$ is stable. Set $\Lambda = \widetilde{D}^{-1}QD^{-1}$. Then with X, Y, $\widetilde{X}$ and $\widetilde{Y}$ the transfer matrices from the coprime factorization (3.9) we have

$$\begin{aligned} \Lambda &= \begin{bmatrix} \widetilde{X} & -\widetilde{Y} \end{bmatrix} \begin{bmatrix} \widetilde{D} \\ \widetilde{N} \end{bmatrix} \Lambda \begin{bmatrix} N & D \end{bmatrix} \begin{bmatrix} -Y \\ X \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{X} & -\widetilde{Y} \end{bmatrix} \begin{bmatrix} \widetilde{D}\Lambda N & \widetilde{D}\Lambda D \\ \widetilde{N}\Lambda N & \widetilde{N}\Lambda D \end{bmatrix} \begin{bmatrix} -Y \\ X \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{X} & -\widetilde{Y} \end{bmatrix} \begin{bmatrix} QG_{22} & Q \\ G_{22}QG_{22} & G_{22}Q \end{bmatrix} \begin{bmatrix} -Y \\ X \end{bmatrix}. \end{aligned}$$

Thus Λ is stable. □

Lemma 3.7. *Assume that G_{22} admits a double coprime factorization* (3.9)*. Then K_0 is a stabilizing controller for G_{22} if and only if there exist $X_0 \in \mathbb{A}^{n_{\mathcal{Y}} \times n_{\mathcal{Y}}}$, $Y_0 \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$, $\widetilde{X}_0 \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{U}}}$ and $\widetilde{Y}_0 \in \mathbb{A}^{n_{\mathcal{U}} \times n_{\mathcal{Y}}}$ with $\det(X_0) \neq 0$, $\det(\widetilde{X}_0) \neq 0$ so that $K_0 = Y_0X_0^{-1} = \widetilde{X}_0^{-1}\widetilde{Y}_0$ and*

$$\begin{bmatrix} D & -N \\ -\widetilde{Y}_0 & \widetilde{X}_0 \end{bmatrix} \begin{bmatrix} X_0 & \widetilde{N} \\ Y_0 & \widetilde{D} \end{bmatrix} = \begin{bmatrix} I_{n_{\mathcal{Y}}} & 0 \\ 0 & I_{n_{\mathcal{U}}} \end{bmatrix}.$$

In particular, $K = YX^{-1} = \widetilde{X}^{-1}\widetilde{Y}$ is a stabilizing controller for G_{22}, where $X, Y, \widetilde{X}, \widetilde{Y}$ come from the double coprime factorization (3.9) *for G_{22}.*

Proof. Note that if K is a stabilizing controller for G_{22}, then, in particular,

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-KG_{22})^{-1} & K(I-G_{22}K)^{-1} \\ (I-G_{22}K)^{-1}G_{22} & (I-G_{22}K)^{-1} \end{bmatrix} \tag{3.10}$$

is stable. The above identity makes sense, irrespectively of K being a stabilizing controller, as long as the left hand side is invertible. Let X, Y, $\widetilde{X}$ and $\widetilde{Y}$ be the transfer matrices from the double coprime factorization. Set $K = \widetilde{X}^{-1}\widetilde{Y} = YX^{-1}$. Then we have

$$\begin{aligned} \begin{bmatrix} \widetilde{X} & -\widetilde{Y} \\ -N & D \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{X} & 0 \\ 0 & D \end{bmatrix} &= \left(\begin{bmatrix} \widetilde{X}^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{X} & -\widetilde{Y} \\ -N & D \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} I & -\widetilde{X}^{-1}\widetilde{Y} \\ -D^{-1}N & I \end{bmatrix} = \begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1}. \end{aligned}$$

Since $\widetilde{X}$, D and $\left[\begin{smallmatrix} \widetilde{X} & -\widetilde{Y} \\ -N & D \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} \widetilde{D} & Y \\ \widetilde{N} & X \end{smallmatrix}\right]$ are stable, it follows that the right-hand side of (3.10) is stable as well. We conclude that $K = \widetilde{X}^{-1}\widetilde{Y} = YX^{-1}$ stabilizes G_{22}.

Now let K_0 be any stabilizing controller for G_{22}. It follows from the first part of the proof that $K = YX^{-1} = \widetilde{X}^{-1}\widetilde{Y}$ is stabilizing for G_{22}. Define V and U by (3.1) and $\widetilde{V}$ and $\widetilde{U}$ by (3.3). Then, using Theorem 3.2 and Lemma 3.6, there exists a $\Lambda \in \mathbb{A}^{n_U \times n_Y}$ so that

$$K_0 = (V+Q)(U+G_{22}Q)^{-1} = (\widetilde{U}+QG_{22})^{-1}(\widetilde{V}+Q),$$

where $Q = \widetilde{D}\Lambda D$. We compute that

$$(I-G_{22}K)^{-1} = (I-D^{-1}NYX^{-1})^{-1} = X(DX-NY)^{-1}D = XD \tag{3.11}$$

and

$$(I-KG_{22})^{-1} = (I-\widetilde{X}^{-1}\widetilde{Y}\widetilde{N}\widetilde{D}^{-1})^{-1} = \widetilde{D}(\widetilde{X}\widetilde{D}-\widetilde{Y}\widetilde{N})^{-1}\widetilde{X} = \widetilde{D}\widetilde{X}. \tag{3.12}$$

Thus

$$V = YD, \quad U = XD, \quad \widetilde{V} = \widetilde{D}\widetilde{Y}, \quad \widetilde{U} = \widetilde{D}\widetilde{X}.$$

Therefore

$$\begin{aligned} K_0 &= (V+Q)(U+G_{22}Q)^{-1} = (YD+\widetilde{D}\Lambda D)(XD+\widetilde{N}\Lambda D)^{-1} \\ &= (Y+\widetilde{D}\Lambda)(X+\widetilde{N}\Lambda)^{-1} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} K_0 &= (\widetilde{U}+QG_{22})^{-1}(\widetilde{V}+Q) = (\widetilde{D}\widetilde{X}+\widetilde{D}\Lambda N)^{-1}(\widetilde{D}\widetilde{Y}+\widetilde{D}\Lambda D) \\ &= (\widetilde{X}+\Lambda N)^{-1}(\widetilde{Y}+\Lambda D). \end{aligned} \tag{3.14}$$

Set

$$Y_0 = (Y+\widetilde{D}\Lambda), \quad X_0 = (X+\widetilde{N}\Lambda), \quad \widetilde{Y}_0 = (\widetilde{Y}+\Lambda D), \quad \widetilde{X}_0 = (\widetilde{X}+\Lambda N).$$

Then certainly $\det X_0 \neq 0$ and $\det \widetilde{X}_0 \neq 0$, and we have

$$\begin{aligned}
&\begin{bmatrix} D & -N \\ -\widetilde{Y}_0 & \widetilde{X}_0 \end{bmatrix}\begin{bmatrix} X_0 & \widetilde{N} \\ Y_0 & \widetilde{D} \end{bmatrix} = \begin{bmatrix} D & -N \\ -\widetilde{Y} - \Lambda D & \widetilde{X} + \Lambda N \end{bmatrix}\begin{bmatrix} X + \widetilde{N}\Lambda & \widetilde{N} \\ Y + \widetilde{D}\Lambda & \widetilde{D} \end{bmatrix} \\
&\quad = \begin{bmatrix} I & 0 \\ -\Lambda & I \end{bmatrix}\begin{bmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{bmatrix}\begin{bmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{bmatrix}\begin{bmatrix} I & 0 \\ \Lambda & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\Lambda & I \end{bmatrix}\begin{bmatrix} I & 0 \\ \Lambda & I \end{bmatrix} \\
&\quad = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
\end{aligned}$$

□

Since any stabilizing controller for G is also a stabilizing controller for G_{22}, the following corollary is immediate.

Corollary 3.8. *Assume that $G \in \mathbb{K}^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$ is a stabilizable and that G_{22} admits a double coprime factorization. Then any stabilizing controller K of G admits a double coprime factorization.*

Lemma 3.9. *Assume that G is stabilizable and that G_{22} admits a double coprime factorization. Then there exists a double coprime factorization* (3.9) *for G_{22} so that DG_{21} and $G_{12}\widetilde{D}$ are stable.*

Proof. Let K be a stabilizing controller for G. Then K is also a stabilizing controller for G_{22}. Thus, according to Lemma 3.7, there exists a double coprime factorization (3.9) for G_{22} so that $K = YX^{-1} = \widetilde{X}\widetilde{Y}^{-1}$. Note that (3.9) implies that $\left[\begin{smallmatrix} X & \widetilde{N} \\ Y & \widetilde{D} \end{smallmatrix}\right]\left[\begin{smallmatrix} D & -N \\ -\widetilde{Y} & \widetilde{X} \end{smallmatrix}\right] = I$. In particular, $\widetilde{D}\widetilde{Y} = YD$ and $\widetilde{N}\widetilde{X} = XN$. Moreover, from the computations (3.11) and (3.12) we see that

$$(I - G_{22}K)^{-1} = XD \quad \text{and} \quad (I - KG_{22})^{-1} = \widetilde{D}\widetilde{X}.$$

Inserting these identities into the formula for $\Theta(G,K)$, and using that K stabilizes G, we find that

$$\Theta(G,K) = \begin{bmatrix} G_{11} + G_{12}YDG_{21} & G_{12}\widetilde{D}\widetilde{X} & G_{12}\widetilde{D}\widetilde{Y} \\ YDG_{21} & \widetilde{D}\widetilde{X} & \widetilde{D}\widetilde{Y} \\ XDG_{21} & \widetilde{N}\widetilde{X} & I + \widetilde{N}\widetilde{Y} \end{bmatrix} \text{ is stable.}$$

In particular $\begin{bmatrix} G_{12}\widetilde{D}\widetilde{X} & G_{12}\widetilde{D}\widetilde{Y} \end{bmatrix}$ is stable, and thus

$$\begin{bmatrix} G_{12}\widetilde{D}\widetilde{X} & G_{12}\widetilde{D}\widetilde{Y} \end{bmatrix}\begin{bmatrix} \widetilde{N} \\ -\widetilde{D} \end{bmatrix} = G_{12}\widetilde{D}(\widetilde{X}\widetilde{N} - \widetilde{Y}\widetilde{D}) = G_{12}\widetilde{D}$$

is stable. Similarly, since $\left[\begin{smallmatrix} YDG_{21} \\ XDG_{21} \end{smallmatrix}\right]$ is stable, we find that

$$\begin{bmatrix} -N & D \end{bmatrix}\begin{bmatrix} YDG_{21} \\ XDG_{21} \end{bmatrix} = (-NY + DX)DG_{21} = DG_{21}$$

is stable. □

We now present an alternative proof of Corollary 3.4 for the case that G_{22} admits a double coprime factorization.

Lemma 3.10. *Assume that G is stabilizable and G_{22} admits a double coprime factorization. Then K stabilizes G if and only if K stabilizes G_{22}.*

Proof. It was already noted that in case K stabilizes G, then K also stabilizes G_{22}. Now assume that K stabilizes G_{22}. Let $Q \in \mathbb{K}^{n_\mathcal{U} \times n_\mathcal{Y}}$ so that K is given by (3.5). It suffices to show that $Q \in \Omega$, with Ω defined by (3.6). Since G is stabilizable, it follows from Lemma 3.9 that there exists a double coprime factorization (3.9) of G_{22} so that DG_{21} and $G_{12}\widetilde{D}$ are stable. According to Lemma 3.6, $Q = \widetilde{D}\Lambda D$ for some $\Lambda \in \mathbb{A}^{n_\mathcal{U} \times n_\mathcal{Y}}$. It follows that

$$\begin{aligned}
\begin{bmatrix} G_{12} \\ I \\ G_{22} \end{bmatrix} Q \begin{bmatrix} G_{21} & G_{22} & I \end{bmatrix} &= \begin{bmatrix} G_{12}\widetilde{D} \\ \widetilde{D} \\ G_{22}\widetilde{D} \end{bmatrix} \Lambda \begin{bmatrix} DG_{21} & DG_{22} & D \end{bmatrix} \\
&= \begin{bmatrix} G_{12}\widetilde{D} \\ \widetilde{D} \\ \widetilde{N} \end{bmatrix} \Lambda \begin{bmatrix} DG_{21} & N & D \end{bmatrix}
\end{aligned}$$

is stable. Hence $Q \in \Omega$. $\square$

Combining the results from the Lemmas 3.6, 3.7 and 3.10 with Theorem 3.2 and the computations (3.13) and (3.14) from the proof of Lemma 3.7 we obtain the Youla-Kučera parametrization of all stabilizing controllers.

Theorem 3.11. *Assume that $G \in \mathbb{K}^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$ is stabilizable and that G_{22} admits a double coprime factorization* (3.9)*. Then the set of all stabilizing controllers is given by*

$$K = (Y + \widetilde{D}\Lambda)(X + \widetilde{N}\Lambda)^{-1} = (\widetilde{X} + \Lambda N)^{-1}(\widetilde{Y} + \Lambda D),$$

where Λ is a free stable parameter from $\mathbb{A}^{n_\mathcal{U} \times n_\mathcal{Y}}$ such that $\det(X + \widetilde{N}\Lambda) \neq 0$ *or equivalently* $\det(\widetilde{X} + \Lambda N) \neq 0$.

3.3. The standard H^∞-problem reduced to model matching.

We now consider the H^∞-problem for a plant $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right] : \mathcal{W} \oplus \mathcal{U} \to \mathcal{Z} \oplus \mathcal{Y}$, i.e., we seek a controller $K \colon \mathcal{Y} \to \mathcal{U}$ so that not only $\Theta(G, K)$ in (2.2) is stable, but also

$$\|G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\|_{op} \leq 1.$$

Assume that the plant G is stabilizable, and that $K_* \colon \mathcal{Y} \to \mathcal{U}$ stabilizes G. Define U_*, V_*, $\widetilde{U}_*$ and $\widetilde{V}_*$ as in Theorem 3.2. We then know that all stabilizing controllers of G are given by

$$K = (V_* + Q)(U_* + G_{22}Q)^{-1} = (\widetilde{U}_* + QG_{22})^{-1}(\widetilde{V}_* + Q)$$

where $Q \in \mathbb{K}^{n_\mathcal{U} \times n_\mathcal{Y}}$ is any element of Ω in (3.6). We can then express the transfer matrices U and V in (3.1) in terms of Q as follows:

$$\begin{aligned} U &= (I - G_{22}K)^{-1} = (I - G_{22}(V_* - Q)(U_* - G_{22}Q)^{-1})^{-1} \\ &= (U_* - G_{22}Q)(U_* - G_{22}Q - G_{22}(V_* - Q))^{-1} \\ &= (U_* - G_{22}Q)(U_* - G_{22}V_*)^{-1} \\ &= (U_* - G_{22}Q), \end{aligned}$$

where we used that $U_* - G_{22}V_* = I$, and

$$V = KU = V_* - Q.$$

Similar computations provide the formulas

$$\widetilde{U} = \widetilde{U}_* + QG_{22} \quad \text{and} \quad \widetilde{V} = \widetilde{V}_* + Q$$

for the transfer matrices $\widetilde{U}$ and $\widetilde{V}$ in (3.3). Now recall that $\Theta(G, K)$ can be expressed in terms of U and V as in (3.2). It then follows that left upper block in $\Theta(G, K)$ is equal to

$$\begin{aligned} G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} &= G_{11} + G_{12}VG_{21} \\ &= G_{11} + G_{12}V_*G_{21} - G_{12}QG_{21}. \end{aligned} \tag{3.15}$$

The fact that K_* stabilizes G implies that $\widetilde{G}_{11} := G_{11} + G_{12}V_*G_{21}$ is stable, and thus $G_{12}QG_{21}$ is stable as well. We are now close to a reformulation of the H^∞-problem as a model matching problem. However, to really formulate it as a model matching problem, we need to apply the change of design parameter $Q \mapsto \Lambda$ defined in Lemma 3.3, or Lemma 3.6 in case G_{22} admits a double coprime factorization. The next two results extend the idea of Theorem 2.2 to this more general setting.

Theorem 3.12. *Assume that $G \in \mathbb{K}^{(n_\mathcal{Z}+n_\mathcal{Y})\times(n_\mathcal{W}+n_\mathcal{U})}$ is stabilizable and let $K \in \mathbb{K}^{n_\mathcal{U} \times n_\mathcal{Y}}$. Then K is a solution to the standard H^∞ problem for G if and only if*

$$K = (V_* + Q)(U_* + G_{22}Q)^{-1} = (\widetilde{U}_* + QG_{22})^{-1}(\widetilde{V}_* + Q)$$

with $Q = \widetilde{L}\Lambda L$, where $\widetilde{L}$ and L are defined by (3.8), so that $\det(U_* + G_{22}Q) \neq 0$, *or equivalently* $\det(\widetilde{U}_* + QG_{22}) \neq 0$, *and $\Lambda \in \mathbb{A}^{(n_\mathcal{U}+n_\mathcal{Y})\times(n_\mathcal{U}+n_\mathcal{Y})}$ is any solution to the model matching problem for $\widetilde{G}_{11}$, $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ defined by*

$$\widetilde{G}_{11} := G_{11} + G_{12}V_*G_{21}, \quad \widetilde{G}_{12} := G_{12}\widetilde{L}, \quad \widetilde{G}_{21} := LG_{21},$$

i.e., so that

$$\|\widetilde{G}_{11} + \widetilde{G}_{12}\Lambda\widetilde{G}_{21}\|_{op} \leq 1.$$

Proof. The statement essentially follows from Theorem 3.5 and the computation (3.15) except that we need to verify that the functions $\widetilde{G}_{11}$, $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ satisfy the conditions to be data for a model matching problem, that is, they should be stable. It was already observed that $\widetilde{G}_{11}$ is stable. The fact that $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ are stable was shown in the proof of Lemma 3.3. $\square$

We have a similar result in case G_{22} admits a double coprime factorization.

Theorem 3.13. *Assume that $G \in \mathbb{K}^{(n_{\mathcal{Z}}+n_{\mathcal{Y}})\times(n_{\mathcal{W}}+n_{\mathcal{U}})}$ is stabilizable and that G_{22} admits a double coprime factorization* (3.9)*. Let $K \in \mathbb{K}^{n_{\mathcal{Y}}\times n_{\mathcal{U}}}$. Then K is a solution to the standard H^∞ problem for G if and only if*

$$K = (Y + \widetilde{D}\Lambda)(X + \widetilde{N}\Lambda)^{-1} = (\widetilde{X} + \Lambda N)^{-1}(\widetilde{Y} + \Lambda D),$$

where $\Lambda \in \mathbb{A}^{n_{\mathcal{U}}\times n_{\mathcal{Y}}}$ so that $\det(X + \widetilde{N}\Lambda) \neq 0$*, or equivalently* $\det(\widetilde{X} + \Lambda N) \neq 0$*, is any solution to the model matching problem for $\widetilde{G}_{11}$, $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ defined by*

$$\widetilde{G}_{11} := G_{11} + G_{12}YDG_{21}, \quad \widetilde{G}_{12} := G_{12}\widetilde{D}, \quad \widetilde{G}_{21} := DG_{21},$$

i.e., so that

$$\|\widetilde{G}_{11} + \widetilde{G}_{12}\Lambda\widetilde{G}_{21}\|_{op} \leq 1.$$

Proof. The same arguments apply as in the proof of Theorem 3.12, except that in this case Lemma 3.9 should be used to show that $\widetilde{G}_{12}$ and $\widetilde{G}_{21}$ are stable. □

3.4. Notes

The development in Section 3.1 on the parametrization of stabilizing controllers without recourse to a double coprime factorization of G_{22} is based on the exposition of Quadrat [120]. It was already observed by Zames-Francis [140] that $Q = K(I - G_{22}K)^{-1}$ can be used as a free stable design parameter in case G_{22} is itself already stable; in case G_{22} is not stable, Q is subject to some additional interpolation conditions. The results of [120] is an adaptation of this observation to the general ring-theoretic setup. The more theoretical papers [118, 122] give module-theoretic interpretations for the structure associated with internal stabilizability. In particular, it comes out that every matrix transfer function G_{22} with entries in $\mathbb{K}$ has a double-coprime factorization if and only if $\mathbb{A}$ is a *Bezout domain*, i.e., every finitely generated ideal in $\mathbb{A}$ is principal; this recovers a result already appearing in the book of Vidyasagar [136]. A new result which came out of this module-theoretic interpretation was that internal stabilizability of a plant G_{22} is equivalent to the existence of a double-coprime factorization for G_{22} exactly when the ring $\mathbb{A}$ is *projective-free*, i.e., every submodule of a finitely generated free module over $\mathbb{A}$ must itself be free. This gives an explanation for the earlier result of Smith [130] that this phenomenon holds for the case where $\mathbb{A}$ is equal H^∞ over the unit disk or right-half plane.

Earlier less complete results concerning parametrization of the set of stabilizing controllers without the assumption of a coprime factorization were obtained by Mori [102] and Sule [132]. Mori [103] also showed that the internal-stabilization problem can be reduced to model matching form for the general case where the plant has the full 2×2-block structure $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$.

Lemma 3.10 for the classical case is Theorem 2 on page 35 in [64]. The proof there relies in a careful analysis of signal-flow diagrams; we believe that our proof is more direct.

4. Feedback control for linear time-invariant multidimensional systems

4.1. Multivariable frequency-domain formulation

The most obvious multivariable analogue of the classical single-variable setting considered in the book of Francis [64] is as follows. We take the underlying field to be the complex numbers $\mathbb{C}$; in the engineering applications, one usually requires that the underlying field be the reals $\mathbb{R}$, but this can often be incorporated at the end by using the characterization of real rational functions as being those complex rational functions which are invariant under the conjugation operator $s(z) \mapsto \overline{s(\overline{z})}$. We let $\mathbb{D}^d = \{z = (z_1, \dots, z_d) \colon |z_k| < 1\}$ be the unit polydisk in the d-dimensional complex space $\mathbb{C}^d$ and we take our ring $\mathbb{A}$ of stable plants to be the ring $\mathbb{C}(z)_s$ of all rational functions $s(z) = \frac{p(z)}{q(z)}$ in d variables (thus, p and q are polynomials in the d variables $z_1, \dots, z_d$ where we set $z = (z_1, \dots, z_d)$) such that $s(z)$ is bounded on the polydisk $\mathbb{D}^d$. The ring $\mathbb{C}[z]$ of polynomials in d variables is a unique factorization domain so we may assume that p and q have no common factor (i.e., that p and q are *relatively coprime*) in the fractional representation $s = \frac{p}{q}$ for any element of $\mathbb{C}(z_1, \dots, z_d)$. Unlike in the single-variable case, for the case $d > 1$ it can happen that p and q have common zeros in $\mathbb{C}^d$ even when they are coprime in $\mathbb{C}[z]$ (see [138] for an early analysis of the resulting distinct notions of coprimeness). It turns out that for $d \geq 3$, the ring $\mathbb{C}(z)_s$ is difficult to work with since the denominator q for a stable ring element depends in a tricky way on the numerator p: if $s \in \mathbb{C}(z)_s$ has coprime fractional representation $s = \frac{p}{q}$, while it is the case that necessarily q has no zeros in the open polydisk $\mathbb{D}^d$, it can happen that the zero variety of q touches the boundary $\partial\mathbb{D}^d$ as long as the zero variety of p also touches the same points on the boundary in such a way that the quotient $s = \frac{p}{q}$ remains bounded on $\mathbb{D}^d$. Note that at such a boundary point ζ, the quotient $s = p/q$ has no well-defined value. In the engineering literature (see e.g. [45, 131, 84]), such a point is known as a *nonessential singularity of the second kind.*

To avoid this difficulty, Lin [92, 93] introduced the ring $\mathbb{C}(z)_{ss}$ of *structured stable* rational functions, i.e., rational functions $s \in \mathbb{C}(z)$ so that the denominator q in any coprime fractional representation $s = \frac{p}{q}$ for s has no zeros in the closed polydisk $\overline{\mathbb{D}}^d$. According to the result of Kharitonov–Torres-Muñoz [84], whenever $s = \frac{p}{q} \in \mathbb{C}(z)_s$ is stable in the first (non-structured) sense, an arbitrarily small perturbation of the coefficients of q may lead to the perturbed q having zeros in the open polydisk $\mathbb{D}^d$ resulting in the perturbed version $s = \frac{p}{q}$ of s being unstable; this phenomenon does does not occur for $s \in \mathbb{C}(z)_{ss}$, and thus *structured stable* can be viewed just as a robust version of *stable* (in the unstructured sense). Hence one can argue that structured stability is the more desirable property from an engineering perspective. In the application to delay systems using the systems-over-rings approach [46, 85, 83], on the other hand, it is the collection $\mathbb{C}(z)_{ss}$ of structurally stable rational functions which comes up in the first place.

As the ring $\mathbb{A} = \mathbb{C}(z)_{ss}$ is a commutative integral domain, we can apply the results of Section 3 to this particular setting. It was proved in connection with work on systems-over-rings rather than multidimensional systems (see [46, 83]) that the ring $\mathbb{C}(z)_{ss}$ is *projective-free*. As pointed out in the notes of Section 3 above, it follows that stabilizability of G_{22} is equivalent to the existence of a double coprime factorization for the plant G_{22} (see [119]), thereby settling a conjecture of Lin [92, 93, 94]. We summarize these results as follows.

Theorem 4.1. *Suppose that we are given a system* $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ *over the quotient field* $Q(\mathbb{C}(z)_{ss})$ *of the ring* $\mathbb{C}(z)_{ss}$ *of structurally stable rational functions in* d *variables. If there exists a controller* $K = YX^{-1} = \widetilde{X}^{-1}\widetilde{Y}$ *which internally stabilizes* G*, then* G_{22} *has a double coprime factorization and all internally stabilizing controllers* K *for* G *are given by the Youla-Kučera parametrization.*

Following Subsection 3.3, the Youla-Kučera parametrization can then be used to rewrite the H^∞-problem in the form of a model-matching problem: *Given* T_1, T_2, T_3 *equal to matrices over* $\mathbb{C}(z)_{ss}$ *of respective sizes* $n_\mathcal{Z} \times n_\mathcal{W}$, $n_\mathcal{W} \times n_\mathcal{U}$ *and* $n_\mathcal{Y} \times n_\mathcal{W}$*, find a matrix* Λ *over* $\mathbb{C}(z)_{ss}$ *of size* $n_\mathcal{U} \times n_\mathcal{Y}$ *so that the affine expression* S *given by*

$$S = T_1 + T_2 \Lambda T_3 \tag{4.1}$$

has supremum norm at most 1, i.e., $\|S\|_\infty = \max\{\|S(z)\| \colon z \in \overline{\mathbb{D}}^d\} \le 1$.

For mathematical convenience we shall now widen the class of admissible solutions and allow $\Lambda_1, \dots, \Lambda_J$ to be in the algebra $H^\infty(\mathbb{D}^d)$ of bounded analytic functions on $\mathbb{D}^d$. The unit ball of $H^\infty(\mathbb{D}^d)$ is the set of all holomorphic functions S mapping the polydisk $\mathbb{D}^d$ into the closed single-variable unit disk $\mathbb{D} \subset \mathbb{C}$; we denote this space by $\mathcal{S}_d$, the *d-variable Schur class.* While T_1, T_2 and T_3 are assumed to be in $\mathbb{C}(z)_{ss}$, we allow Λ in (4.1) to be in $H^\infty(\mathbb{D}^d)$.

Just as in the classical one-variable case, it is possible to give the model-matching form (4.1) an interpolation interpretation, at least for special cases (see [73, 74, 32]). One such case is where $n_\mathcal{W} = n_\mathcal{Z} = n_\mathcal{Y} = 1$ while $n_\mathcal{U} = J$. Then T_1 and T_3 are scalar while $T_2 = \left[\begin{smallmatrix} T_{2,1} & \cdots & T_{2,J} \end{smallmatrix}\right]$ is a row. Assume in addition that $T_3 = 1$. Then the model-matching form (4.1) collapses to

$$S = T_1 + T_{21}\Lambda_1 + \cdots + T_{2J}\Lambda_J \tag{4.2}$$

where $\Lambda_1, \dots \Lambda_J$ are J free stable scalar functions. Under the assumption that the intersection of the zero varieties of $T_{2,1}, \dots, T_{2,J}$ within the closed polydisk $\overline{\mathbb{D}}^d$ consists of finitely many (say N) points

$$z_1 = (z_{1,1}, \dots, z_{1,d}), \cdots, z_N = (z_{N,1}, \dots, z_{N,d})$$

and if we let $w_1, \dots, w_N$ be the values of T_1 at these points

$$w_1 = T_1(z_1), \dots, w_N = T_1(z_N),$$

then it is not hard to see that a function $S \in \mathbb{C}(z)_{ss}$ has the form (4.2) if and only if it satisfies the interpolation conditions

$$S(z_i) = w_i \text{ for } i = 1, \dots, N. \tag{4.3}$$

In this case the model-matching problem thus becomes the following finite-point Nevanlinna-Pick interpolation problem over $\mathbb{D}^d$: *find* $S \in \mathbb{C}(z)_{ss}$ *subject to* $|S(z)| \leq 1$ *for all* $z \in \mathbb{D}^d$ *which satisfies the interpolation conditions* (4.3). Then the d-variable H^∞-Model-Matching problem becomes: *find* $S \in \mathcal{S}_d$ *so that* $S(z_1) = w_1$ *for* $i = 1, \dots, N$.

A second case (see [32]) where the polydisk Model-Matching Problem can be reduced to an interpolation problem is the case where T_2 and T_3 are square (so $n_{\mathcal{Z}} = n_{\mathcal{U}}$ and $n_{\mathcal{Y}} = n_{\mathcal{W}}$) with invertible values on the distinguished boundary of the polydisk; under these assumptions it is shown in [32] (see Theorem 3.5 there) how the model-matching problem is equivalent to a *bitangential Nevanlinna-Pick interpolation problem along a subvariety*, i.e., bitangential interpolation conditions are specified along all points of a codimension-1 subvariety of $\mathbb{D}^d$ (namely, the union of the zero sets of $\det T_2$ and $\det T_3$ intersected with $\mathbb{D}^d$). For $d = 1$, codimension-1 subvarieties are isolated points in the unit disk; thus the codimension-1 interpolation problem is a direct generalization of the bitangential Nevanlinna-Pick interpolation problem studied in [26, 58, 62]. However for the case where the number of variables d is at least 3, there is no theory with results parallel to those of the classical case.

Nevertheless, if we change the problem somewhat there is a theory parallel to the classical case. To formulate this adjustment, we define the *d-variable Schur-Agler class $\mathcal{SA}_d$* to consist of those functions S analytic on the polydisk for which the operator $S(X_1, \dots, X_d)$ has norm at most 1 for any collection $X_1, \dots, X_d$ of d commuting strict contraction operators on a separable Hilbert space $\mathcal{K}$; here $S(X_1, \dots, X_d)$ can be defined via the formal power series for S:

$$S(X_1, \dots, X_d) = \sum_{n \in \mathbb{Z}_+^d} s_n X^n, \quad \text{if } S(z) = \sum_{n \in \mathbb{Z}_+^d} s_n z^n$$

where we use the standard multivariable notation

$$n = (n_1, \dots, n_d) \in \mathbb{Z}_+^d, \quad X^n = X_1^{n_1} \cdots X_d^{n_d} \text{ and } z^n = z_1^{n_1} \cdots z_d^{n_d}.$$

For the cases $d = 1, 2$, it turns out, as a consequence of the von Neumann inequality or the Sz.-Nagy dilation theorem for $d = 1$ and of the Andô dilation theorem [17] for $d = 2$ (see [109, 34] for a full discussion), that the Schur-Agler class $\mathcal{SA}_d$ and the Schur class $\mathcal{S}_d$ coincide, while, due to an explicit example of Varopoulos, the inclusion $\mathcal{SA}_d \subset \mathcal{S}_d$ is strict for $d \geq 3$.

There is a result due originally to Agler [1] and developed and refined in a number of directions since (see [3, 35] and [4] for an overview) which parallels the one-variable case; for the case of a simple set of interpolation conditions (4.3) the result is as follows: *there exists a function S in the Schur-Agler class $\mathcal{SA}_d$ which satisfies the set of interpolation conditions $S(z_i) = w_i$ for $i = 1, \dots, N$ if and only if there exist d positive semidefinite matrices $\mathbb{P}^{(1)}, \dots, \mathbb{P}^{(d)}$ of size $N \times N$ so that*

$$1 - w_i \overline{w_j} = \sum_{k=1}^{d} (1 - z_{i,k} \overline{z_{j,k}}) \mathbb{P}_{i,j}^{(k)}.$$

For the case $d = 1$, the Pick matrix $\mathbb{P} = \left[\frac{1-w_i\overline{w}_j}{1-z_i\overline{z}_j} \right]_{i,j=1}^N$ is the unique solution of this equation, and we recover the classical criterion $\mathbb{P} \geq 0$ for the existence of solutions to the Nevanlinna-Pick problem. There is a later realization result of Agler [2] (see also [3, 35]): a given holomorphic function S is in the Schur-Agler class $\mathcal{SA}_d(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ if and only if S has a contractive Givone-Roesser realization: $S(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : (\oplus_{k=1}^d \mathcal{X}_k \oplus \mathcal{U}) \to (\oplus_{k=1}^d \mathcal{X}_k \oplus \mathcal{Y})$ is contractive with $Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}_1} & & \\ & \ddots & \\ & & z_d I_{\mathcal{X}_d} \end{bmatrix}$.

Direct application of the Agler result to the bitangential Nevanlinna-Pick interpolation problem along a subvariety, however, gives a solution criterion involving an infinite Linear Matrix Inequality (where the unknown matrices have infinitely many rows and columns indexed by the points of the interpolation-node subvariety)—see [32, Theorem 4.1]. Alternatively, one can use the polydisk Commutant Lifting Theorem from [31] to get a solution criterion involving a Linear Operator Inequality [32, Theorem 5.2]. Without further massaging, either approach is computationally unattractive; this is in contrast with the state-space approach discussed below. In that setting there exists computable sufficient conditions, in terms of a pair of LMIs and a coupling condition, that in general are only sufficient, unless one works with a more conservative notion of stability and performance.

4.2. Multidimensional state-space formulation

The starting point in this subsection is a quadruple $\{A, B, C, D\}$ consisting of operators A, B, C and D so that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}\oplus\mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Z}\oplus\mathcal{Y} \end{smallmatrix}\right]$ and a partitioning $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ of the space $\mathcal{X}$. Associate with such a quadruple $\{A, B, C, D\}$ is a linear state-space system Σ of Givone-Roesser type (see [67]) that evolves over $\mathbb{Z}_+^d$ and is given by the system of equations

$$\Sigma := \left\{ \begin{array}{l} \begin{bmatrix} x_1(n+e_1) \\ \vdots \\ x_d(n+e_d) \end{bmatrix} = A \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix} + Bu(n) \\ \qquad y(n) = Cx(n) + Du(n) \end{array} \right. \qquad (n \in \mathbb{Z}_+^d), \tag{4.4}$$

with initial conditions a specification of the state values $x_k(\sum_{j\neq k} t_j e_j)$ for $t = (t_1, \dots, t_d) \in \mathbb{Z}_+^d$ subject to $t_k = 0$ where $k = 1, \dots, d$. Here e_k stands for the k-th unit vector in $\mathbb{C}^d$ and $x(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_d(n) \end{bmatrix}$. We call $\mathcal{X}$ the *state-space* and A the *state operator*. Moreover, the block operator matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is referred to as the *system matrix.*

Following [81], the Givone-Roesser system (4.4) is said to be *asymptotically stable* in case, for zero input $u(n) = 0$ for $n \in \mathbb{Z}_+^d$ and initial conditions with the property

$$\sup_{t\in\mathbb{Z}_+^d\,:\, t_k=0} \|x_k(\sum_{j=1}^d t_j e_j)\| < \infty \text{ for } k = 1, \dots, d,$$

the state sequence x satisfies

$$\sup_{n\in\mathbb{Z}_+^d} \|x(n)\| < \infty \quad \text{and} \quad \lim_{n\to\infty} \|x(n)\| = 0,$$

where $n \to \infty$ is to be interpreted as $\min\{n_1, \ldots n_d\} \to \infty$ when $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$.

With the Givone-Roesser system (4.4) we associate the transfer function $G(z)$ given by

$$G(z) = D + C(I - Z(z)A)^{-1}Z(z)B, \tag{4.5}$$

defined al least for $z \in \mathbb{C}^d$ with $\|z\|$ sufficiently small, where

$$Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}_1} & & \\ & \ddots & \\ & & z_d I_{\mathcal{X}_d} \end{bmatrix} \quad (z \in \mathbb{C}^d). \tag{4.6}$$

We then say that $\{A, B, C, D\}$ is a *(state-space) realization* for the function G, or if G is not specified, just refer to $\{A, B, C, D\}$ as a realization. The realization $\{A, B, C, D\}$, or just the operator A, is said to be *Hautus-stable* in case the pencil $I - Z(z)A$ is invertible on the closed polydisk $\overline{\mathbb{D}}^d$.

Here we only consider the case that $\mathcal{X}$ is finite dimensional; then the entries of the transfer function G are in the quotient field $Q(\mathbb{C}(z)_{ss})$ of $\mathbb{C}(z)_{ss}$ and are analytic at 0, and it is straightforward to see that G is structurally stable in case G admits a Hautus-stable realization. For the case $d = 2$ it has been asserted in the literature [81, Theorem 4.8] that asymptotic stability and Hautus stability are equivalent; presumably this assertion continues to hold for general $d \geq 1$ but we do not go into details here.

Given a realization $\{A, B, C, D\}$ where the decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ is understood, our main interest will be in Hautus-stability; hence we shall say simply that A is *stable* rather than *Hautus-stable*.

As before we consider controllers K in $Q(\mathbb{C}(z)_{ss})$ of size $n_{\mathcal{Y}} \times n_{\mathcal{U}}$ that we also assume to be given by a state-space realization

$$K(z) = D_K + C_K(I - Z_K(z)A_K)^{-1}Z_K(z)B_K \tag{4.7}$$

with system matrix $\left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X}_K \\ \mathcal{Y} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X}_K \\ \mathcal{U} \end{smallmatrix}\right]$, a decomposition of the state-space $\mathcal{X}_K = \mathcal{X}_{1,K} \oplus \cdots \oplus \mathcal{X}_{d,K}$ and $Z_K(z)$ defined analogous to $Z(z)$ but with respect to the decomposition of $\mathcal{X}_K$. We now further specify the matrices B, C and D from the realization $\{A, B, C, D\}$ as

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \tag{4.8}$$

compatible with the decompositions $\mathcal{Z} \oplus \mathcal{Y}$ and $\mathcal{W} \oplus \mathcal{U}$. We can then form the closed loop system $G_{cl} = \Sigma(G, K)$ of the two transfer functions. The closed loop system $G_{cl} = \Sigma(G, K)$ corresponds to the feedback connection

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \to \begin{bmatrix} \widetilde{x} \\ z \\ y \end{bmatrix}, \quad \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} : \begin{bmatrix} x_K \\ u_K \end{bmatrix} \to \begin{bmatrix} \widetilde{x}_K \\ y_K \end{bmatrix}$$

subject to

$$x = Z(z)\widetilde{x}, \quad x_K = Z_K(z)\widetilde{x}_K, \quad u_K = y \quad \text{and} \quad y_K = u.$$

This feedback loop is well-posed exactly when $I - D_{22}D_K$ is invertible. Since, under the assumption of well posedness, one can always arrange via a change of variables that $D_{22} = 0$ (cf., [78]), we shall assume that $D_{22} = 0$ for the remainder of the paper. In that case well-posedness is automatic and the closed loop system G_{cl} admits a state-space realization

$$G_{cl}(z) = D_{cl} + C_{cl}(I - Z_{cl}(z)A_{cl})^{-1}Z_{cl}(z)C_{cl} \tag{4.9}$$

with system matrix

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2D_KC_2 & B_2C_K & B_1 + B_2D_KD_{21} \\ B_KC_2 & A_K & B_KD_{21} \\ \hline C_1 + D_{12}D_KC_2 & D_{12}C_K & D_{11} + D_{12}D_KD_{21} \end{array}\right] \tag{4.10}$$

and

$$Z_{cl}(z) = \begin{bmatrix} Z(z) & 0 \\ 0 & Z_K(z) \end{bmatrix} \quad (z \in \mathbb{C}^d).$$

The *state-space (internal) stabilizability problem* then is: Given the realization $\{A, B, C, D\}$ find a compatible controller K with realization $\{A_K, B_K, C_K, D_K\}$ so that the closed-loop realization $\{A_{cl}, B_{cl}, C_{cl}, D_{cl}\}$ is stable, i.e., so that $I - Z_{cl}(z)A_{cl}$ is invertible on the closed polydisk $\overline{\mathbb{D}}^d$. We also consider the *strict state-space H^∞-problem*: Given the realization $\{A, B, C, D\}$, find a compatible controller K with realization $\{A_K, B_K, C_K, D_K\}$ so that the closed loop realization $\{A_{cl}, B_{cl}, C_{cl}, D_{cl}\}$ is stable and the closed-loop system G_{cl} satisfies $\|G_{cl}(z)\| < 1$ for all $z \in \mathbb{D}^d$.

State-space stabilizability. In the fractional representation setting of Section 3 it took quite some effort to derive the result: "If G is stabilizable, then K stabilizes G if and only if K stabilizes G_{22}" (see Corollary 3.4 and Lemma 3.10). For the state-space stabilizability problem this result is obvious, and what is more, one can drop the assumption that G needs to be stabilizable. Indeed, G_{22} admits the realization $\{A, B_2, C_2, 0\}$ (assuming $D_{22} = 0$), so that the closed-loop realization for $\Sigma(G_{22}, K)$ is equal to $\{A_{cl}, 0, 0, 0\}$. In particular, both closed-loop realizations have the same state operator A_{cl}, and thus K with realization $\{A_K, B_K, C_K, D_K\}$ stabilizes G_{22} if and only if K stabilizes G, without any assumption on the stabilizability of G.

The state-space stabilizability problem does not have a clean solution; To discuss the partial results which exist, we first introduce some terminology.

Let $\{A, B, C, D\}$ be a given realization as above with decomposition of B, C and D as in (4.8). The *Givone-Roesser output pair* $\{C_2, A\}$ is said to be *Hautus-detectable* if the block-column matrix $\left[\begin{smallmatrix} I - Z(z)A \\ C_2 \end{smallmatrix}\right]$ is of maximal rank $n_{\mathcal{X}}$ (i.e., is left invertible) for all z in the closed polydisk $\overline{\mathbb{D}}^d$. We say that $\{C_2, A\}$ is *operator-detectable* in case there exists an output-injection operator $L\colon \mathcal{Y} \to \mathcal{X}$ so that $A + LC_2$ is stable. Dually, the *Givone-Roesser input pair* $\{A, B_2\}$ is called *Hautus-stabilizable* if it is the case that the block-row matrix $\begin{bmatrix} I - AZ(z) & B_2 \end{bmatrix}$ has maximal rank $n_{\mathcal{X}}$ (i.e., is right invertible) for all $z \in \overline{\mathbb{D}}^d$, and *operator-stabilizable* if there is a state-feedback operator $F\colon \mathcal{X} \to \mathcal{U}$ so that $A + B_2F$ is stable. Notice that both Hautus-detectability and operator-detectability for the pair (C, A) reduce to stability of A in case $C = 0$. A similar remark applies to stabilizability for an input pair (A, B).

We will introduce yet another notion of detectability and stabilizability shortly, but in order to do this we need a stronger notion of stability. We first define $\mathcal{D}$ to be the set

$$\mathcal{D} = \left\{ \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{bmatrix} : X_i : \mathcal{X}_i \to \mathcal{X}_i,\ i = 1, , \dots, d \right\}, \tag{4.11}$$

which is also equal to the commutant of $\{Z(z)\colon z \in \mathbb{Z}^d\}$ in the C^*-algebra of bounded operators on $\mathcal{X}$. We then say that the realization $\{A, B, C, D\}$, or just A, is *scaled stable* in case there exists an invertible operator $Q \in \mathcal{D}$ so that $\|Q^{-1}AQ\| < 1$, or, equivalently, if there exists a positive definite operator X (notation $X > 0$) in $\mathcal{D}$ so that $AXA^* - X < 0$. To see that the two definitions coincide, take either $X = QQ^* \in \mathcal{D}$, or, when starting with $X > 0$, factor X as $X = QQ^*$ for some $Q \in \mathcal{D}$. It is not hard to see that scaled stability implies stability. Indeed, assume there exists an invertible $Q \in \mathcal{D}$ so that $\|Q^{-1}AQ\| < 1$. Then $Z(z)Q^{-1}AQ = Q^{-1}Z(z)AQ$ is a strict contraction for each $z \in \overline{\mathbb{D}}^d$, and thus $Q^{-1}(I - Z(z)A)Q = I - Z(z)Q^{-1}AQ$ is invertible on $\overline{\mathbb{D}}^d$. But then $I - Z(z)A$ is invertible on $\overline{\mathbb{D}}^d$ as well, and A is stable. The converse direction, even though asserted in [111, 95], turns out not to be true in general, as shown in [16] via a concrete example. The output pair $\{C_2, A\}$ is then said to be *scaled-detectable* if there exists an output-injection operator $L\colon \mathcal{Y} \to \mathcal{X}$ so that $A + LC_2$ is scaled stable, and the input pair $\{A, B_2\}$ is called *scaled-stabilizable* if there exists a state-feedback operator $F\colon \mathcal{X} \to \mathcal{U}$ so that $A + B_2F$ is scaled stable.

While a classical result for the 1-D case states that operator, Hautus and scaled detectability, as well as operator, Hautus and scaled stabilizability, are equivalent, in the multidimensional setting considered here only one direction is clear.

Proposition 4.2. *Let $\{A, B, C, D\}$ be a given realization as above with decomposition of B, C and D as in* (4.8).

1. *If the output pair $\{C_2, A\}$ is scaled-detectable, then $\{C_2, A\}$ is also operator-detectable. If the output pair $\{C_2, A\}$ is operator-detectable, then $\{C_2, A\}$ is also Hautus-detectable.*
2. *If the input pair $\{A, B_2\}$ is scaled-stabilizable, then $\{A, B_2\}$ is also operator-stabilizable. If the input pair $\{A, B_2\}$ is operator-stabilizable, then $\{A, B_2\}$ is also Hautus-stabilizable.*

Proof. Since scaled stability is a stronger notion than stability, the first implications of both (1) and (2) are obvious. Suppose that $L\colon \mathcal{Y} \to \mathcal{X}$ is such that $A+LC_2$ is stable. Then

$$\begin{bmatrix} I & -Z(z)L \end{bmatrix} \begin{bmatrix} I - Z(z)A \\ C_2 \end{bmatrix} = I - Z(z)(A + LC_2)$$

is invertible for all $z \in \overline{\mathbb{D}}^d$ from which it follows that $\{C_2, A\}$ is Hautus-detectable. The last assertion concerning stabilizability follows in a similar way by making use of the identity

$$\begin{bmatrix} I - AZ(z) & B_2 \end{bmatrix} \begin{bmatrix} I \\ -FZ(z) \end{bmatrix} = I - (A + B_2F)Z(z).$$

□

The combination of operator-detectability together with operator-stabilizability is strong enough for stabilizability of the realization $\{A, B, C, D\}$ and we have the following weak analogue of Theorem 2.4

Theorem 4.3. *Let $\{A, B, C, D\}$ be a given realization as above with decomposition of B, C and D as in* (4.8) *(with $D_{22} = 0$). Assume that $\{C_2, A\}$ is operator-detectable and $\{A, B_2\}$ is operator-stabilizable. Then $\{A, B, C, D\}$ is stabilizable. Moreover, in this case one stabilizing controller is $K \sim \{A_K, B_K, C_K, D_K\}$ where*

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} = \begin{bmatrix} A + B_2F + LC_2 & -L \\ F & 0 \end{bmatrix} \tag{4.12}$$

where $L : \mathcal{Y} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{U}$ are any operators chosen such that $A + LC_2$ and $A + FB_2$ are stable.

Proof. It is possible to motivate these formulas with some observability theory (see [57]) but, once one has the formulas, it is a simple direct check that

$$\begin{aligned} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} &= \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \\ &= \begin{bmatrix} A & B_2F \\ -LC_2 & A + B_2F + LC_2 \end{bmatrix}. \end{aligned}$$

It is now a straightforward exercise to check that this last matrix can be put in the triangular form $\left[\begin{smallmatrix} A+LC_2 & 0 \\ -LC_2 & A+B_2F \end{smallmatrix}\right]$ via a sequence of block-row/block-column similarity transformations, from which we conclude that A_{cl} is stable as required.

□

Remark 4.4. A result for the systems-over-rings setting that is analogous to that of Theorem 4.3 is given in [85]. There the result is given in terms of a Hautus-type stabilizable/detectable condition; in the systems-over-rings setting, Hautus-detectability/stabilizability is equivalent to operator-detectability/stabilizability (see Theorem 3.2 in [83]) rather than merely sufficient as in the present setting (see Proposition 4.2 above). The Hautus-type notions of detectability and stabilizability in principle are checkable using methods from [80]: see the discussion in [83, page 161]. The weakness of Theorem 4.3 for our multidimensional setting is that there are no checkable criteria for when $\{C_2, A\}$ and $\{A, B_2\}$ are operator-detectable and operator-stabilizable since the Hautus test is in general only necessary.

An additional weakness of Theorem 4.3 is that it goes in only one direction: we do not assert that operator-detectability of $\{C_2, A\}$ and operator-stabilizability for $\{A, B_2\}$ is necessary for stabilizability of $\{A, B, C, D\}$. These weaknesses probably explain why apparently this result does not appear explicitly in the control literature.

While there are no tractable necessary and sufficient conditions for solving the state-space stabilizability problem available, the situation turns out quite differently when working with the more conservative notion of scaled stability. The following is a more complete analogue of Theorem 2.4 combined with Theorem 2.3.

Theorem 4.5. *Let $\{A, B, C, D\}$ be a given realization. Then $\{A, B, C, D\}$ is scaled-stabilizable, i.e., there exists a controller K with realization $\{A_K, B_K, C_K, D_K\}$ so that the closed loop state operator A_{cl} is scaled stable, if and only if the input pair $\{A, B_2\}$ is scaled operator-stabilizable and the output pair $\{C_2, A\}$ is scaled operator-detectable, i.e., there exist matrices F and L so that $A+B_2F$ and $A+LC_2$ are scaled stable. In this case the controller K given by* (4.12) *solves the scaled-stabilization problem for $\{A, B, C, D\}$. Moreover:*

1. *The following conditions concerning the input pair are equivalent:*
   (a) *$\{A, B_2\}$ is scaled operator-stabilizable.*
   (b) *There exists $Y \in \mathcal{D}$ satisfying the LMIs:*

$$B_{2,\perp}(AYA^* - Y)B_{2,\perp}^* < 0, \quad Y > 0 \tag{4.13}$$

   where $B_{2,\perp}$ any injective operator with range equal to $\operatorname{Ker} B_2$.
   (c) *There exists $Y \in \mathcal{D}$ satisfying the LMIs*

$$AYA^* - Y - B_2B_2^* < 0, \quad Y > 0. \tag{4.14}$$

2. *The following conditions concerning the output pair are equivalent:*
   (a) *$\{C_2A, \}$ is scaled operator-detectable.*
   (b) *There exists $X \in \mathcal{D}$ satisfying the LMIs:*

$$C_{2,\perp}^*(A^*XA - X)C_{2,\perp} < 0, \quad X > 0. \tag{4.15}$$

   where $C_{2,\perp}$ any injective operator with range equal to $\operatorname{Ker} C_2$.

(c) *There exists $X \in \mathcal{D}$ satisfying the LMIs*

$$A^*XA - X - C_2^*C_2 < 0, \quad X > 0. \tag{4.16}$$

One of the results we shall use in the proof of Theorem 4.5 is known as Finsler's lemma [61], which also plays a key role in [98, 78]. This result can be interpreted as a refinement of the Douglas lemma [51] which is well known in the operator theory community.

Lemma 4.6 (Finsler's lemma). *Assume R and H are given matrices of appropriate size with $H = H^*$. Then there exists a $\mu > 0$ so that $\mu R^*R > H$ if and only if $R_\perp^* H R_\perp < 0$ where $R_\perp$ is any injective operator with range equal to* $\ker R$.

Finsler's lemma can be seen as a special case of another important result, which we shall refer to as Finsler's lemma II. This is one of the main underlying tools in the proof of the solution to the H^∞-problem obtained in [66, 18].

Lemma 4.7 (Finsler's lemma II). *Given matrices R, S and H of appropriate sizes with $H = H^*$, the following are equivalent:*

(i) *There exists a matrix J so that* $H + \begin{bmatrix} R^* & S^* \end{bmatrix} \begin{bmatrix} 0 & J^* \\ J & 0 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix} < 0$,

(ii) $R_\perp^* H R_\perp < 0$ *and* $S_\perp^* H S_\perp < 0$*, where $R_\perp$ and $S_\perp$ are injective operators with ranges equal to* $\ker R$ *and* $\ker S$*, respectively.*

The proof of Finsler's Lemma II given in [66] uses only basic linear algebra and is based on a careful administration of the kernels and ranges from the various matrices. In particular, the matrices J in statement (i) can actually be constructed from the data. We show here how Finsler's lemma follows from the extended version.

Proof of lemma 4.6 *using Lemma* 4.7. Apply Lemma 4.7 with $R = S$. Then (ii) reduces to $R_\perp^* H R_\perp < 0$, which is equivalent to the existence of a matrix J so that $K = -(J^* + J)$ satisfies $R^*KR > H$. Since for such a matrix K we have $K^* = K$, it follows that $R^*\widetilde{K}R > H$ holds for $\widetilde{K} = \mu I$ as long as $\mu I > K$. □

With these results in hand we can proof Theorem 4.5.

Proof of Theorem 4.5. We shall first prove that scaled stabilizability of $\{A, B, C, D\}$ is equivalent to the existence of solutions X and Y in $\mathcal{D}$ for the LMIs (4.15) and (4.13). Note that A_{cl} can be written in the following affine way:

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}. \tag{4.17}$$

Now let $X_{cl} : \mathcal{X} \oplus \mathcal{X}_K$ be an invertible matrix in $\mathcal{D}_{cl}$, where $\mathcal{D}_{cl}$ stands for the commutant of $\{Z_{cl}(z) \colon z \in \mathbb{Z}^d\}$. Let X be the compression of X_{cl} to $\mathcal{X}$ and Y the compression of X_{cl}^{-1} to $\mathcal{X}$. Then $X, Y \in \mathcal{D}$. Assume that $X_{cl} > 0$. Thus, in particular, $X > 0$ and $Y > 0$. Then $A_{cl}X_{cl}A_{cl} - X_{cl} < 0$ if and only if

$$\begin{bmatrix} -X_{cl}^{-1} & A_{cl} \\ A_{cl}^* & -X_{cl} \end{bmatrix} < 0. \tag{4.18}$$

Now define

$$H = \begin{bmatrix} -X_{cl}^{-1} & \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} A^* & 0 \\ 0 & 0 \end{bmatrix} & -X_{cl} \end{bmatrix}, \quad R^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & C_2^* \\ I & 0 \end{bmatrix}, \quad S^* = \begin{bmatrix} 0 & I \\ B_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}.$$

Note that H, R and S are determined by the problem data, while J amounts to the system matrix of the controller to be designed. Then

$$\begin{bmatrix} -X_{cl}^{-1} & A_{cl} \\ A_{cl}^* & -X_{cl} \end{bmatrix} = H + \begin{bmatrix} R^* & S^* \end{bmatrix} \begin{bmatrix} 0 & J^* \\ J & 0 \end{bmatrix} \begin{bmatrix} R \\ S \end{bmatrix}. \tag{4.19}$$

Thus, by Finsler's lemma II, the inequality (4.18) holds for some $J = \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$ if and only if $R_\perp^* H R_\perp < 0$ and $S_\perp^* H S_\perp < 0$, where without loss of generality we can take

$$R_\perp = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & C_{2,\perp} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S_\perp = \begin{bmatrix} 0 & 0 & 0 \\ B_{2,\perp} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

with $C_{2,\perp}$ and $B_{2,\perp}$ as described in part (b) of statements 1 and 2. Writing out $R_\perp^* H R_\perp$ we find that $R_\perp^* H R_\perp < 0$ if and only if

$$\begin{bmatrix} -X_{cl}^{-1} & \begin{bmatrix} AC_{2,\perp} \\ 0 \end{bmatrix} \\ \begin{bmatrix} C_{2,\perp}^* A^* & 0 \end{bmatrix} & -C_{2,\perp}^* X C_{2,\perp} \end{bmatrix} < 0$$

which, after taking a Schur complement, turns out to be equivalent to

$$C_{2,\perp}^*(A^*XA - X)C_{2,\perp} = \begin{bmatrix} C_{2,\perp}^* A^* & 0 \end{bmatrix} X_{cl} \begin{bmatrix} AC_{2,\perp} \\ 0 \end{bmatrix} - C_{2,\perp}^* X C_{2,\perp} < 0.$$

A similar computation shows that $S_\perp^* H S_\perp < 0$ is equivalent to $B_{2,\perp}(AYA^* - Y)B_{2,\perp}^* < 0$. This proves the first part of our claim.

For the converse direction assume we have X and Y in $\mathcal{D}$ satisfying (4.15)–(4.13). Most of the implications in the above argumentation go both ways, and it suffices to prove that there exists an operator X_{cl} on $\mathcal{X} \oplus \mathcal{X}_K$ in $\mathcal{D}_{cl}$, with $\mathcal{X}_K$ an arbitrary finite dimensional Hilbert space with some partitioning $\mathcal{X}_K = \mathcal{X}_{K,1} \oplus \cdots \oplus \mathcal{X}_{K,d}$, so that $X_{cl} > 0$ and X and Y are the compressions to $\mathcal{X}$ of X_{cl} and X_{cl}^{-1}, respectively. Since (4.15)–(4.13) hold with X and Y replaced by ρX and ρY for any positive number ρ, we may assume without loss of generality that $\left[\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right] > 0$. The existence of the required matrix X_{cl} can then be derived from Lemma 7.9 in [57] (with $n_K = n$). To enforce the fact that X_{cl} be in $\mathcal{D}_{cl}$ we decompose $X = \operatorname{diag}(X_1, \ldots, X_d)$ and $Y = \operatorname{diag}(Y_1, \ldots, Y_d)$ as in (4.11) and complete X_i and Y_i to positive definite matrices so that $\left[\begin{smallmatrix} X_i & * \\ * & * \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} Y_i & * \\ * & * \end{smallmatrix}\right]$.

To complete the proof it remains to show the equivalences of parts (a), (b) and (c) in both statements 1 and 2. The equivalences of the parts (b) and (c) follows immediately from Finsler's lemma with $R = B_2$ (respectively, $R = C_2^*$) and $H = AYA^* - Y$ (respectively, $H = A^*XA - X$), again using that X in (4.15) can be replaced with μX (respectively, Y in (4.13) can be replaced with μY) for any positive number μ.

We next show that (a) is equivalent to (b) for statement 1; for statement 2 the result follows with similar arguments. Let $F : \mathcal{X} \to \mathcal{U}$, and let $X \in \mathcal{D}$ be positive definite. Taking a Schur complement it follows that

$$(A^* + F^*B_2^*)X(A + B_2F) - X < 0 \tag{4.20}$$

if and only if

$$\begin{bmatrix} -X^{-1} & A + B_2F \\ A^* + F^*B_2^* & -X \end{bmatrix} < 0.$$

Now write

$$\begin{bmatrix} -X^{-1} & A + B_2F \\ A^* + F^*B_2^* & -X \end{bmatrix} = \\ \begin{bmatrix} -X^{-1} & A \\ A^* & -X \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & F \\ F^* & 0 \end{bmatrix} \begin{bmatrix} B_2^* & 0 \\ 0 & I \end{bmatrix}.$$

Thus, applying Finsler's lemma II with

$$H = \begin{bmatrix} -X^{-1} & A \\ A^* & -X \end{bmatrix}, \quad R = \begin{bmatrix} B_2^* & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & I \end{bmatrix} \quad \text{and} \quad J = F, \tag{4.21}$$

we find that there exists an F so that (4.20) holds if and only if

$$R_\perp^* H R_\perp < 0 \quad \text{and} \quad S_\perp^* H S_\perp < 0$$

with now $R_\perp = \left[\begin{smallmatrix} B_{2,\perp} & 0 \\ 0 & I \end{smallmatrix}\right]$ and $S_\perp = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$. The latter inequality is the same as $-X^{-1} < 0$ and thus vacuous. The first inequality, after writing out $R_\perp^* H R_\perp$, turns out to be

$$\begin{bmatrix} -B_{2,\perp}^* X^{-1} B_{2,\perp} & B_{2,\perp}^* A \\ A^* B_{2,\perp} & -X \end{bmatrix} < 0,$$

which, after another Schur complement, is equivalent to $B_{2,\perp}^*(AX^{-1}A^* - X^{-1})B_{2,\perp}$. $\square$

Since scaled stability implies stability, it is clear that finding operators F and L wit $A+B_2F$ and $A+LC_2$ scaled-stable implies that $A+B_2F$ and $A+LC_2$ are also stable. In particular, having such operators F and L we find the coprime factorization of G_{22} via the functions in Theorem 4.3. While there are no known tractable necessary and sufficient conditions for operator-detectability/stabilizability, the LMI criteria in parts (iii) and (iv) of Theorem 4.5 for the scaled versions are considered computationally tractable. Moreover, an inspection of the last part of the proof shows how operators F and L so that $A+B_2F$ and $A+LC_2$ are scaled stable can be constructed from the solutions X and Y from the LMIs in (4.13)–(4.16):

Assume we have $X, Y \in \mathcal{D}$ satisfying (4.13)–(4.16). Define H, R and S as in (4.21), and determine a J so that $H + \left[\begin{smallmatrix} R^* & S^* \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & J^* \\ J & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} R \\ S \end{smallmatrix}\right] < 0$; this is possible as the proof of Finsler's lemma II is essentially constructive. Then take $F = J$. In a similar way one can construct L using the LMI solution Y.

Stability versus scaled stability, μ versus $\widehat{\mu}$. We observed above that the notion of scaled stability is stronger, and more conservative than the more intuitive notions of stability in the Hautus or asymptotic sense. This remains true in a more general setting that has proved useful in the study of robust control [98, 57, 107] and that we will encounter later in the paper.

Let A be a bounded linear operator on a Hilbert space $\mathcal{X}$. Assume that in addition we are given a unital C^*-algebra $\boldsymbol{\Delta}$ which is realized concretely as a subalgebra of $\mathcal{L}(\mathcal{X})$, the space of bounded linear operators on $\mathcal{X}$. The complex structured singular value $\mu_{\boldsymbol{\Delta}}(A)$ of A (with respect to the structure $\boldsymbol{\Delta}$) is defined as

$$\mu_{\boldsymbol{\Delta}}(A) = \frac{1}{\inf\{\sigma(\Delta)\colon \Delta \in \boldsymbol{\Delta},\ I - \Delta A \text{ is not invertible}\}}. \tag{4.22}$$

Here $\sigma(M)$ stands for the largest singular value of the operator M. Note that this contains two standard measures for A: the operator norm $\|A\|$ if we take $\boldsymbol{\Delta} = \mathcal{L}(\mathcal{X})$, and $\rho(A)$, the spectral radius of A, if we take $\boldsymbol{\Delta} = \{\lambda I_{\mathcal{X}}\colon \lambda \in \mathbb{C}\}$; it is not hard to see that for any unital C^*-algebra $\boldsymbol{\Delta}$ we have $\rho(A) \leq \mu_{\boldsymbol{\Delta}}(A) \leq \|A\|$. See [107] for a tutorial introduction on the complex structured singular value and [60] for the generalization to algebras of operators on infinite dimensional spaces.

The C^*-algebra that comes up in the context of stability for the N-D systems studied in this section is $\boldsymbol{\Delta} = \{Z(z)\colon z \in \mathbb{C}^d\}$. Indeed, note that for this choice of $\boldsymbol{\Delta}$ we have that A is stable if and only if $\mu_{\boldsymbol{\Delta}}(A) < 1$.

In order to introduce the more conservative measure for A in this context, we write $\mathcal{D}_{\boldsymbol{\Delta}}$ for the commutant of the C^*-algebra $\boldsymbol{\Delta}$ in $\mathcal{L}(\mathcal{X})$. We then define

$$\begin{aligned} \widehat{\mu}_{\boldsymbol{\Delta}}(A) &= \inf\{\gamma\colon \|Q^{-1}AQ\| < \gamma \text{ for some invertible } Q \in \mathcal{D}_{\boldsymbol{\Delta}}\} \\ &= \inf\{\gamma\colon AXA^* - \gamma X < 0 \text{ for some } X \in \mathcal{D}_{\boldsymbol{\Delta}}, X > 0\}. \end{aligned} \tag{4.23}$$

The equivalence of the two definitions again goes through the relation between X and Q via $X = Q^*Q$. It is immediate that with $\boldsymbol{\Delta} = \{Z(z)\colon z \in \mathbb{C}^d\}$ we find $\mathcal{D}_{\boldsymbol{\Delta}} = \mathcal{D}$ as in (4.11), and that A is scaled stable if and only if $\widehat{\mu}_{\boldsymbol{\Delta}}(A) < 1$.

The state-space H^∞-problem. The problems of finding tractable necessary and sufficient conditions for the strict state-space H^∞-problem are similar to that for the state-space stabilizability problem. Here one also typically resorts to a more conservative 'scaled' version of the problem.

We say that the realization $\{A, B, C, D\}$ with decomposition (4.8) has *scaled performance* whenever there exists an invertible $Q \in \mathcal{D}$ so that

$$\left\| \begin{bmatrix} Q^{-1} & 0 \\ 0 & I_{\mathcal{Z}\oplus\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} \right\| < 1, \tag{4.24}$$

or, equivalently, if there exists an $X > 0$ in $\mathcal{D}$ so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix} < 0. \tag{4.25}$$

The equivalence of the two definitions goes as for the scaled stability case through the relation $X = QQ^*$. Looking at the left upper entry in (4.25) it follows that scaled performance of $\{A, B, C, D\}$ implies scaled stability. Moreover, if (4.24) holds for $Q \in \mathcal{D}$, then it is not hard to see that the transfer function $G(z)$ in (4.5) is also given by

$$G(z) = D + C'(I - Z(z)A')^{-1}Z(z)B'$$

where the system matrix

$$\begin{bmatrix} A' & B' \\ C' & D \end{bmatrix} = \begin{bmatrix} Q^{-1} & 0 \\ 0 & I_{\mathcal{Z}\oplus\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_{\mathcal{W}\oplus\mathcal{U}} \end{bmatrix}$$

is equal to a strict contraction. It then follows from a standard fact on feedback connections (see e.g. Corollary 1.3 page 434 of [62] for a very general formulation) that $\|G(z)\| < 1$ for $z \in \overline{\mathbb{D}}^d$, i.e., G has strict performance. The *scaled H^∞-problem* is then to find a controller K with realization $\{A_K, B_K, C_K, D_K\}$ so that the closed loop system $\{A_{cl}, B_{cl}, C_{cl}, D_{cl}\}$ has scaled performance. The above analysis shows that solving the scaled H^∞-problem implies solving that state-space H^∞-problem. The converse is again not true in general. Further elaboration of the same techniques as used in the proof of Theorem 4.5 yields the following result for the scaled H^∞-problem; see [18, 66]. For the connections between the Theorems 4.8 and 4.5, in the more general setting of LFT models with structured uncertainty, we refer to [25]. Note that the result collapses to Theorem 2.5 given in the Introduction when we specialize to the single-variable case $d = 1$.

Theorem 4.8. *Let $\{A, B, C, D\}$ be a given realization. Then there exists a solution for the scaled H^∞-problem associated with $\{A, B, C, D\}$ if and only if there exist $X, Y \in \mathcal{D}$ satisfying LMIs:*

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AYA^* - Y & AYC_1^* & B_1 \\ C_1YA^* & C_1YC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0, \quad Y > 0, \tag{4.26}$$

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*XA - X & A^*XB_1 & C_1^* \\ B_1^*XA & B_1^*XB_1 - I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0, \quad X > 0, \tag{4.27}$$

and the coupling condition

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{4.28}$$

Here N_c and N_o are matrices chosen so that

$$N_c \text{ is injective and } \operatorname{Im} N_c = \operatorname{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \text{ and}$$
$$N_o \text{ is injective and } \operatorname{Im} N_o = \operatorname{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}.$$

Note that Theorem 4.8 does not require that the problem be first brought into model-matching form; thus this solution bypasses the Nevanlinna-Pick-interpolation interpretation of the H^∞-problem.

4.3. Equivalence of frequency-domain and state-space formulations

In this subsection we suppose that we are given a transfer matrix G of size $(n_{\mathcal{Z}} + n_{\mathcal{Y}}) \times (n_{\mathcal{W}} + n_{\mathcal{U}})$ with coefficients in $Q(\mathbb{C}(z)_{ss})$ as in Section 4.1 with a given state-space realization as in Subsection 4.2:

$$G(z) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - Z(z)A)^{-1} Z(z) \begin{bmatrix} B_1 & B_2 \end{bmatrix} \tag{4.29}$$

where $Z(z)$ is as in (4.6). We again consider the problem of finding stabilizing controllers K, also equipped with a state-space realization

$$K(z) = D_K + C_K (I - Z_K(z) A_K)^{-1} Z_K(z) B_K, \tag{4.30}$$

in either the state-space stability or in the frequency-domain stability sense. A natural question is whether the frequency-domain H^∞-problem with formulation in state-space coordinates is the same as the state-space H^∞-problem formulated in Section 4.2.

For simplicity in the computations to follow, we shall always assume that the plant G has been normalized so that $D_{22} = 0$. In one direction the result is clear. Suppose that $K(z) = D_K + C_K(I - Z(z)A_K)^{-1} Z(z) B_K$ is a stabilizing controller for $G(z)$ in the state-space sense. It follows that the closed-loop state matrix

$$A_{cl} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \tag{4.31}$$

is stable, i.e., $I - Z_{cl}(z) A_{cl}$ is invertible for all z in the closed polydisk $\overline{\mathbb{D}}^d$, with $Z_{cl}(z)$ as defined in Subsection 4.2. On the other hand one can compute that the transfer matrix $\Theta(G_{22}, K) := \left[\begin{smallmatrix} I & -K(z) \\ -G_{22}(z) & I \end{smallmatrix}\right]^{-1}$ has realization

$$\widetilde{W}(z) = \begin{bmatrix} I & I_{D_K} \\ 0 & I \end{bmatrix} + \begin{bmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{bmatrix} (I - Z_{cl}(z) A_{cl})^{-1} Z_{cl}(z) \begin{bmatrix} B_2 & B_2 D_K \\ 0 & B_K \end{bmatrix}. \tag{4.32}$$

As the resolvent expression $(I - Z_{cl}(z) A_{cl})^{-1}$ has no singularities in the closed polydisk $\overline{\mathbb{D}}^d$, it is clear that $\widetilde{W}(z)$ has matrix entries in $\mathbb{C}(z)_{ss}$, and it follows that K stabilizes G_{22} in the frequency-domain sense. Under the assumption that G is internally stabilizable (frequency-domain sense), it follows from Corollary 3.4 that K also stabilizes G (frequency-domain sense).

We show that the converse direction holds under an additional assumption. The early paper [88] of Kung-Lévy-Morf-Kailath introduced the notion of modal controllability and modal observability for 2-D systems. We extend these notions to N-D systems as follows. Given a Givone-Roesser output pair $\{C, A\}$, we say that $\{C, A\}$ is *modally observable* if the block-column matrix $\left[\begin{smallmatrix} I - Z(z)A \\ C \end{smallmatrix}\right]$ has maximal rank $n_{\mathcal{X}}$ for a generic point z on each irreducible component of the variety $\det(I -$

$Z(z)A) = 0$. Similarly we say that the Givone-Roesser input pair $\{A, B\}$ is *modally controllable* if the block-row matrix $\left[\begin{smallmatrix} I-AZ(z) & B \end{smallmatrix}\right]$ has maximal rank $n_{\mathcal{X}}$ for a generic point on each irreducible component of the variety $\det(I - AZ(z)) = \det(I - Z(z)A) = 0$. Then the authors of [88] define the realization $\{A, B, C, D\}$ to be *minimal* if both $\{C, A\}$ is modally observable and $\{A, B\}$ is modally controllable. While this is a natural notion of minimality, unfortunately it is not clear that an arbitrary realization $\{A, B, C, D\}$ of a given transfer function $S(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ can be reduced to a minimal realization $\{A_0, B_0, C_0, D_0\}$ of the same transfer function $S(z) = D_0 + C_0(I - Z(z)A_0)^{-1}Z(z)B_0$.

As a natural modification of the notions of modally observable and modally controllable, we now introduce the notions of modally detectable and modally stabilizable as follows. For $\{C, A\}$ a Givone-Roesser output pair, we say that $\{C, A\}$ is *modally detectable* if the column matrix $\left[\begin{smallmatrix} I-Z(z)A \\ C \end{smallmatrix}\right]$ has maximal rank $n_{\mathcal{X}}$ for a generic point z on each irreducible component of the variety $\det(I - Z(z)A) = 0$ which enters into the polydisk $\overline{\mathbb{D}}^d$. Similarly, we say that the Givone-Roesser input pair $\{A, B\}$ is *modally stabilizable* if the row matrix $\left[\begin{smallmatrix} I-AZ(z) & B \end{smallmatrix}\right]$ has maximal rank $n_{\mathcal{X}}$ for a generic point z on each irreducible component of the variety $\det(I - Z(z)A) = 0$ which has nonzero intersection with the closed polydisk $\overline{\mathbb{D}}^d$. We then have the following partial converse of the observation made above that state-space internal stabilization implies frequency-domain internal stabilization; this is an N-D version of Theorem 2.6 in the Introduction.

Theorem 4.9. *Let* (4.29) *and* (4.30) *be given realizations for* $G\colon \left[\begin{smallmatrix} \mathcal{W} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{Y} \end{smallmatrix}\right]$ *and* $K\colon \mathcal{Y} \to \mathcal{U}$. *Assume that* $\{C_2, A\}$ *and* $\{C_K, A_K\}$ *are modally detectable and* $\{A, B_2\}$ *and* $\{A_K, B_K\}$ *are modally stabilizable. Then* K *internally stabilizes* G_{22} *in the state-space sense (and thus state-space stabilizes* G*) if and only if* K *stabilizes* G_{22} *in the frequency-domain sense (and* G *if* G *is stabilizable in the frequency-domain sense).*

Remark 4.10. As it is not clear that a given realization can be reduced to a modally observable and modally controllable realization for a given transfer function, it is equally not clear whether a given transfer function has a modally detectable and modally stabilizable realization. However, in the case that $d = 1$, such realizations always exists and Theorem 4.9 recovers the standard 1-D result (Theorem 2.6 in the Introduction).

The proof of Theorem 4.9 will make frequent use of the following basic result from the theory of holomorphic functions in several complex variables. For the proof we refer to [128, Theorem 4 page 176]; note that if the number of variables d is 1, then the only analytic set of codimension at least 2 is the empty set and the theorem is vacuous; the theorem has content only when the number of variables is at least 2.

Theorem 4.11. Principle of Removal of Singularities *Suppose that the complex-valued function* φ *is holomorphic on a set* S *contained in* $\mathbb{C}^d$ *of the form* $S = \mathcal{D} - \mathcal{E}$

where $\mathcal{D}$ is an open set in $\mathbb{C}^d$ and $\mathcal{E}$ is the intersection with $\mathcal{D}$ of an analytic set of codimension at least 2. Then φ has analytic continuation to a function holomorphic on all of $\mathcal{D}$.

We shall also need some preliminary lemmas.

Lemma 4.12. 1. *Modal detectability is invariant under output injection, i.e., given a Givone-Roesser output pair $\{C, A\}$ (where $A\colon \mathcal{X} \to \mathcal{X}$ and $C\colon \mathcal{X} \to \mathcal{Y}$) together with an output injection operator $L\colon \mathcal{Y} \to \mathcal{X}$, then the pair $\{C, A\}$ is modally detectable if and only if the pair $\{C, A+LC\}$ is modally detectable.*

2. *Modal stabilizability is invariant under state feedback, i.e., given a Givone-Roesser input pair $\{A, B\}$ (where $A\colon \mathcal{X} \to \mathcal{X}$ and $B\colon \mathcal{U} \to \mathcal{X}$) together with a state-feedback operator $F\colon \mathcal{X} \to \mathcal{U}$, then the pair $\{A, B\}$ is modally stabilizable if and only if the pair $\{A + BF, B\}$ is modally stabilizable.*

Proof. To prove the first statement, note the identity

$$\begin{bmatrix} I & -Z(z)L \\ 0 & I \end{bmatrix} \begin{bmatrix} I - Z(z)A \\ C \end{bmatrix} = \begin{bmatrix} I - Z(z)(A + LC) \\ C \end{bmatrix}.$$

Since the factor $\left[\begin{smallmatrix} I & -Z(z)L \\ 0 & I \end{smallmatrix}\right]$ is invertible for all z, we conclude that, for each $z \in \mathbb{C}^d$, $\left[\begin{smallmatrix} I-Z(z)A \\ C \end{smallmatrix}\right]$ has maximal rank exactly when $\left[\begin{smallmatrix} I-Z(z)(A+LC) \\ C \end{smallmatrix}\right]$ has maximal rank, and hence, in particular, the modal detectability for $\{C, A\}$ holds exactly when modal detectability for $\{C, A + LC\}$ holds.

The second statement follows in a similar way from the identity

$$\begin{bmatrix} I - AZ(z) & B \end{bmatrix} \begin{bmatrix} I & 0 \\ -FZ(z) & I \end{bmatrix} = \begin{bmatrix} I - (A + BF)Z(z) & B \end{bmatrix}.$$

$\square$

Lemma 4.13. *Suppose that the function $W(z)$ is stable (i.e., all matrix entries of W are in $\mathbb{C}(z)_{ss}$) and suppose that*

$$W(z) = D + C(I - Z(z)A)^{-1}Z(z)B \tag{4.33}$$

is a realization for W which is both modally detectable and modally stabilizable. Then the matrix A is stable, i.e., $(I - Z(z)A)^{-1}$ exists for all z in the closed polydisk $\overline{\mathbb{D}}^d$.

Proof. As W is stable and $Z(z)B$ is trivially stable, then certainly

$$\begin{bmatrix} I - Z(z)A \\ C \end{bmatrix} (I - Z(z)A)^{-1}Z(z)B = \begin{bmatrix} Z(z)B \\ W(z) - D \end{bmatrix} \tag{4.34}$$

is stable (i.e., holomorphic on $\overline{\mathbb{D}}^d$). Trivially $\left[\begin{smallmatrix} I-Z(z)A \\ C \end{smallmatrix}\right]$ has maximal rank $n_{\mathcal{X}}$ for all $z \in \overline{\mathbb{D}}^d$ where $\det(I - Z(z)A) \neq 0$. By assumption, $\left[\begin{smallmatrix} I-Z(z)A \\ C \end{smallmatrix}\right]$ has maximal rank generically on each irreducible component of the zero variety of $\det(I - Z(z)A)$ which intersects $\overline{\mathbb{D}}^d$. We conclude that $\left[\begin{smallmatrix} I-Z(z)A \\ C \end{smallmatrix}\right]$ has maximal rank $n_{\mathcal{X}}$ at all points of $\overline{\mathbb{D}}^d$ except those in an exceptional set $\mathcal{E}$ which is contained in a subvariety, each irreducible component of which has codimension at least 2. In a neighborhood of each such point $z \in \overline{\mathbb{D}}^d - \mathcal{E}$, $\left[\begin{smallmatrix} I - Z(z)A \\ C \end{smallmatrix}\right]$ has a holomorphic left inverse; combining this fact with the identity (4.34), we see that $(I - Z(z)A)^{-1}Z(z)B$ is holomorphic on $\overline{\mathbb{D}}^d - \mathcal{E}$. By Theorem 4.11, it follows that $(I - Z(z)A)^{-1}Z(z)B$ has analytic continuation to all of $\overline{\mathbb{D}}^d$.

We next note the identity

$$\begin{bmatrix} Z(z) & (I - Z(z)A)^{-1}Z(z)B \end{bmatrix} = Z(z)(I - AZ(z))^{-1} \begin{bmatrix} I - AZ(z) & B \end{bmatrix} \tag{4.35}$$

where the quantity on the left-hand side is holomorphic on $\overline{\mathbb{D}}^d$ by the result established above. By assumption $\{A, B\}$ is modally stabilizable; by an argument analogous to that used above for the modally detectable pair $\{C, A\}$, we see that the pencil $\begin{bmatrix} I - AZ(z) & B \end{bmatrix}$ has a holomorphic right inverse in the neighborhood of each point z in $\overline{\mathbb{D}}^d - \mathcal{E}'$ where the exception set $\mathcal{E}'$ is contained in a subvariety each irreducible component of which has codimension at least 2. Multiplication of the identity (4.35) on the right by this right inverse then tells us that $Z(z)(I - Z(z)A)^{-1}$ is holomorphic on $\overline{\mathbb{D}}^d - \mathcal{E}'$. Again by Theorem 4.11, we conclude that in fact $Z(z)(I - Z(z)A)^{-1}$ is holomorphic on all of $\overline{\mathbb{D}}^d$.

We show that $(I - Z(z)A)^{-1}$ is holomorphic on $\overline{\mathbb{D}}^d$ as follows. Let $E_j : \mathcal{X} \to \mathcal{X}_j$ be the projection on the j-th component of $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$. Note that the first block row of $(I - Z(z)A)^{-1}$ is equal to $z_1 E_1 (I - Z(z)A)^{-1}$. This is holomorphic on the closed polydisk $\overline{\mathbb{D}}^d$. For z in a sufficiently small polydisk $|z_i| < \rho$ for $i = 1, \dots, d$, $(I - Z(z)A)^{-1}$ is analytic and hence $z_1 E_1 (I - Z(z)A)^{-1}|_{z_1 = 0} = 0$. By analytic continuation, it then must hold that $z_1 (E_1 (I - Z(z)A)^{-1} = 0$ for all $z = (0, z_2, \dots, z_d)$ with $|z_i| \leq 1$ for $i = 2, \dots, d$. For each fixed $(z_2, \dots, z_d)$, we may use the single-variable result that one can divide out zeros to conclude that $E_1(I - Z(z)A)^{-1}$ is holomorphic in z_1 at $z_1 = 0$. As the result is obvious for $z_1 \neq 0$, we conclude that $E_1(I - Z(z)A)^{-1}$ is holomorphic on the whole closed polydisk $\overline{\mathbb{D}}^d$. In a similar way working with the variable z_i, one can show that $E_i(I - Z(z)A)^{-1}$ is holomorphic on the whole closed polydisk, and it follows that $(I - Z(z)A)^{-1} = \left[\begin{smallmatrix} E_1 \\ \vdots \\ E_d \end{smallmatrix}\right] (I - Z(z)A)^{-1}$ is holomorphic on the whole closed polydisk as wanted. $\square$

We are now ready for the proof of Theorem 4.9.

Proof of Theorem 4.9. Suppose that K stabilizes G_{22} in the frequency-domain sense. This simply means that the transfer function $\widetilde{W}$ given by (4.32) is holomorphic on the closed polydisk $\overline{\mathbb{D}}^d$. To show that A_{cl} is stable, by Lemma 4.13 it suffices to show that $\left\{ \left[\begin{smallmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{smallmatrix}\right], A_{cl} \right\}$ is modally detectable and that $\left\{ A_{cl}, \left[\begin{smallmatrix} B_2 & B_2 D_K \\ 0 & B_K \end{smallmatrix}\right] \right\}$ is modally stabilizable.

To prove that $\left\{ \left[\begin{smallmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{smallmatrix}\right], A_{cl} \right\}$ is modally detectable, from the definition (4.17) we note that

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{bmatrix}.$$

By Lemma 4.12 we see that modal detectability of $\left\{ \left[\begin{smallmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{smallmatrix}\right], A_{cl} \right\}$ is equivalent to modal detectability of $\left\{ \left[\begin{smallmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & 0 \\ 0 & A_K \end{smallmatrix}\right] \right\}$. As $\left[\begin{smallmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} D_K & I \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} C_2 & 0 \\ 0 & C_K \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} D_K & I \\ I & 0 \end{smallmatrix}\right]$ invertible, it is easily seen that modal detectability of the input pair $\left\{ \left[\begin{smallmatrix} D_K C_2 & C_K \\ C_2 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & 0 \\ 0 & A_K \end{smallmatrix}\right] \right\}$ is equivalent to modal detectability of $\left\{ \left[\begin{smallmatrix} C_2 & 0 \\ 0 & C_K \end{smallmatrix}\right], \left[\begin{smallmatrix} A & 0 \\ 0 & A_K \end{smallmatrix}\right] \right\}$. But the modal detectability of this last pair in turn follows from its diagonal form and the assumed modal detectability of $\{C_2, A\}$ and $\{C_K, A_K\}$.

The modal stabilizability of $\left\{ A_{cl}, \left[\begin{smallmatrix} B_2 & B_2 D_K \\ 0 & B_K \end{smallmatrix}\right] \right\}$ follows in a similar way by making use of the identities

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 D_K & B_2 \\ B_K & 0 \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & C_K \end{bmatrix}, \quad \begin{bmatrix} B_2 D_K & B_2 \\ B_K & 0 \end{bmatrix} = \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} D_K & I \\ I & 0 \end{bmatrix}$$

and noting that $\left[\begin{smallmatrix} D_K & I \\ I & 0 \end{smallmatrix}\right]$ is invertible. □

In both the frequency-domain setting of Section 4.1 and the state-space setting of Section 4.2, the true H^∞-problem is intractable and we resorted to some compromise: the Schur-Agler-class reformulation in Section 4.1 and the scaled-H^∞-problem reformulation in Section 4.2. We would now like to compare these compromises for the setting where they both apply, namely, where we are given both the transfer function G and the state-space representation $\{A, B, C, D\}$ for the plant.

Theorem 4.14. *Suppose that $G(z) = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & 0 \end{smallmatrix}\right]$ is in model-matching form with state-space realization $G(z) = D + C(I - Z(z)A)^{-1}Z(z)B$ as in* (4.29)*. Suppose that the controller $K(z) = D_K + C_K(I - Z_K(z)A_K)^{-1}Z_K(z)B_K$ solves the scaled H^∞-problem. Then the transfer function $\widetilde{W}(z)$ as in* (4.32) *is a Schur-Agler-class solution of the Model-Matching problem.*

Proof. Simply note that, under the assumptions of the theorem, $\widetilde{W}(z)$ has a realization $\widetilde{W} = D_{cl} + C_{cl}(I - Z_{cl}(z)A_{cl})^{-1}Z_{cl}(z)B_{cl}$ for which there is a state-space change of coordinates $Q \in \mathcal{D}$ transforming the realization to a contraction:

$$\left\| \begin{bmatrix} A' & B' \\ C' & D \end{bmatrix} \right\| < 1 \text{ where } \begin{bmatrix} A' & B' \\ C' & D \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

Thus we also have $\widetilde{W}(z) = D + C'(I - Z_{cl}(z)A')Z_{cl}(z)B'$ from which it follows that W is in the strict Schur-Agler class, i.e., $\|\widetilde{W}(X)\| < 1$ for any d-tuple $X = (X_1, \dots, X_d)$ of contraction operators X_j on a separable Hilbert space $\mathcal{X}$. By construction $\widetilde{W}$ necessarily has the model matching form $\widetilde{W} = G_{11} + G_{12}\Lambda G_{21}$ with Λ stable. □

Remark 4.15. In general a Schur-Agler function $S(z)$ can be realized with a colligation matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ which is not of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tag{4.36}$$

with $\left[\begin{smallmatrix} A' & B' \\ C' & D \end{smallmatrix}\right]$ equal to a strict contraction and $Q \in \mathcal{D}$ invertible. As an example, let A be the block 2×2 matrix given by Anderson-et-al in [16]. This matrix has the property that $I - Z(z)A$ is invertible for all $z \in \overline{\mathbb{D}}^2$, but there is no $Q \in \mathcal{D}$ so that $\|Q^{-1}AQ\| < 1$. Here $Z(z)$ and $\mathcal{D}$ are compatible with the block decomposition of A. Then for $\gamma > 0$ sufficiently small the function $S(z) = \gamma(I - Z(z)A)^{-1}$ has $\|S(z)\| \le \rho < 1$ for some $0 < \rho < 1$ and all $z \in \overline{\mathbb{D}}^2$. Hence S is a strict Schur-class function. As mentioned in Section 4.1, a consequence of the Andô dilation theorem [17] is that the Schur class and the Schur-Agler class coincide for $d = 2$; it is not hard to see that this equality carries over to the strict versions and hence S is in the strict Schur-Agler class. As a consequence of the strict Bounded-Real-Lemma in [29], S admits a strictly contractive state-space realization $\left[\begin{smallmatrix} A' & B' \\ C' & D \end{smallmatrix}\right]$. However, the realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & A \\ \gamma I & \gamma I \end{smallmatrix}\right]$ of S, obtained from the fact that

$$S(z) = \gamma(I - Z(z)A)^{-1} = \gamma I + \gamma(I - Z(z)A)^{-1}Z(z)A,$$

cannot relate to $\left[\begin{smallmatrix} A' & B' \\ C' & D \end{smallmatrix}\right]$ as in (4.36) since that would imply the existence of an invertible $Q \in \mathcal{D}$ so that $Q^{-1}AQ = A'$ is a strict contraction.

Remark 4.16. Let us assume that the $G(z)$ in Theorem 4.14 is such that G_{12} and G_{21} are square and invertible on the distinguished boundary $\mathbb{T}^d$ of the polydisk $\mathbb{D}^d$ so that the Model-Matching problem can be converted to a polydisk bitangential Nevanlinna-Pick interpolation problem along a subvariety as in [32]. As we have seen, the solution criterion using the Agler interpolation theorem of [1, 35] then involves an LOI (Linear Operator Inequality or infinite LMI). On the other hand, if we assume that we are given a stable state-space realization $\{A, B, C, D\}$ for $G(z) = \left[\begin{smallmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & 0 \end{smallmatrix}\right]$, we may instead solve the associated scaled H^∞-problem associated with this realization data-set. The associated solution criterion in Theorem 4.8 remarkably involves only finite LMIs. A disadvantage of this state-space approach, however, is that in principle one would have to sweep all possible (similarity equivalence classes of) realizations of $G(z)$; while each non-equivalent realization gives a distinct H^∞-problem, the associated frequency-domain Model-Matching/bitangential variety-interpolation problem remains the same.

4.4. Notes

In [92] Lin conjectured the result stated in Theorem 4.1 that G_{22}-stabilizability is equivalent to the existence of a stable coprime factorization for G_{22}. This conjecture was settled by Quadrat (see [122, 117, 120]) who obtained the equivalence of this property with projective-freeness of the underlying ring and noticed the applicability of the results from [46, 83] concerning the projective-freeness of $\mathbb{C}(z)_{ss}$.

For the general theory of the N-D systems, in particular for $N=2$, considered in Subsection 4.2 we refer to [81, 55].

The sufficiency of scaled stability for asymptotic/Hautus-stability goes back to [59]. Theorem 4.5 was proved in [98] for the more general LFT models in the context of robust control with structured uncertainty. The proof given here is based on the extended Finsler's lemma (Lemma 4.7), and basically follows the proof from [66] for the solution to the scaled H^∞-problem (Theorem 4.8). As pointed out in [66], one of the advantages of the LMI-approach to the state-space H^∞ problem, even in the classical setting, is that it allows one to seek controllers that solve the scaled H^∞-problem with a given maximal order. Indeed, it is shown in [66, 18] (see also [57]) that certain additional rank constraints on the solutions X and Y of the LMIs (4.26) and (4.27) enforce the existence of a solution with a prescribed maximal order. However, these additional constraints destroy the convexity of the solution criteria, and are therefore usually not considered as a desirable addition.

An important point in the application of Finsler's lemma in the derivation of the LMI solution criteria in Theorems 4.5 and 4.8 is that the closed-loop system matrix A_{cl} in (4.31) has an affine expression in terms of the unknown design parameters $\{A_K, B_K, C_K, D_K\}$. This is the key point where the assumption $D_{22} = 0$ is used. A parallel simplification occurs in the frequency-domain setting where the assumption $G_{22} = 0$ leads to the Model-Matching form. The distinction however is that the assumption $G_{22} = 0$ is considered unattractive from a physical point of view while the parallel state-space assumption $D_{22} := G_{22}(0) = 0$ is considered innocuous.

There is a whole array of lemmas of Finsler type; we have only mentioned the form most suitable for our application. It turns out that these various Finsler lemmas are closely connected with the theory of plus operators and Pesonen operators on an indefinite inner product space (see [44]). An engaging historical survey on all the Finsler's lemmas is the paper of Uhlig [135].

The notions of modally detectable and modally stabilizable introduced in Subsection 4.3 along with Theorem 4.9 seem new, though of somewhat limited use because it is not known if every realization can be reduced to a modally detectable and modally stabilizable realization. We included the result as an illustration of the difficulties with realization theory for N-D transfer functions.

We note that the usual proof of Lemma 4.13 for the classical 1-D case uses the pole-shifting characterization of stabilizability/detectability (see [57, Exercise 2.19]). The proof here using the Hautus characterization of stabilizability/detectability provides a different proof for the 1-D case.

5. Robust control with structured uncertainty: the commutative case

In the analysis of 1-D control systems, an issue is the uncertainty in the plant parameters. As a control goal, one wants the control to achieve internal stability (and perhaps also performance) not only for the nominal plant G but also for a whole prescribed family of plants containing the nominal plant G.

A question then is whether the controller can or cannot have (online) access to the uncertainty parameters. In a state-space context it is possible to find sufficient conditions for the case that the controller cannot access the uncertainty parameters, with criteria that are similar to those found in Theorems 4.5 and 4.8 but additional rank constraints need to be imposed as well, which destroys the convex character of the solution criterion. The case where the controller can have access to the uncertainty parameters is usually given the interpretation of gain-scheduling, and fits better with the multidimensional system problems discussed in Section 4. In this section we discuss three formulations of 1-D control systems with uncertainty in the plant parameters, two of which can be given gain-scheduling interpretation, i.e., the controller has access to the uncertainty parameters, and one where the controller is not allowed to use the uncertainty parameters.

5.1. Gain-scheduling in state-space coordinates

Following [106], we suppose that we are given a standard linear time-invariant input/state/output system

$$\Sigma\colon \left\{ \begin{array}{rcl} x(t+1) & = & A_M(\delta_U)x(t) + B_{M1}(\delta_U)w(t) + B_{M2}(\delta_U)u(t) \\ z(t) & = & C_{M1}(\delta_U)x(t) + D_{M11}(\delta_U)w(t) + D_{M12}(\delta_U)u(t) \\ y(t) & = & C_{M2}(\delta_U)x(t) + D_{M21}(\delta_U)w(t) + D_{M22}(\delta_U)u(t) \end{array} \right. \quad (t \in \mathbb{Z}_+) \tag{5.1}$$

but where the system matrix

$$\begin{bmatrix} A_M(\delta_U) & B_{M1}(\delta_U) & B_{M2}(\delta_U) \\ C_{M1}(\delta_U) & D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ C_{M2}(\delta_U) & D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Z} \\ \mathcal{Y} \end{bmatrix}$$

is not known exactly but depends on some *uncertainty* parameters $\delta_U = (\delta_1, \dots, \delta_d)$ in $\mathbb{C}^d$. Here the quantities δ_i are viewed as uncertain parameters which the controller can measure and use in real time. The goal is to design a controller Σ_K (independent of δ_U) off-line so that the closed-loop system (with the controller accessing the current values of the varying parameters $\delta_1, \dots, \delta_d$ as well as the value of the measurement signal y from the plant) has desirable properties for all admissible values of δ_U, usually normalized to be $|\delta_k| \le 1$ for $k = 1, \dots, d$.

The transfer function for the uncertainty parameter δ_U can be expressed as

$$\begin{aligned} G(\delta) = & \begin{bmatrix} D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{bmatrix} \\ & + \lambda \begin{bmatrix} C_{M1}(\delta_U) \\ C_{M2}(\delta_U) \end{bmatrix} (I_{\mathcal{X}} - \lambda A_M(\delta_U))^{-1} \begin{bmatrix} B_{M1}(\delta_U) & B_{M2}(\delta_U) \end{bmatrix} \end{aligned} \tag{5.2}$$

where we have introduced the aggregate variable

$$\delta = (\delta_U, \lambda) = (\delta_1, \dots, \delta_d, \lambda).$$

It is not too much of a restriction to assume in addition that the functional dependence on δ_U is given by a linear fractional map (where the subscript U suggests *uncertainty* and the subscript S suggests *shift*)

$$\begin{bmatrix} A_M(\delta_U) & B_{M1}(\delta_U) & B_{M2}(\delta_U) \\ C_{M1}(\delta_U) & D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ C_{M2}(\delta_U) & D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{bmatrix} = \begin{bmatrix} A_{SS} & B_{S1} & B_{S2} \\ C_{1S} & D_{11} & D_{12} \\ C_{2S} & D_{21} & D_{22} \end{bmatrix} + \\ + \begin{bmatrix} A_{SU} \\ C_{1U} \\ C_{2U} \end{bmatrix} (I - Z(\delta_U)A_{UU})^{-1} Z(\delta_U) \begin{bmatrix} A_{US} & B_{U1} & B_{U2} \end{bmatrix},$$

where $Z(\delta_U)$ is defined analogously to $Z(z)$ in (4.6) relative to a given decomposition of the "uncertainty" state-space $\mathcal{X}_U = \mathcal{X}_{U,1} \oplus \cdots \oplus \mathcal{X}_{U,d}$ on which that state operator A_{UU} acts. In that case the transfer function $G(\delta)$ admits a state-space realization

$$G(\delta) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (I - Z(\delta)A)^{-1} Z(\delta) \begin{bmatrix} B_1 & B_2 \end{bmatrix} \tag{5.3}$$

with system matrix given by

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A_{UU} & A_{US} & B_{U1} & B_{U2} \\ A_{SU} & A_{SS} & B_{S1} & B_{S2} \\ \hline C_{1U} & C_{1S} & D_{11} & D_{12} \\ C_{2U} & C_{2S} & D_{21} & D_{22} \end{array}\right]. \tag{5.4}$$

Here $Z(\delta)$ is again defined analogously to (4.6) but now on the extended state-space $\mathcal{X}_{ext} = \mathcal{X}_U \oplus \mathcal{X}$.

We can then consider this gain-scheduling problem as a problem of the constructed N-D system (with $N = d+1$), and seek for a controller K with a state-space realization

$$K(\delta) = D_K + C_K(I - Z_K(\delta)A_K)Z_K(\delta)B_K \tag{5.5}$$

so that the closed loop system has desirable properties from a gain-scheduling perspective. Making a similar decomposition of the system matrix for the controller K as in (5.4), we note that $K(\delta)$ can also be written as

$$K(\delta) = D_{M,K}(\delta_U) + \lambda C_{M,K}(\delta_U)(I - \lambda A_{M,K}(\delta_U))^{-1} B_{M,K}(\delta_U),$$

where $A_{M,K}(\delta_U)$, $B_{M,K}(\delta_U)$, $C_{M,K}(\delta_U)$ and $D_{M,K}(\delta_U)$ appear as the transfer functions of N-D systems (with $N = d$), that is, $K(\delta)$ can be seen as the transfer function of a linear time-invariant input/state/output system

$$\Sigma_K : \begin{cases} x_K(t+1) &= A_{M,K}(\delta_U)x_K(t) + B_{M,K}(\delta_U)u(t) \\ u(t) &= C_{M,K}(\delta_U)x_K(t) + D_{M,K}(\delta_U)y(t) \end{cases} \quad (n \in \mathbb{Z}_+)$$

depending on the same uncertainty parameters $\delta_U = (\delta_1, \ldots, \delta_d)$ as the system Σ.

Similarly, the transfer function $G_{cl}(\delta)$ of the closed-loop system with system matrix $\left[\begin{smallmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{smallmatrix}\right]$ as defined in (4.10) also can be written as a transfer matrix

$$G_{cl}(\delta) = D_{M,cl}(\delta_U) + \lambda C_{M,cl}(\delta_U)(I - \lambda A_{M,cl}(\delta_U))^{-1} B_{M,cl}(\delta_U)$$

with $A_{M,cl}(\delta_U)$, $B_{M,cl}(\delta_U)$, $C_{M,cl}(\delta_U)$ and $D_{M,cl}(\delta_U)$ transfer functions of N-D systems (with $N = d$), and the corresponding linear time-invariant input/state/output system

$$\Sigma_{cl} : \left\{ \begin{array}{rcl} x(t+1) & = & A_{M,cl}(\delta_U)x(t) + B_{M,cl}(\delta_U)w(t) \\ z(t) & = & C_{M,cl}(\delta_U)x(t) + D_{M,cl}(\delta_U)w(t) \end{array} \right. \qquad (n \in \mathbb{Z}_+)$$

also appears as the closed-loop system of Σ and Σ_K.

It then turns out that stability of A_{cl}, that is, $I - Z_{cl}(\delta)A_{cl}$ invertible for all δ in $\overline{\mathbb{D}}^{d+1}$ (with Z_{cl} as defined in Subsection 4.2) corresponds precisely to *robust stability of* Σ_{cl}, i.e., the spectral radius of $A_{M,cl}(\delta_U)$ is less than 1 for all $\delta_U = (\delta_1, \dots, \delta_d)$ so that $|\delta_k| \leq 1$ for $k = 1, \dots, d$, and K with realization (5.5) solves the state-space H^∞-problem for G with realization (5.3) means that the closed loop system Σ_{cl} has *robust performance*, i.e., Σ_{cl} is robustly stable and the transfer function G_{cl} satisfies

$$\|G_{cl}(\delta)\| \leq 1 \text{ for all } \delta = (\delta_1, \dots, \delta_d, \lambda) \in \overline{\mathbb{D}}^{d+1}.$$

We may thus see the state-space formulation of the gain-scheduling problems considered in this subsection as a special case of the *N-D* system stabilization and H^∞-problems of Subsection 4.2. In particular, the sufficiency analysis given there, and the results of Theorem 4.5 and 4.8, provide practical methods for obtaining solutions. As the conditions are only sufficient, solutions obtained in principle may be conservative.

5.2. Gain-scheduling: a pure frequency-domain formulation

In the approach of Helton (see [73, 74]), one eschews transfer functions and state-space coordinates completely and supposes that one is given a plant G whose frequency response depends on a load with frequency function $\delta(z)$ at the discretion of the user; when the load δ is loaded onto G, the resulting frequency-response function has the form $G(z, \delta(z))$ where $G = G(\cdot, \cdot)$ is a function of two variables. The control problem (for the company selling this device G to a user) is to design the controller $K = K(\cdot, \cdot)$ so that $K(\cdot, \delta(\cdot))$ solves the H^∞-problem for the plant $G(\cdot, \delta(\cdot))$. The idea here is that once the user loads δ onto G with known frequency-response function, he is also to load δ onto the controller K (designed off-line); in this way the same controller works for many customers using many different δ's. When the dust settles, this problem reduces to the frequency-domain problem posed in Section 4.1 with $d = 2$; an application of the Youla-Kučera parametrization (or simply using the function $Q(z) = K(z)(I - G_{22}(z)K(z))^{-1}$ if the plant G itself is stable) reduces the problem of designing the control K to a Nevanlinna-Pick-type interpolation problem on the bidisk.

5.3. Robust control with a hybrid frequency-domain/state-space formulation

We now consider a hybrid frequency-domain/state-space formulation of the problem considered in Subsection 5.1; the main difference is that in this case the controller is not granted access to the uncertainty parameters.

Assume we are given a 1-D-plant $G(\lambda)$ that depends on uncertainty parameters $\delta_U = (\delta_1, \dots, \delta_d)$ via the linear fractional representation

$$G(\delta_U, \lambda) = \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix} + \\ + \begin{bmatrix} G_{1U}(\lambda) \\ G_{2U}(\lambda) \end{bmatrix} (I - Z(\delta_U)G_{UU}(\lambda))^{-1} Z(\delta_U) \begin{bmatrix} G_{U1}(\lambda) & G_{U2}(\lambda) \end{bmatrix} \quad (5.6)$$

with $Z(\delta_U)$ as defined in Subsection 5.1, and where the coefficients are 1-D-plants independent of δ_U:

$$G_{aug}(\lambda) = \begin{bmatrix} G_{UU}(\lambda) & G_{U1}(\lambda) & G_{U2}(\lambda) \\ G_{1U}(\lambda) & G_{11}(\lambda) & G_{12}(\lambda) \\ G_{2U}(\lambda) & G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix} : \begin{bmatrix} \mathcal{X}_U \\ \mathcal{W} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_U \\ \mathcal{Z} \\ \mathcal{Y} \end{bmatrix}.$$

In case $G_{aug}(\lambda)$ is also given by a state-space realization, we can write $G(\delta_U, \lambda)$ as in (5.3) with $\delta = (\delta_U, \lambda)$ and $Z(\delta)$ acting on the extended state-space $\mathcal{X}_{ext} = \mathcal{X}_U \oplus \mathcal{X}$.

For this variation of the gain-scheduling problem we seek to design a controller $K(\lambda)$ with matrix values representing operators from $\mathcal{Y}$ to $\mathcal{U}$ so that K solves the H^∞-problem for $G(\delta_U, \lambda)$ for every δ_U with $\|Z(\delta_U)\| \le 1$, i.e., $|\delta_j| \le 1$ for $j = 1, \dots, d$. For the sequel it is convenient to assume that $\mathcal{Z} = \mathcal{W}$. In that case, using the Main Loop Theorem [141, Theorem 11.7 page 284], it is easy to see that this problem can be reformulated as: *Find a single-variable transfer matrix $K(\cdot)$ so that $\Theta(\widetilde{G}, K)$ given by (2.2), with $\widetilde{G} = \left[\begin{smallmatrix} \widetilde{G}_{11} & \widetilde{G}_{12} \\ \widetilde{G}_{21} & \widetilde{G}_{22} \end{smallmatrix}\right]$ in (2.2) taken to be*

$$\begin{bmatrix} \widetilde{G}_{11}(\lambda) & \widetilde{G}_{12}(\lambda) \\ \widetilde{G}_{21}(\lambda) & \widetilde{G}_{22}(\lambda) \end{bmatrix} = \left[\begin{array}{cc|c} G_{UU}(\lambda) & G_{U1}(\lambda) & G_{U2}(\lambda) \\ G_{1U}(\lambda) & G_{11}(\lambda) & G_{12}(\lambda) \\ \hline G_{2U}(\lambda) & G_{21}(\lambda) & G_{22}(\lambda) \end{array}\right],$$

is stable and such that

$$\mu_{\mathbf{\Delta}}\left(\widetilde{G}_{11}(\lambda) + \widetilde{G}_{12}(\lambda)(I - K(\lambda)\widetilde{G}_{22}(\lambda))^{-1}K(\lambda)\widetilde{G}_{21}(\lambda)\right) < 1.$$

Here $\mu_{\mathbf{\Delta}}$ is as defined in (4.22) with $\mathbf{\Delta}$ the C^*-algebra

$$\mathbf{\Delta} = \left\{ \begin{bmatrix} Z(\delta_U) & 0 \\ 0 & T \end{bmatrix} : \delta_U \in \mathbb{C}^d,\ T \in \mathcal{L}(\mathcal{Z}) \right\} \subset \mathcal{L}(\mathcal{X}_U \oplus \mathcal{Z}).$$

Application of the Youla-Kučera parametrization of the controllers K that stabilize $\Theta(\widetilde{G}, K)$ as in Subsection 3.3 converts the problem to the following: *Given stable 1-variable transfer functions $T_1(\lambda)$, $T_2(\lambda)$, and $T_3(\lambda)$ with matrix values representing operators in the respective spaces*

$$\mathcal{L}(\mathcal{X}_U \oplus \mathcal{W}, \mathcal{X}_U \oplus \mathcal{Z}), \quad \mathcal{L}(\mathcal{X}_U \oplus \mathcal{U}, \mathcal{X}_U \oplus \mathcal{Z}), \quad \mathcal{L}(\mathcal{X}_U \oplus \mathcal{W}, \mathcal{X}_U \oplus \mathcal{Y}),$$

find a stable 1-variable transfer function $\Lambda(\lambda)$ with matrix values representing operators in $\mathcal{L}(\mathcal{X}_U \oplus \mathcal{Y}, \mathcal{X}_U \oplus \mathcal{U})$ so that the transfer function $S(\lambda)$ given by

$$S(\lambda) = T_1(\lambda) + T_2(\lambda)\Lambda(\lambda)T_3(\lambda) \quad (5.7)$$

has $\mu_{\mathbf{\Delta}}(S(\lambda)) < 1$ *for all* $\lambda \in \mathbb{D}$. If $T_2(\zeta)$ and $T_3(\zeta)$ are square and invertible for ζ on the boundary $\mathbb{T}$ of the unit disk $\mathbb{D}$, the model-matching form (5.7) can be converted to bitangential interpolation conditions (see e.g. [26]); for simplicity, say that these interpolation conditions have the form

$$x_i S(\lambda_i) = y_i, \quad S(\lambda'_j)u_j = v_j \text{ for } i = 1, \dots, k, \quad j = 1, \dots, k' \tag{5.8}$$

for given distinct points λ_i, λ'_j in $\mathbb{D}$, row vectors x_i, y_i and column vectors u_j, v_j. Then the robust H^∞-problem (H^∞ rather than rational version) can be converted to the μ-Nevanlinna-Pick problem: *find holomorphic function* S *on the unit disk with matrix values representing operators in* $\mathcal{L}(\mathcal{X}_U \oplus \mathcal{W}, \mathcal{X}_U \oplus \mathcal{Z})$ *satisfying the interpolation conditions* (5.8) *such that also*

$$\mu_{\mathbf{\Delta}}(S(\lambda)) < 1 \text{ for all } \lambda \in \mathbb{D}.$$

It is this μ-version of the Nevanlinna-Pick interpolation problem which has been studied from various points of view (including novel variants of the Commutant Lifting Theorem) by Bercovici-Foias-Tannenbaum (see [38, 39, 40, 41]) and Agler-Young (see [5, 7, 9, 11] and Huang-Marcantognini-Young [77]). These authors actually study only very special cases of the general control problem as formulated here; hence the results at this stage are not particularly practical for actual control applications. However this work has led to interesting new mathematics in a number of directions: we mention in particular the work of Agler-Young on new types of dilation theory and operator-model theory (see [6, 9]), new kinds of realization theorems [10], the complex geometry of new kinds of domains in $\mathbb{C}^d$ (see [8, 12, 13]), and a multivariable extension of the Bercovici-Foias-Tannenbaum spectral commutant lifting theorem due to Popescu [114].

5.4. Notes

In the usual formulation of μ (see [107, 141]), in addition to the scalar blocks $\delta_i I_{n_i}$ in $Z(\delta)$, it is standard to also allow some of the blocks to be full blocks of the form $\Delta_i = \begin{bmatrix} \delta_{11}^{(i)} & \cdots & \delta_{1n_i}^{(i)} \\ \vdots & & \vdots \\ \delta_{n_i 1}^{(i)} & \cdots & \delta_{n_i n_i}^{(i)} \end{bmatrix}$. The resulting transfer functions then have domains equal to be (reducible) Cartan domains which are more general than the unit polydisk. The theory of the Schur-Agler class has been extended to this setting in [15, 20]. More generally, it is natural also to allow non-square blocks. A formalism for handling this is given in [29]; for this setting one must work with the intertwining space of Δ rather than the commutant of Δ in the definition of $\widehat{\mu}$ in (4.23). With a formalism for such a non-square uncertainty structure available, one can avoid the awkward assumption in Subsection 5.3 and elsewhere that $\mathcal{W} = \mathcal{Z}$.

6. Robust control with dynamic time-varying structured uncertainty

6.1. The state-space LFT-model formulation

Following [97, 98, 96, 108], we now introduce a variation on the gain-scheduling problem discussed in Section 5.1 where the uncertainty parameters $\delta_U = (\delta_1, \dots, \delta_d)$ become operators on ℓ^2, the space of square-summable sequences of complex numbers indexed by the integers $\mathbb{Z}$, and are to be interpreted as dynamic, time-varying uncertainties. To make the ideas precise, we suppose that we are given a system matrix as in (5.4). We then tensor all operators with the identity operator I_{ℓ^2} on ℓ^2 to obtain an enlarged system matrix

$$\mathbf{M} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \otimes I_{\ell^2} = \left[\begin{array}{cc|cc} A_{UU} & A_{US} & B_{U1} & B_{U2} \\ A_{SU} & A_{SS} & B_{S1} & B_{S2} \\ \hline C_{1U} & C_{1S} & D_{11} & D_{12} \\ C_{2U} & C_{2S} & D_{21} & D_{22} \end{array}\right] \otimes I_{\ell^2}, \tag{6.1}$$

which we also write as

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} : \begin{bmatrix} (\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2 \\ \mathcal{W} \otimes \ell^2 \\ \mathcal{U} \otimes \ell^2 \end{bmatrix} \to \begin{bmatrix} (\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2 \\ \mathcal{Z} \otimes \ell^2 \\ \mathcal{Y} \otimes \ell^2 \end{bmatrix}. \tag{6.2}$$

Given a decomposition $\mathcal{X}_U = \mathcal{X}_{U1} \oplus \cdots \oplus \mathcal{X}_{Ud}$ of the uncertainty state space $\mathcal{X}_U$, we define the matrix pencil $\mathbf{Z}_U(\boldsymbol{\delta}_U)$ with argument equal to a d-tuple $\boldsymbol{\delta}_U = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$ of (not necessarily commuting) operators on ℓ^2 by

$$\mathbf{Z}_U(\boldsymbol{\delta}_U) = \begin{bmatrix} I_{\mathcal{X}_{U1}} \otimes \boldsymbol{\delta}_1 & & \\ & \ddots & \\ & & I_{\mathcal{X}_{Ud}} \otimes \boldsymbol{\delta}_d \end{bmatrix},$$

In addition we let $\mathbf{S}$ denote the bilateral shift operator on ℓ^2; we sometimes will also view $\mathbf{S}$ as an operator on the space ℓ of all sequences of complex numbers or on the subspace ℓ^2_{fin} of ℓ^2 that consists of all sequences in ℓ^2 with finite support. We obtain an uncertain linear system of the form

$$\mathbf{\Sigma} \colon \left\{ \begin{array}{ccl} \mathbf{S}^* \vec{x} & = & A_M(\boldsymbol{\delta}_U)\vec{x} + B_{M1}(\boldsymbol{\delta}_U)\vec{w} + B_{M2}(\boldsymbol{\delta}_U)\vec{u} \\ \vec{z} & = & C_{M1}(\boldsymbol{\delta}_U)\vec{x} + D_{M11}(\boldsymbol{\delta}_U)\vec{w} + D_{M12}(\boldsymbol{\delta}_U)\vec{u} \\ \vec{y} & = & C_{M2}(\boldsymbol{\delta}_U)\vec{x} + D_{M21}(\boldsymbol{\delta}_U)\vec{w} + D_{M22}(\boldsymbol{\delta}_U)\vec{u} \end{array} \right. \tag{6.3}$$

where the system matrix

$$\begin{bmatrix} A_M(\boldsymbol{\delta}_U) & B_{M1}(\boldsymbol{\delta}_U) & B_{M2}(\boldsymbol{\delta}_U) \\ C_{M1}(\boldsymbol{\delta}_U) & D_{M11}(\boldsymbol{\delta}_U) & D_{M12}(\boldsymbol{\delta}_U) \\ C_{M2}(\boldsymbol{\delta}_U) & D_{M21}(\boldsymbol{\delta}_U) & D_{M22}(\boldsymbol{\delta}_U) \end{bmatrix} : \begin{bmatrix} \mathcal{X}_S \otimes \ell^2_{\text{fin}} \\ \mathcal{W} \otimes \ell^2_{\text{fin}} \\ \mathcal{U} \otimes \ell^2_{\text{fin}} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_S \otimes \ell \\ \mathcal{Z} \otimes \ell \\ \mathcal{Y} \otimes \ell \end{bmatrix}$$

is obtained from the feedback connection

$$\begin{bmatrix} \vec{\tilde{x}}_U \\ \vec{\tilde{x}}_S \\ \vec{z} \\ \vec{y} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \vec{x}_U \\ \vec{x}_S \\ \vec{z} \\ \vec{y} \end{bmatrix}, \quad \text{subject to} \quad \vec{x}_U = \mathbf{Z}_U(\boldsymbol{\delta}_U)\vec{\tilde{x}}_U,$$

that is,

$$\begin{bmatrix} \mathbf{A}_M(\boldsymbol{\delta}_U) & \mathbf{B}_{M1}(\boldsymbol{\delta}_U) & \mathbf{B}_{M2}(\boldsymbol{\delta}_U) \\ \mathbf{C}_{M1}(\boldsymbol{\delta}_U) & \mathbf{D}_{M11}(\boldsymbol{\delta}_U) & \mathbf{D}_{M12}(\boldsymbol{\delta}_U) \\ \mathbf{C}_{M2}(\boldsymbol{\delta}_U) & \mathbf{D}_{M21}(\boldsymbol{\delta}_U) & \mathbf{D}_{M22}(\boldsymbol{\delta}_U) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{SS} & \mathbf{B}_{S1} & \mathbf{B}_{S2} \\ \mathbf{C}_{1S} & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_{2S} & \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} + \\ + \begin{bmatrix} \mathbf{A}_{SU} \\ \mathbf{C}_{1U} \\ \mathbf{C}_{2U} \end{bmatrix} (I - \mathbf{Z}_U(\boldsymbol{\delta}_U)\mathbf{A}_{UU})^{-1}\mathbf{Z}_U(\boldsymbol{\delta}_U) \begin{bmatrix} \mathbf{A}_{US} & \mathbf{B}_{U1} & \mathbf{B}_{U2} \end{bmatrix}. \tag{6.4}$$

As this system is time-varying, due to the presence of the time-varying uncertainty parameters $\boldsymbol{\delta}_U$, it is not convenient to work with a transfer-function acting on the frequency-domain; instead we stay in the time-domain and work with the input-output operator which has the form

$$\mathbf{G}(\boldsymbol{\delta}) = \begin{bmatrix} \mathbf{D}_{M11}(\boldsymbol{\delta}_U) & \mathbf{D}_{M12}(\boldsymbol{\delta}_U) \\ \mathbf{D}_{M21}(\boldsymbol{\delta}_U) & \mathbf{D}_{M22}(\boldsymbol{\delta}_U) \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{M1}(\boldsymbol{\delta}_U) \\ \mathbf{C}_{M2}(\boldsymbol{\delta}_U) \end{bmatrix} \times \\ \times (I_{\mathcal{X}_S \otimes \ell^2} - (I_{\mathcal{X}_S} \otimes \mathbf{S})\mathbf{A}_M(\boldsymbol{\delta}_U))^{-1}(I_{\mathcal{X}_S} \otimes \mathbf{S}) \begin{bmatrix} \mathbf{B}_{M1}(\boldsymbol{\delta}_U) & \mathbf{B}_{M2}(\boldsymbol{\delta}_U) \end{bmatrix}, \tag{6.5}$$

Now write $\boldsymbol{\delta}$ for the collection $(\boldsymbol{\delta}_U, \mathbf{S})$ of $d+1$ operators on ℓ^2. Then the input-output operator $\mathbf{G}(\boldsymbol{\delta})$ given by (6.5) has the noncommutative transfer-function realization

$$\mathbf{G}(\boldsymbol{\delta}) = \begin{bmatrix} \mathbf{G}_{11}(\boldsymbol{\delta}) & \mathbf{G}_{12}(\boldsymbol{\delta}) \\ \mathbf{G}_{21}(\boldsymbol{\delta}) & \mathbf{G}_{22}(\boldsymbol{\delta}) \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} (I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta}) \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \tag{6.6}$$

with system matrix as in (6.1) and $\mathbf{Z}(\boldsymbol{\delta}) = \left[\begin{smallmatrix} \mathbf{Z}_U(\boldsymbol{\delta}_U) & 0 \\ 0 & I_{\mathcal{X}_S} \otimes \mathbf{S} \end{smallmatrix}\right]$. In the formulas (6.4)-(6.6) the inverses may have to be interpreted as the algebraic inverses of the corresponding infinite block matrices; in that way, the formulas make sense at least for the nominal plant, i.e., with $\boldsymbol{\delta}_U = (0, \dots, 0)$.

More generally, the transfer-function $\mathbf{G}$ can be extended to a function of $d+1$ variables in $\mathcal{L}(\ell^2)$ by replacing $\mathbf{S}$ with another variable $\delta_{d+1} \in \mathcal{L}(\ell^2)$. In that case, the transfer-function can be viewed as an LFT-model with structured uncertainty, as studied in [98, 57]. However, as a consequence of the Sz.-Nagy dilation theory, without loss of generality it is possible in this setting of LFT-models to fix one of the variables to be the shift operator $\mathbf{S}$; in this way the LFT-model results developed for $d+1$ free variable contractions apply equally well to the case of interest where one of the variables is fixed to be the shift operator.

Such an input/state/output system $\mathbf{\Sigma}$ with structured dynamic time-varying uncertainty $\boldsymbol{\delta}_U$ is said to be *robustly stable* (with respect to the dynamic time-varying uncertainty structure $\mathbf{Z}_U(\boldsymbol{\delta}_U)$) if the state-matrix $\mathbf{A}_M(\boldsymbol{\delta}_U)$ is stable for all choices of $\boldsymbol{\delta}_U$ subject to $\|\mathbf{Z}_U(\boldsymbol{\delta}_U)\| \leq 1$, that is, if $I_{\mathcal{X}_S \otimes \ell^2} - (I_{\mathcal{X}_S} \otimes \mathbf{S})\mathbf{A}_M(\boldsymbol{\delta}_U)$ is invertible as an operator on $\mathcal{X}_S \oplus \ell^2$ for all $\boldsymbol{\delta}_U$ with $\|\mathbf{Z}_U(\boldsymbol{\delta}_U)\| \leq 1$. Since

$$\mathbf{A}_M(\boldsymbol{\delta}_U) = \mathbf{A}_{SS} + \mathbf{A}_{SU}(I - \mathbf{Z}_U(\boldsymbol{\delta}_U)\mathbf{A}_{UU})^{-1}\mathbf{Z}(\boldsymbol{\delta}_U)\mathbf{A}_{US},$$

it follows from the Main Loop Theorem [141, Theorem 11.7 page 284], that this condition in turn reduces to:

$$I_{\boldsymbol{\mathcal{X}}} - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A} \text{ is invertible for all } \boldsymbol{\delta} = (\boldsymbol{\delta}_U, \mathbf{S}) \text{ with } \|\mathbf{Z}(\boldsymbol{\delta})\| \leq 1. \tag{6.7}$$

Note that this condition amounts to a noncommutative version of the Hautus-stability criterion for the matrix A (where $\mathbf{A} = A \otimes I_{\ell^2}$). We shall therefore call the state matrix $\mathbf{A}$ *nc-Hautus-stable* if (6.7) is satisfied (with nc indicating that we are in the noncommutative setting). The input/state/output system $\boldsymbol{\Sigma}$ is said to have *nc-performance* (with respect to the dynamic time-varying uncertainty structure $\mathbf{Z}_U(\boldsymbol{\delta}_U)$) if it is robustly stable (with respect to this dynamic time-varying uncertainty structure) and in addition the input-output operator $G(\boldsymbol{\delta})$ has norm strictly less than 1 for all choices of $\boldsymbol{\delta} = (\boldsymbol{\delta}_U, \mathbf{S})$ with $\|\mathbf{Z}(\boldsymbol{\delta})\| \leq 1$.

One of the key results from the thesis of Paganini [108] which makes the noncommutative setting of this section more in line with the 1-D case is that, contrary to what is the case in Subsection 4.2, for operators $\mathbf{A} = A \oplus I_{\ell^2}$ on $\mathcal{X} \oplus \ell^2$ we do have $\mu_{\boldsymbol{\Delta}}(\mathbf{A}) = \widehat{\mu}_{\boldsymbol{\Delta}}(\mathbf{A})$ when we take $\boldsymbol{\Delta}$ to be the C^*-algebra

$$\boldsymbol{\Delta} = \left\{ \begin{bmatrix} \mathbf{Z}(\boldsymbol{\delta}_U) & 0 \\ 0 & I_{\mathcal{X}_S} \otimes \boldsymbol{\delta}_{d+1} \end{bmatrix} : \boldsymbol{\delta}_U = (\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_d),\ \boldsymbol{\delta}_j \in \mathcal{L}(\ell^2),\ j = 1, \ldots, d+1 \right\}. \tag{6.8}$$

Write $\boldsymbol{\mathcal{D}}$ for the commutant of $\boldsymbol{\Delta}$ in $\mathcal{L}((\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2)$. Then the main implication of the fact that $\mu_{\boldsymbol{\Delta}}(\mathbf{A}) = \widehat{\mu}_{\boldsymbol{\Delta}}(\mathbf{A})$ is that nc-Hautus-stability of $\mathbf{A}$ is now the same as the existence of an invertible operator $\mathbf{Q} \in \boldsymbol{\mathcal{D}}$ so that $\|\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\| < 1$ or, equivalently, the existence of a solution $\mathbf{X} \in \boldsymbol{\mathcal{D}}$ to the LMIs $\mathbf{A}^*\mathbf{X}\mathbf{A} - \mathbf{A} < 0$ and $\mathbf{X} > 0$. However, it is not hard to see that $\mathbf{X}$ is an element of $\boldsymbol{\mathcal{D}}$ if and only if $\mathbf{X} = X \otimes I_{\ell^2}$ with X being an element of the C^*-algebra $\mathcal{D}$ in (4.11). Thus, in fact, we find that $\mathbf{A} = A \oplus I_{\ell^2}$ is nc-Hautus-stable precisely when A is scaled stable, i.e., when there exists a solution $X \in \mathcal{D}$ to the LMIs $A^*XA - A < 0$ and $X > 0$.

These observations can also be seen as a special case (when $C_2 = 0$ and $B_2 = 0$) of the following complete analogue of Theorem 2.3 for this noncommutative setting due to Paganini [108].

Proposition 6.1. *Given a system matrix as in* (6.1)-(6.2)*, then:*

(i) *The output pair* $\{\mathbf{C}_2, \mathbf{A}\}$ *is nc-Hautus-detectable, that is, for every* $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{d+1})$*, with* $\boldsymbol{\delta}_j \in \mathcal{L}(\ell^2)$ *for* $j = 1, \ldots, d+1$*, so that* $\|\mathbf{Z}(\boldsymbol{\delta})\| \leq 1$ *the operator*

$$\begin{bmatrix} I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A} \\ \mathbf{C}_2 \end{bmatrix} : (\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2 \to \begin{bmatrix} (\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2 \\ \mathcal{Y} \oplus \ell^2 \end{bmatrix}$$

has a left inverse, if and only if $\{\mathbf{C}_2, \mathbf{A}\}$ *is nc-operator-detectable, i.e., there exists an operator* $\mathbf{L} = L \otimes I_{\ell^2}$*, with* $L : \mathcal{Y} \to \mathcal{X}$*, so that* $\mathbf{A} + \mathbf{L}\mathbf{C}_2$ *is nc-Hautus-stable, if and only if there exists a solution* $X \in \mathcal{D}$ *to the LMIs*

$$A^* X A - X - C_2^* C_2 < 0, \quad X > 0. \tag{6.9}$$

(ii) *The input pair* $\{\mathbf{A}, \mathbf{B}_2\}$ *is nc-Hautus-stabilizable, that is, for every* $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{d+1})$*, with* $\boldsymbol{\delta}_j \in \mathcal{L}(\ell^2)$ *for* $j = 1, \dots, d+1$*, so that* $\|\mathbf{Z}(\boldsymbol{\delta})\| \le 1$ *the operator*

$$\begin{bmatrix} I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A} & \mathbf{B}_2 \end{bmatrix} : \begin{bmatrix} (\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2 \\ \mathcal{U} \oplus \ell^2 \end{bmatrix} \to (\mathcal{X}_U \oplus \mathcal{X}_S) \otimes \ell^2$$

has a left inverse, if and only if $\{\mathbf{A}, \mathbf{B}_2\}$ *is nc-operator-stabilizable, i.e., there exists an operator* $\mathbf{F} = F \otimes I_{\ell^2}$*, with* $F : \mathcal{X} \to \mathcal{U}$*, so that* $\mathbf{A} + \mathbf{B}_2\mathbf{F}$ *is nc-Hautus-stable, which happens if and only if there exists a solution* $Y \in \mathcal{D}$ *to the LMIs*

$$A Y A^* - Y - B_2 B_2^* < 0, \quad Y > 0. \tag{6.10}$$

In case the input/state/output system $\boldsymbol{\Sigma}$ is not stable and/or does not have performance, we want to remedy this by means of a feedback with a controller $\mathbf{K}$, which we assume has on-line access to the structured dynamic time-varying uncertainty operators $\boldsymbol{\delta}_U$ in addition to being dynamic, i.e., $\mathbf{K} = \mathbf{K}(\boldsymbol{\delta}) = \mathbf{K}(\boldsymbol{\delta}_U, \mathbf{S})$. More specifically, we shall restrict to controllers of the form

$$\mathbf{K}(\boldsymbol{\delta}) = \mathbf{D}_K + \mathbf{C}_K (I - \mathbf{Z}_K(\boldsymbol{\delta})\mathbf{A}_K)^{-1} \mathbf{Z}_K(\boldsymbol{\delta}) \mathbf{B}_K \tag{6.11}$$

where

$$\mathbf{Z}_K(\boldsymbol{\delta}) = \begin{bmatrix} \mathbf{Z}_{KU}(\boldsymbol{\delta}_U) & 0 \\ 0 & I_{\mathcal{X}_{KS}} \otimes \mathbf{S} \end{bmatrix}, \; \mathbf{Z}_{KU}(\boldsymbol{\delta}_U) = \begin{bmatrix} I_{\mathcal{X}_{K1}} \otimes \boldsymbol{\delta}_1 & & \\ & \ddots & \\ & & I_{\mathcal{X}_{Kd}} \otimes \boldsymbol{\delta}_d \end{bmatrix},$$

with system matrix $\mathbf{M}_{\mathbf{K}}$ of the form

$$\mathbf{M}_K = \begin{bmatrix} \mathbf{A}_K & \mathbf{B}_K \\ \mathbf{C}_K & \mathbf{D}_K \end{bmatrix} : \begin{bmatrix} (\mathcal{X}_{KU} \oplus \mathcal{X}_{KS}) \otimes \ell^2 \\ \mathcal{Y} \otimes \ell^2 \end{bmatrix} \to \begin{bmatrix} (\mathcal{X}_{KU} \oplus \mathcal{X}_{KS}) \otimes \ell^2 \\ \mathcal{U} \otimes \ell^2 \end{bmatrix} \tag{6.12}$$

where $\mathcal{X}_{KU} = \mathcal{X}_{KU1} \oplus \cdots \oplus \mathcal{X}_{KUd}$, and where the matrix entries in turn have a tensor-factorization

$$\begin{bmatrix} \mathbf{A}_K & \mathbf{B}_K \\ \mathbf{C}_K & \mathbf{D}_K \end{bmatrix} = \begin{bmatrix} A_K \otimes I_{\ell^2} & B_K \otimes I_{\ell^2} \\ C_K \otimes I_{\ell^2} & D_K \otimes I_{\ell^2} \end{bmatrix}. \tag{6.13}$$

If such a controller $\mathbf{K}(\boldsymbol{\delta})$ is put in feedback connection with $\mathbf{G}(\boldsymbol{\delta})$, where we impose the usual assumption $D_{22} = 0$ to guarantee well-posedness, the resulting closed-loop system input-output operator $\mathbf{G}_{cl}(\boldsymbol{\delta})$, as a function of the operator uncertainty parameters $\boldsymbol{\delta}_U = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$ and the shift $\mathbf{S}$, has a realization which is formally exactly as in (4.9), that is

$$\mathbf{G}_{cl}(\boldsymbol{\delta}) = \mathbf{D}_{cl} + \mathbf{C}_{cl}(I - \mathbf{Z}_{cl}(\boldsymbol{\delta})\mathbf{A}_{cl})^{-1} \mathbf{Z}_{cl}(\boldsymbol{\delta}) \mathbf{C}_{cl}$$

with system matrix

$$\begin{bmatrix} \mathbf{A}_{cl} & \mathbf{B}_{cl} \\ \mathbf{C}_{cl} & \mathbf{D}_{cl} \end{bmatrix} = \left[\begin{array}{cc|c} \mathbf{A} + \mathbf{B}_2\mathbf{D}_K\mathbf{C}_2 & \mathbf{B}_2\mathbf{C}_K & \mathbf{B}_1 + \mathbf{B}_2\mathbf{D}_K\mathbf{D}_{21} \\ \mathbf{B}_K\mathbf{C}_2 & \mathbf{A}_K & \mathbf{B}_K\mathbf{D}_{21} \\ \hline \mathbf{C}_1 + \mathbf{D}_{12}\mathbf{D}_K\mathbf{C}_2 & \mathbf{D}_{12}\mathbf{C}_K & \mathbf{D}_{11} + \mathbf{D}_{12}\mathbf{D}_K\mathbf{D}_{21} \end{array}\right], \tag{6.14}$$

which is the same as the system matrix (4.10) tensored with I_{ℓ^2}, and

$$\mathbf{Z}_{cl}(\boldsymbol{\delta}) = \begin{bmatrix} \mathbf{Z}(\boldsymbol{\delta}) & 0 \\ 0 & \mathbf{Z}_K(\boldsymbol{\delta}) \end{bmatrix} \quad \text{where} \quad \boldsymbol{\delta} = (\boldsymbol{\delta}_U, \mathbf{S}). \tag{6.15}$$

The *state-space nc-stabilization problem* (with respect to the given dynamic time-varying uncertainty structure $\boldsymbol{\delta}_U$) then is to design a controller $\mathbf{K}$ with state-space realization $\{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ as above so that the closed-loop system $\boldsymbol{\Sigma}_{cl}$ defined by the system matrix (6.14) is robustly stable. The *state-space nc-H^∞-problem* is to design a controller $\mathbf{K}$ with state-space realization $\{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ as above so that the closed-loop system $\boldsymbol{\Sigma}_{cl}$ also has robust performance.

Since the closed-loop state-operator $\mathbf{A}_{cl}$ is equal to $A_{cl} \otimes I_{\ell^2}$ with A_{cl} defined by (4.10), it follows as another implication of the fact that $\mu_{\boldsymbol{\Delta}}$ is equal to $\widehat{\mu}_{\boldsymbol{\Delta}}$ for operators that are tensored with I_{ℓ^2} (with respect to the appropriate C^*-algebra $\boldsymbol{\Delta}$) that $\mathbf{A}_{cl}$ is nc-Hautus-stable precisely when A_{cl} is scaled stable, i.e., we have the following result.

Proposition 6.2. *Let $\boldsymbol{\Sigma}$ and Σ be the systems given by* (6.3) *and* (5.1)*, respectively, corresponding to a given system matrix* (5.4)*. Then $\boldsymbol{\Sigma}$ is nc-Hautus-stabilizable if and only if Σ is scaled-stabilizable.*

Thus, remarkably, the solution criterion given in Section 4.2 for the scaled state-space stabilization problem turns out to be necessary and sufficient for the solution of the dynamic time-varying structured-uncertainty version of the problem.

Theorem 6.3. *Let $\boldsymbol{\Sigma}$ be the system given by* (6.3) *corresponding to a given system matrix* (6.1)*. Then $\boldsymbol{\Sigma}$ is nc-Hautus-stabilizable if and only if the output pair $\{\mathbf{C}_2, \mathbf{A}\}$ is nc-Hautus-detectable and the input pair $\{\mathbf{A}, \mathbf{B}_2\}$ is nc-Hautus-stabilizable, i.e., if there exist solutions $X, Y \in \mathcal{D}$, with $\mathcal{D}$ the C^*-algebra given in* (4.11)*, to the LMIs* (6.9) *and* (6.10)*. In this case $\mathbf{K} \sim \left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right] \otimes I_{\ell^2}$ with $\left[\begin{smallmatrix} A_K & B_K \\ C_K & D_K \end{smallmatrix}\right]$ as in* (4.12) *is a controller solving the nc-Hautus stabilization problem for $\boldsymbol{\Sigma}$.*

In a similar way, the state-space nc-H^∞-problem corresponds to the scaled H^∞-problem of Subsection (4.2).

Theorem 6.4. *Let $\boldsymbol{\Sigma}$ be the system given by* (6.3) *for a given system matrix* (6.1)*. Then there exists a solution $\mathbf{K}$, with realization* (6.11)*, to the state-space nc-H^∞-problem for the non-commutative system $\boldsymbol{\Sigma}$ if and only if there exist $X, Y \in \mathcal{D}$ that satisfy the LMIs* (4.27) *and* (4.26) *and the coupling condition* (4.28)*.*

Proof. Let $\boldsymbol{\Sigma}$ and Σ be the systems given by (6.3) and (5.1), respectively, corresponding to a given system matrix (5.4). Using the strict bounded real lemma from

[29] in combination with similar arguments as used above for the nc-stabilizability problem, it follows that a transfer-function $\mathbf{K}$ with realization (6.11)-(6.13) is a solution to the state-space nc-H^∞-problem for $\mathbf{\Sigma}$ if and only if the transfer function K with realization (4.7) is a solution to the scaled H^∞-problem for the system Σ. The statement then follows from Theorem 4.8. $\square$

6.2. A noncommutative frequency-domain formulation

In this subsection we present a frequency-domain version of the noncommutative state-space setup of the previous subsection used to model linear input/state/output systems with LFT-model for dynamic time-varying structured uncertainty. The frequency-domain setup here is analogous to that of Section 4.1 but the unit polydisk $\overline{\mathbb{D}}^d$ is replaced by the noncommutative polydisk $\overline{\mathbb{D}}^d_{nc}$ consisting of all d-tuples $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$ of contraction operators on a fixed separable infinite-dimensional Hilbert space $\mathcal{K}$.

We need a few preliminary definitions. We define $\mathcal{F}_d$ to be the free semigroup consisting of all words $\alpha = i_N \cdots i_1$ in the letters $\{1, \dots, d\}$. When $\alpha = i_N \cdots i_1$ we write $N = |\alpha|$ for the number of letters in the word α. The multiplication of two words is given by concatenation:

$$\alpha \cdot \beta = i_N \cdots i_1 j_M \cdots j_1 \text{ if } \alpha = i_N \cdots i_1 \text{ and } \beta = j_M \cdots j_1.$$

The unit element of $\mathcal{F}_d$ is the *empty word* denoted by $\emptyset$ with $|\emptyset| = 0$. In addition, we let $z = (z_1, \dots, z_d)$ stands for a d-tuple of *noncommuting* indeterminates, and for any $\alpha = i_N \cdots i_1 \in \mathcal{F}_d - \{\emptyset\}$, we let z^α denote the noncommutative monomial $z^\alpha = z_{i_N} \cdots z_{i_1}$, while $z^\emptyset = 1$. If α and β are two words in $\mathcal{F}_d$, we multiply the associated monomials z^α and z^β in the natural way:

$$z^\alpha \cdot z^\beta = z^{\alpha \cdot \beta}.$$

Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we let $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ denote the collection of all noncommutative formal power series $S(z)$ of the form $S(z) = \sum_{\alpha \in \mathcal{F}_d} S_\alpha z^\alpha$ where the coefficients S_α are operators in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ for each $\alpha \in \mathcal{F}_d$. Given a formal power series $S(z) = \sum_{\alpha \in \mathcal{F}_d} S_\alpha z^\alpha$ together with a d-tuple of linear operators $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$ acting on ℓ^2, we define $S(\boldsymbol{\delta})$ by

$$S(\boldsymbol{\delta}) = \lim_{N \to \infty} \sum_{\alpha \in \mathcal{F}_d \colon |\alpha| = N} S_\alpha \otimes \boldsymbol{\delta}^\alpha \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$$

whenever the limit exists in the operator-norm topology; here we use the notation $\boldsymbol{\delta}^\alpha$ for the operator

$$\boldsymbol{\delta}^\alpha = \boldsymbol{\delta}_{i_N} \cdots \boldsymbol{\delta}_{i_1} \text{ if } \alpha = i_N \cdots i_1 \in \mathcal{F}_d - \{\emptyset\} \text{ and } \boldsymbol{\delta}^\emptyset = I_{\mathcal{K}}.$$

We define the *noncommutative Schur-Agler class* $\mathcal{SA}_{nc,d}(\mathcal{U}, \mathcal{Y})$ (*strict noncommutative Schur-Agler class* $\mathcal{SA}^o_{nc,d}(\mathcal{U}, \mathcal{Y})$) to consist of all formal power series in $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ such that $\|S(\boldsymbol{\delta}))\| \leq 1$ ($\|S(\boldsymbol{\delta})\| < 1$) whenever $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$ is a

d-tuple of operators on $\mathcal{K}$ with $\|\boldsymbol{\delta}_j\| < 1$ ($\|\boldsymbol{\delta}_j\| \le 1$) for $j = 1, \dots, d$. Let

$$\mathbb{D}_{nc,d} := \{\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d) \colon \boldsymbol{\delta}_j \in \mathcal{L}(\mathcal{K}),\ \|\boldsymbol{\delta}_j\| < 1,\ j = 1, \dots, d\},$$
$$\overline{\mathbb{D}}_{nc,d} := \{\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d) \colon \boldsymbol{\delta}_j \in \mathcal{L}(\mathcal{K}),\ \|\boldsymbol{\delta}_j\| \le 1,\ j = 1, \dots, d\}.$$

We then define the *strict noncommutative H^∞-space* $H^{\infty,o}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ to consist of all functions F from $\overline{\mathbb{D}}_{nc,d}$ to $\mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ which can be expressed in the form

$$F(\boldsymbol{\delta}) = S(\boldsymbol{\delta})$$

for all $\boldsymbol{\delta} \in \overline{\mathbb{D}}_{nc,d}$ where $\rho^{-1}S$ is in the strict noncommutative Schur-Agler class $\mathcal{SA}^o_{nc,d}(\mathcal{U}, \mathcal{Y})$ for some real number $\rho > 0$. We write $H^\infty_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ for the set of functions G from $\mathbb{D}_{nc,d}$ to $\mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ that are also of the form $G(\boldsymbol{\delta}) = S(\boldsymbol{\delta})$, but now for $\boldsymbol{\delta} \in \mathbb{D}_{nc,d}$ and $\rho^{-1}S$ in $\mathcal{SA}_{nc,d}(\mathcal{U}, \mathcal{Y})$ for some $\rho > 0$. Note that $\mathcal{SA}_{nc,d}(\mathcal{U}, \mathcal{Y})$ amounts to $\mathcal{SA}_{nc,d}(\mathbb{C}, \mathbb{C}) \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$. In the sequel we abbreviate the notation $\mathcal{SA}_{nc,d}(\mathbb{C}, \mathbb{C})$ for the scalar Schur-Agler class to simply $\mathcal{SA}_{nc,d}$. Similarly, we simply write $\mathcal{SA}^o_{nc,d}$, $H^{\infty,o}_{nc,d}$ and $H^\infty_{nc,d}$ instead of $\mathcal{SA}^o_{nc,d}(\mathbb{C}, \mathbb{C})$, $H^{\infty,o}_{nc,d}(\mathbb{C}, \mathbb{C})$ and $H^\infty_{nc,d}(\mathbb{C}, \mathbb{C})$, respectively. Thus we also have $H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y})) = H^{\infty,o}_{nc,d} \otimes \mathcal{L}(\mathcal{U}, \mathcal{Y})$, etc. We shall be primarily interested in the strict versions $\mathcal{SA}^o_{nc,d}$ and $H^{\infty,o}_{nc,d}$ of the noncommutative Schur-Agler class and H^∞-space.

We have the following characterization of the space $H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$. For the definition of *completely positive kernel* and more complete details, we refer to [30]. The formulation given here does not have the same form as in Theorem 3.6(2) of [30], but one can use the techniques given there to convert to the form given in the following theorem.

Theorem 6.5. *The function $F \colon \overline{\mathbb{D}}_{nc,d} \to \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ is in the strict noncommutative H^∞-space $H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ if and only if there are d strictly completely positive kernels*

$$K_k \colon (\overline{\mathbb{D}}_{nc,d} \times \overline{\mathbb{D}}_{nc,d}) \times \mathcal{L}(\mathcal{K}) \to \mathcal{L}(Y \otimes \mathcal{K}) \text{ for } k = 1, \dots, d$$

and a positive real number ρ so that the following Agler decomposition holds:

$$\rho^2 \cdot (I \otimes B) - S(\boldsymbol{\delta})\,(I \otimes B)\,S(\boldsymbol{\tau})^* = \sum_{k=1}^d K_k(\boldsymbol{\delta}, \boldsymbol{\tau})[B - \boldsymbol{\delta}_k B \boldsymbol{\tau}_k^*]$$

for all $B \in \mathcal{L}(\mathcal{K})$ and $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$, $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_d)$ in $\overline{\mathbb{D}}_{nc,d}$.

One of the main results of [28] is that the noncommutative Schur-Agler class has a contractive Givone-Roesser realization.

Theorem 6.6. *(See* [28, 29]*.) A given function $F \colon \overline{\mathbb{D}}_{nc,d} \to \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ is in the strict noncommutative Schur-Agler class $\mathcal{SA}^o_{nc,d}(\mathcal{U}, \mathcal{Y})$ if and only if there exists a strictly contractive colligation matrix*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \oplus_{j=1}^d \mathcal{X}_j \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{j=1}^d \mathcal{X}_j \\ \mathcal{Y} \end{bmatrix}$$

for some Hilbert state space $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_d$ *so that the evaluation of* F *at* $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d) \in \overline{\mathbb{D}}_{nc,d}$ *is given by*

$$F(\boldsymbol{\delta}) = D \otimes I_{\mathcal{K}} + (C \otimes I_{\mathcal{K}}((I - \mathbf{Z}(\boldsymbol{\delta})(A \otimes I_{\mathcal{K}}))^{-1}\mathbf{Z}(\boldsymbol{\delta})(B \otimes I_{\mathcal{K}}) \tag{6.16}$$

where

$$\mathbf{Z}(\boldsymbol{\delta}) = \begin{bmatrix} I_{\mathcal{X}_1} \otimes \boldsymbol{\delta}_1 & & \\ & \ddots & \\ & & I_{\mathcal{X}_d} \otimes \boldsymbol{\delta}_d \end{bmatrix}.$$

Hence a function $F \colon \overline{\mathbb{D}}_{nc,d} \to \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K})$ *is in the strict noncommutative* H^∞*-space* $H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ *if and only if there is a bounded linear operator*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \oplus_{k=1}^d \mathcal{X}_k \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{k=1}^d \mathcal{X}_k \\ \mathcal{Y} \end{bmatrix}$$

such that

$$\left\| \begin{bmatrix} A & B \\ \rho^{-1}C & \rho^{-1}D \end{bmatrix} \right\| < 1 \text{ for some } \rho > 0$$

so that F *is given as in* (6.16).

If $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional Hilbert spaces, we may view $\mathcal{SA}^o_{nc,d}(\mathcal{U}, \mathcal{Y})$ and $H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$ as matrices over the respective scalar-valued classes $\mathcal{SA}^o_{nc,d}$ and $H^{\infty,o}_{nc,d}$. When this is the case, it is natural to define *rational* versions of $\mathcal{SA}^o_{nc,d}$ and $H^{\infty,o}_{nc,d}$ to consist of those functions in $\mathcal{SA}^o_{nc,d}$ (respectively, $H^{\infty,o}_{nc,d}$) for which the realization (6.16) can be taken with the state spaces $\mathcal{X}_1, \dots, \mathcal{X}_d$ also finite-dimensional; we denote the rational versions of $\mathcal{SA}^o_{nc,d}$ and $H^{\infty,o}_{nc,d}$ by $\mathcal{RSA}^o_{nc,d}$ and $\mathcal{R}H^{\infty,o}_{nc,d}$, respectively. We remark that as a consequence of Theorem 11.1 in [27], this rationality assumption on a given function F in $H^{\infty,o}_{nc,d}$ can be expressed intrinsically in terms of the finiteness of rank for a finite collection of Hankel matrices formed from the power-series coefficients F_α of F, i.e., the operators $F_\alpha \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that

$$F(\boldsymbol{\delta}) = \sum_{\alpha \in \mathcal{F}_d} F_\alpha \otimes \boldsymbol{\delta}^\alpha.$$

In general, the embedding of a noncommutative integral domain into a skew-field is difficult (see e.g. [75, 82]). For the case of $\mathcal{R}H^{\infty,o}$, the embedding issue becomes tractable if we restrict to denominator functions $D(\boldsymbol{\delta}) \in H^{\infty,o} \in \mathcal{L}(\mathcal{U})$ for which $D(0)$ is invertible. If D is given in terms of a strictly contractive realization $D(\boldsymbol{\delta}) = \mathbf{D} + \mathbf{C}(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}$ (where $\mathbf{A} = A \otimes I_{\mathcal{K}}$ and similarly for $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$), then $D(\boldsymbol{\delta})^{-1}$ can be calculated, at least for $\|\mathbf{Z}(\boldsymbol{\delta})\|$ small enough, via the familiar cross-realization formula for the inverse:

$$D(\boldsymbol{\delta})^{-1} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\mathbf{C}(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A}^\times)^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}\mathbf{D}^{-1}$$

where $\mathbf{A}^\times = A^\times \otimes I_{\mathcal{K}}$ with $A^\times = A - BD^{-1}C$. We define $Q(\mathcal{R}H^{\infty,o}_{nc,d})(\mathcal{L}(\mathcal{U}, \mathcal{Y}))_0$ to be the smallest linear space of functions from some neighborhood of 0 in $\overline{\mathbb{D}}_{nc,d}$ (with respect to the Cartesian product operator-norm topology on $\overline{\mathbb{D}}_{nc,d} \subset \mathcal{L}(\mathcal{K})^d$)

to $\mathcal{L}(\mathcal{U},\mathcal{Y})$ which is invariant under multiplication on the left by elements of $\mathcal{R}H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{Y}))$ and by inverses of elements of $\mathcal{R}H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{Y}))$ having invertible value at 0, and invariant under multiplication on the right by the corresponding set of functions with $\mathcal{U}$ in place of $\mathcal{Y}$. Note that the final subscript 0 in the notation $Q(\mathcal{R}H^{\infty,o}_{nc,d})(\mathcal{L}(\mathcal{U},\mathcal{Y}))_0$ is suggestive of the requirement that functions of this class are required to be analytic in a neighborhood of the origin $0 \in \mathbb{D}_{nc,d}$.

Let us denote by $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ the space of functions defined as follows: we say that the function G defined on a neighborhood of the origin in $\mathbb{D}_{nc,d}$ with values in $\mathcal{L}(\mathcal{U},\mathcal{Y})$ is in the space $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ if G has a realization of the form

$$G(\boldsymbol{\delta}) = \mathbf{D} + \mathbf{C}(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}$$

for a colligation matrix $\mathbf{M} := \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$ of the form $\mathbf{M} = M \otimes I_{\mathcal{K}}$ where

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \oplus_{k=1}^{d} \mathcal{X}_k \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \oplus_{k=1}^{d} \mathcal{X}_k \\ \mathcal{Y} \end{bmatrix}$$

for some finite-dimensional state-spaces $\mathcal{X}_1, \dots, \mathcal{X}_d$. Unlike the assumptions in the case of a realization for a Schur-Agler-class function in Theorem 6.6, there is no assumption that M be contractive or that A be stable. It is easily seen that $Q(\mathcal{R}H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y})))_0$ is a subset of $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$; whether these two spaces are the same or not we leave as an open question. We also note that the class $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ has an intrinsic characterization: F is in $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ if and only if some rescaled version $\widetilde{F}(\boldsymbol{\delta}) = F(r\boldsymbol{\delta})$ (where $r\boldsymbol{\delta} = (r\boldsymbol{\delta}_1, \dots, r\boldsymbol{\delta}_d)$ if $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_d)$) is in the rational noncommutative H^∞-class $\mathcal{R}H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ for some $r > 0$ and hence has the intrinsic characterization in terms of a completely positive Agler decomposition and finite-rankness of a finite collection of Hankel matrices as described above for the class $\mathcal{R}H^{\infty,o}_{nc,c}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$.

We may then pose the following control problems:

Noncommutative polydisk internal-stabilization/H^∞-control problem: We suppose that we are given finite-dimensional spaces $\mathcal{W}$, $\mathcal{U}$, $\mathcal{Z}$, $\mathcal{Y}$ and a block-matrix $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right]$ in $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{W} \oplus \mathcal{U}, \mathcal{Z} \oplus \mathcal{Y}))$. We seek to find a controller K in $\mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{Y},\mathcal{U}))$ which solves the (1) *internal stabilization problem*, i.e. so that the closed-loop system is *internally stable* in the sense that all matrix entries of the block matrix $\Theta(G,K)$ given by (2.2) are in $\mathcal{R}H^{\infty,o}_{nc,d}$, and which possibly also solves the (2) *H^∞-problem*, i.e., in addition to internal stability, the closed-loop system has *performance* in the sense that $T_{zw} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$ is in the rational strict noncommutative Schur-Agler class $\mathcal{RSA}^o_{nc,d}(\mathcal{W},\mathcal{Z})$.

Even though our algebra of scalar plants $\mathcal{RO}^0_{nc,d}$ is noncommutative, the parameterization result Theorem 3.5 still goes through in the following form; we leave it to the reader to check that the same algebra as used for the commutative case leads to the following noncommutative analogue.

Theorem 6.7. *Assume that $G \in \mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{W} \oplus \mathcal{U}, \mathcal{Z} \oplus \mathcal{Y}))$ is given and that G has at least one stabilizing controller K_*. Define $U_* = (I - G_{22}K_*)^{-1}$, $V_* =$*

$K_*(I-G_{22}K_*)^{-1}$, $\widetilde{U}_* = (I-K_*G_{22})^{-1}$ and $\widetilde{V}_* = (I-K_*G_{22})^{-1}K_*$. Then the set of all stabilizing controllers K for G is given by either of the two formulas

$$K = (V_* + Q)(U_* + G_{22}Q)^{-1} \text{ subject to } (U_* + G_{22}Q)(0) \text{ is invertible,}$$

$$K = (\widetilde{U}_* + QG_{22})^{-1}(\widetilde{V}_* + Q) \text{ subject to } (\widetilde{U}_* + QG_{22})(0) \text{ is invertible,}$$

where in addition Q has the form $Q = \widetilde{L}\Lambda L$ where $\widetilde{L}$ and L are given by (3.8) *and Λ is a free stable parameter in $H^{\infty,o}_{nc,d}(\mathcal{L}(\mathcal{Y}\oplus\mathcal{U},\mathcal{U}\oplus\mathcal{Y}))$. Moreover, if $Q = \widetilde{L}\Lambda L$ with Λ stable, then $(U_* + G_{22}Q)(0)$ is invertible if and only if $(\widetilde{U}_* + QG_{22})(0)$ is invertible, and both formulas give rise to the same controller K.*

Given a transfer matrix $G_{22} \in \mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$, we say that G_{22} has a *stable double coprime factorization* if there exist transfer matrices $D(\boldsymbol{\delta})$, $N(\boldsymbol{\delta})$, $X(\boldsymbol{\delta})$, $Y(\boldsymbol{\delta})$, $\widetilde{D}(\boldsymbol{\delta})$, $\widetilde{N}(\boldsymbol{\delta})$, $\widetilde{X}(\boldsymbol{\delta})$, and $\widetilde{Y}(\boldsymbol{\delta})$ of compatible sizes with stable matrix entries (i.e., with matrix entries in $\mathcal{R}H^{\infty,o}_{nc,d}$) subject also to

$$D(0),\ \widetilde{D}(0),\ X(0),\ \widetilde{X}(0) \text{ all invertible}$$

so that the noncommutative version of condition (3.9) holds:

$$\begin{gathered} G_{22}(\boldsymbol{\delta}) = D(\boldsymbol{\delta})^{-1}N(\boldsymbol{\delta}) = \widetilde{N}(\boldsymbol{\delta})\widetilde{D}^{-1}(\boldsymbol{\delta}), \\ \begin{bmatrix} D(\boldsymbol{\delta}) & -N(\boldsymbol{\delta}) \\ -\widetilde{Y}(\boldsymbol{\delta}) & \widetilde{X}(\boldsymbol{\delta}) \end{bmatrix} \begin{bmatrix} X(\boldsymbol{\delta}) & \widetilde{N}(\boldsymbol{\delta}) \\ Y(\boldsymbol{\delta}) & \widetilde{D}(\boldsymbol{\delta}) \end{bmatrix} = \begin{bmatrix} I_{n_\mathcal{Y}} & 0 \\ 0 & I_{n_\mathcal{U}} \end{bmatrix}. \end{gathered} \tag{6.17}$$

Then we leave it to the reader to check that the same algebra as used for the commutative case leads to the following noncommutative version of Theorem 3.11.

Theorem 6.8. *Assume that $G \in \mathcal{RO}^0_{nc,d}$ is stabilizable and that G_{22} admits a double coprime factorization* (6.17)*. Then the set of all stabilizing controllers is given by*

$$\begin{aligned} K(\boldsymbol{\delta}) &= (Y(\boldsymbol{\delta}) + \widetilde{D}(\boldsymbol{\delta})\Lambda(\boldsymbol{\delta}))(X(\boldsymbol{\delta}) + \widetilde{N}(\boldsymbol{\delta})\Lambda(\boldsymbol{\delta}))^{-1} \\ &= (\widetilde{X}(\boldsymbol{\delta}) + \Lambda(\boldsymbol{\delta})N(\boldsymbol{\delta}))^{-1}(\widetilde{Y}(\boldsymbol{\delta}) + \Lambda(\boldsymbol{\delta})D(\boldsymbol{\delta})), \end{aligned}$$

where Λ is a free stable parameter from $H^{\infty,0}_{nc,d}(\mathcal{L}(\mathcal{U},\mathcal{Y})$ such that $X(0) - \widetilde{N}(0)\Lambda(0)$ is invertible and $\widetilde{X}(0) + \Lambda(0)N(0)$ is invertible.

Just as in the commutative case, consideration of the H^∞-control problem for a given transfer matrix $G \in \mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{W}\oplus\mathcal{U},\mathcal{Z}\oplus\mathcal{Y}))$ after the change of the design parameter from the controller K to the free-stable parameter Λ in either of the two parameterizations of Theorems 6.7 and 6.8 leads to the following noncommutative version of the Model-Matching problem; we view this problem as a noncommutative version of a Sarason interpolation problem.

Noncommutative-polydisk Sarason interpolation problem: Given matrices T_1, T_2, T_3 of compatible sizes over $\mathcal{R}H^{\infty,o}_{nc,d}$, find a matrix Λ (of appropriate size) over $\mathcal{R}H^{\infty,o}_{nc,d}$ so that the matrix $S = T_1 + T_2\Lambda T_3$ is in the strict rational noncommutative Schur-Agler class $\mathcal{RSA}^o_{nc,d}(\mathcal{W},\mathcal{Z})$.

While there has been some work on left-tangential Nevanlinna-Pick-type interpolation for the noncommutative Schur-Agler class (see [22]), there does not seem to have been any work on a Commutant Lifting theorem for this setup or on how to convert a Sarason problem as above to an interpolation problem as formulated in [22]. We leave this area to future work.

6.3. Equivalence of state-space noncommutative LFT-model and noncommutative frequency-domain formulation

In order to make the connections between the results in the previous two subsections, we consider functions as in Subsection 6.2, but we normalize the infinite dimensional Hilbert space $\mathcal{K}$ to be ℓ^2 and work with $d+1$ variables $\boldsymbol{\delta} = (\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_{d+1})$ in $\mathcal{L}(\ell^2)$ instead of d. As pointed out in Subsection 6.1, we may without loss of generality assume that the last variable $\boldsymbol{\delta}_{d+1}$ is fixed to be the shift operator $\mathbf{S}$ on ℓ^2.

The following is an improved analogue of Lemma 4.13 for the noncommutative setting.

Theorem 6.9. *Suppose that the matrix function $W \in \mathcal{RO}^0_{nc,d+1}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ has a finite-dimensional realization*

$$W(\boldsymbol{\delta}) = \mathbf{D} + \mathbf{C}(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B},$$

where

$$\mathbf{A} = A \otimes I_{\ell^2}, \quad \mathbf{B} = B \otimes I_{\ell^2}, \quad \mathbf{C} = C \otimes I_{\ell^2}, \quad \mathbf{D} = D \otimes I_{\ell^2},$$

which is both nc-Hautus-detectable and nc-Hautus-observable. Then W is stable in the noncommutative frequency-domain sense (i.e., all matrix entries of W are in $H^{\infty,o}_{nc,d+1}$) if and only if W is stable in the state-space sense, i.e., the matrix $\mathbf{A}$ is nc-Hautus-stable.

Proof. If the matrix $\mathbf{A}$ is nc-Hautus-stable, it is trivial that then all matrix entries of W are in $H^{\infty,o}_{nc,d+1}$. We therefore assume that all matrix entries of W are in $H^{\infty,o}_{nc,d+1}$. It remains to show that, under the assumption that $\{C, A\}$ is nc-Hautus detectable and that $\{A, B\}$ is nc-Hautus stabilizable, it follows that A is nc-Hautus stable.

The first step is to observe the identity

$$S_1(\boldsymbol{\delta}) := \begin{bmatrix} I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A} \\ \mathbf{C} \end{bmatrix} (I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B} = \begin{bmatrix} \mathbf{Z}(\boldsymbol{\delta})\mathbf{B} \\ W(\boldsymbol{\delta}) - \mathbf{D} \end{bmatrix}. \tag{6.18}$$

Since $W(\boldsymbol{\delta}) - \mathbf{D}$ is in $H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ by assumption and trivially $\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}$ is in $H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{U},\mathcal{X}))$, it follows that $S_1(\boldsymbol{\delta})$ is in $H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{U},\mathcal{X} \oplus \mathcal{Y}))$. By the detectability assumption and Proposition 6.1 it follow that there exists an operator $\mathbf{L} = L \otimes I_{\ell^2}$ with $L : \mathcal{Y} \to \mathcal{X}$ so that $\mathbf{A} + \mathbf{LC}$ is nc-Hautus-stable. Thus

$$F_1(\boldsymbol{\delta}) = (I - \mathbf{Z}(\boldsymbol{\delta})(\mathbf{A} + \mathbf{LC}))^{-1} \begin{bmatrix} I & -\mathbf{Z}(\boldsymbol{\delta})\mathbf{L} \end{bmatrix}$$

is in $H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{X}\oplus\mathcal{Y},\mathcal{X}))$. Note that $F_1(\boldsymbol{\delta})S_1(\boldsymbol{\delta}) = (I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}$. The fact that both F_1 and S_1 are transfer-functions over $H^{\infty,o}_{nc,d+1}$ implies that $S_2(\boldsymbol{\delta}) = (I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}$ is in $H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{U},\mathcal{X}))$.

We next use the identity

$$\begin{aligned}\begin{bmatrix}\mathbf{Z}(\boldsymbol{\delta}) & S_2(\boldsymbol{\delta})\end{bmatrix} :=& \begin{bmatrix}\mathbf{Z}(\boldsymbol{\delta}) & (I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}\end{bmatrix}\\ =&\mathbf{Z}(\boldsymbol{\delta})(I-\mathbf{A}\mathbf{Z}(\boldsymbol{\delta}))^{-1}\begin{bmatrix}I-\mathbf{A}\mathbf{Z}(\boldsymbol{\delta}) & \mathbf{B}\end{bmatrix}.\end{aligned} \tag{6.19}$$

Now the nc-Hautus-stabilizability assumption and the second part of Proposition 6.1 imply in a similar way that $S_3(\boldsymbol{\delta}) = \mathbf{Z}(\boldsymbol{\delta})(I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}$ is in $H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{X},\mathcal{X}))$. Note that S_3 in turn has the trivial realization

$$S_3(\boldsymbol{\delta}) = \mathbf{D}' + \mathbf{C}'(I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A}')^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}'$$

where $\left[\begin{smallmatrix}\mathbf{A}' & \mathbf{B}'\\ \mathbf{C}' & \mathbf{D}'\end{smallmatrix}\right] = \left[\begin{smallmatrix}A' & B'\\ C' & D'\end{smallmatrix}\right]\otimes I_{\ell^2}$ and $\left[\begin{smallmatrix}A' & B'\\ C' & D'\end{smallmatrix}\right] = \left[\begin{smallmatrix}A & I\\ I & 0\end{smallmatrix}\right]$. Thus $(A',B',C',D') = (A,I,I,0)$ is trivially GR-controllable and GR-observable in the sense of [27]. On the other hand, by Theorem 6.6 there exists a strictly contractive matrix $\left[\begin{smallmatrix}A'' & B''\\ C'' & 0\end{smallmatrix}\right]$ so that

$$S_3(\boldsymbol{\delta}) = r''\mathbf{C}''(I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A}'')^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}''$$

for some $r<\infty$. Moreover, by the Kalman decomposition for noncommutative GR-systems given in [27], we may assume without loss of generality that $(A'',B'',C'',0)$ is GR-controllable and GR-observable. Then, by the main result of Alpay–Kaliuzhnyi-Verbovetskyi in [14], it is known that the function $S(\boldsymbol{\delta}) = \sum_{\alpha\in\mathcal{F}_d} S_\alpha\otimes\boldsymbol{\delta}^\alpha$ uniquely determines the formal power series $S(z) = \sum_{\alpha\in\mathcal{F}_d} S_\alpha z^\alpha$. It now follows from the State-Space Similarity Theorem for noncommutative GR-systems in [27] that there is an invertible block diagonal similarity transform $Q\in\mathcal{L}(\mathcal{X}',\mathcal{X}'')$ so that

$$\begin{bmatrix}A & I\\ I & 0\end{bmatrix} := \begin{bmatrix}A' & B'\\ C' & 0\end{bmatrix} = \begin{bmatrix}Q^{-1} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A'' & B''\\ C'' & 0\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & I\end{bmatrix}.$$

In particular, $A = Q^{-1}A''Q$ where A'' is a strict contraction and Q is a structured similarity from which it follows that A is also nc-Hautus-stable as wanted. □

We can now obtain the equivalence of the frequency-domain and state-space formulations of the internal stabilization problems for the case where the state-space internal stabilization problem is solvable.

Theorem 6.10. *Suppose that we are given a realization*

$$G(\boldsymbol{\delta}) = \begin{bmatrix}G_{11}(\boldsymbol{\delta}) & G_{12}(\boldsymbol{\delta})\\ G_{21}(\boldsymbol{\delta}) & G_{22}(\boldsymbol{\delta})\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{D}_{12}\\ \mathbf{D}_{21} & 0\end{bmatrix} + \begin{bmatrix}\mathbf{C}_1\\ \mathbf{C}_2\end{bmatrix}(I-\mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\begin{bmatrix}\mathbf{B}_1 & \mathbf{B}_2\end{bmatrix}$$

for an element $G\in\mathcal{RO}^0_{nc,d+1}(\mathcal{L}(\mathcal{W}\oplus\mathcal{U},\mathcal{Z}\oplus\mathcal{Y}))$ such that the state-space internal stabilization problem has a solution. Suppose also that we are given a controller $K\in\mathcal{RO}^0_{nc,d+1}(\mathcal{L}(\mathcal{Y},\mathcal{U}))$ with state-space realization

$$K(\boldsymbol{\delta}) = \mathbf{D}_K + \mathbf{C}_K(I-\mathbf{Z}_K(\boldsymbol{\delta})\mathbf{A}_K)^{-1}\mathbf{Z}_K(\boldsymbol{\delta})\mathbf{B}_K.$$

which is both nc-Hautus-stabilizable and nc-Hautus-detectable. Then the controller $K \sim \{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ *solves the state-space internal stabilization problem associated with* $\{\mathbf{A}, \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}, \left[\begin{smallmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & 0 \end{smallmatrix}\right]\}$ *if and only if* $K(\boldsymbol{\delta})$ *solves the noncommutative frequency-domain internal stabilization problem associated with*

$$G(\boldsymbol{\delta}) = \left[\begin{smallmatrix} G_{11}(\boldsymbol{\delta}) & G_{12}(\boldsymbol{\delta}) \\ G_{21}(\boldsymbol{\delta}) & G_{22}(\boldsymbol{\delta}) \end{smallmatrix}\right].$$

Proof. By Theorem 6.3, the assumption that that the state-space internal stabilization problem is solvable means that $\{\mathbf{C}_2, \mathbf{A}\}$ is nc-Hautus-detectable and $\{\mathbf{A}, \mathbf{B}_2\}$ is nc-Hautus-stabilizable. We shall use this form of the standing assumption. Moreover, in this case, a given controller $K \sim \{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ solves the state-space internal stabilization problem if and only if K stabilizes G_{22}.

Suppose now that $K \sim \{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ solves the state-space internal stabilization problem, i.e., the state operator $\mathbf{A}_{cl}$ in (6.14) is nc-Hautus-stable. Note that the 3×3 noncommutative transfer matrix $\boldsymbol{\Theta}(G, K)$ has realization $\boldsymbol{\Theta}(G, K) = \mathbf{D}_\Theta + \mathbf{C}_\Theta (I - \mathbf{Z}_\Theta(\boldsymbol{\delta}) \mathbf{A}_\Theta)^{-1} \mathbf{Z}_\Theta(\boldsymbol{\delta}) \mathbf{B}_\Theta$ with $\mathbf{Z}_\Theta(\boldsymbol{\delta}) = \mathbf{Z}_{cl}(\boldsymbol{\delta})$ as in (6.15) where

$$\begin{bmatrix} \mathbf{A}_\Theta & \mathbf{B}_\Theta \\ \mathbf{C}_\Theta & \mathbf{D}_\Theta \end{bmatrix} = \begin{bmatrix} A_\Theta & B_\Theta \\ C_\Theta & D_\Theta \end{bmatrix} \otimes I_{\ell^2}$$

with

$$A_\Theta = \begin{bmatrix} A + B_2 D_K C_2 & B_w C_K \\ B_K C_2 & A_K \end{bmatrix}, \quad B_\Theta = \begin{bmatrix} B_1 + B_2 D_K D_2 & B_2 & B_2 D_K \\ B_K D_{21} & 0 & B_K \end{bmatrix},$$
$$C_\Theta = \begin{bmatrix} C_1 + D_{12} D_K C_2 & D_{12} C_K \\ D_K C_2 & C_K \\ C_2 & 0 \end{bmatrix}, \quad D_\Theta = \begin{bmatrix} D_1 + D_{12} D_K D_{21} & D_{12} & D_{12} D_K \\ D_K D_{21} & I & K_K \\ D_{21} & 0 & I \end{bmatrix}. \tag{6.20}$$

Now observe that $\mathbf{A}_\Theta$ is equal to $\mathbf{A}_{cl}$, so that all nine transfer matrices in $\boldsymbol{\Theta}(G, K)$ have a realization with state operator $\mathbf{A}_\Theta = \mathbf{A}_{cl}$ nc-Hautus-stable. Hence all matrix entries of $\boldsymbol{\Theta}(G, K)$ are in $H^{\infty, o}_{nc, d+1}$.

Suppose that $K(\boldsymbol{\delta})$ with realization $K \sim \{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ internally stabilizes G in the frequency-domain sense. This means that all nine transfer matrices in $\Theta(G, K)$ are stable. In particular, the 2×2 transfer matrix $\widetilde{W} := \boldsymbol{\Theta}(G_{22}, K) - \boldsymbol{\Theta}(G_{22}, K)(0)$ is stable. From (6.20) we read off that $\widetilde{W}$ has realization

$$\widetilde{W}(\boldsymbol{\delta}) = \begin{bmatrix} \mathbf{D}_K \mathbf{C}_2 & \mathbf{C}_K \\ \mathbf{C}_2 & 0 \end{bmatrix} (I - \mathbf{Z}_\Theta(\boldsymbol{\delta}) \mathbf{A}_\Theta)^{-1} \begin{bmatrix} \mathbf{B}_2 & \mathbf{B}_2 \mathbf{D}_K \\ 0 & \mathbf{B}_K \end{bmatrix}.$$

By Theorem 6.9, to show that $\mathbf{A}_{cl} = \mathbf{A}_\Theta$ is nc-Hautus-stable, it suffices to show that $\left\{ \left[\begin{smallmatrix} \mathbf{D}_K \mathbf{C}_2 & \mathbf{C}_K \\ \mathbf{C}_2 & 0 \end{smallmatrix}\right], \mathbf{A}_{cl} \right\}$ is nc-Hautus-detectable and that $\left\{ \mathbf{A}_{cl}, \left[\begin{smallmatrix} \mathbf{B}_2 & \mathbf{B}_2 \mathbf{D}_K \\ 0 & \mathbf{B}_K \end{smallmatrix}\right] \right\}$ is nc-Hautus-stabilizable. By using our assumption that $\{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ is both nc-Hautus-detectable and nc-Hautus-stabilizable, one can now follow the argument in the proof of Theorem 4.9 to deduce that $\left\{ \left[\begin{smallmatrix} \mathbf{D}_K \mathbf{C}_2 & \mathbf{C}_K \\ \mathbf{C}_2 & 0 \end{smallmatrix}\right], \mathbf{A}_{cl} \right\}$ is noncommutative detectable and that $\left\{ \mathbf{A}_{cl}, \left[\begin{smallmatrix} \mathbf{B}_2 & \mathbf{B}_2 \mathbf{D}_K \\ 0 & \mathbf{B}_K \end{smallmatrix}\right] \right\}$ is noncommutative Hautus-stabilizable as needed. $\square$

We do not know as of this writing whether any given controller K in the space $\mathcal{RO}^0_{nc,d+1}(\mathcal{L}(\mathcal{Y},\mathcal{U}))$ has a nc-Hautus-detectable/stabilizable realization (see the discussion in the Notes below). However, for the Model-Matching problem, internal stabilizability in the frequency-domain sense means that all transfer matrices T_1, T_2, T_3 are stable (i.e., have all matrix entries in $H^{\infty,o}_{nc,d+1}$) and hence the standard plant matrix $G = \left[\begin{smallmatrix} T_{11} & T_{12} \\ T_{22} & 0 \end{smallmatrix}\right]$ has a stable realization. A given controller K solves the internal stabilization problem exactly when it is stable; thus we may work with realizations $K \sim \{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ with $\mathbf{A}_K$ nc-Hautus-stable, and hence a fortiori with both $\{\mathbf{C}_K, \mathbf{A}_K\}$ nc-Hautus-detectable and $\{\mathbf{A}_K, \mathbf{B}_K\}$ nc-Hautus-stabilizable. In this scenario Theorem 6.10 tells us that a controller $K(\boldsymbol{\delta})$ solves the frequency-domain internal stabilization problem exactly when any stable realization $K \sim \{\mathbf{A}_K, \mathbf{B}_K, \mathbf{C}_K, \mathbf{D}_K\}$ solves the state-space internal stabilization problem. Moreover, the frequency-domain performance measure matches with the state-space performance measure, namely: that the closed-loop transfer matrix $T_{zw} = G_{11} + G_{12}(I - KG_{22})^{-1}KG_{21}$ be in the strict noncommutative Schur-Agler class $\mathcal{SA}^o_{nc,d+1}(\mathcal{W},\mathcal{Z})$. In this way we arrive at a solution of the noncommutative Sarason interpolation problem posed in Section 6.2.

Theorem 6.11. *Suppose that we are given a transfer matrix of the form* $G = \left[\begin{smallmatrix} T_1 & T_2 \\ T_3 & 0 \end{smallmatrix}\right] \in H^{\infty,o}_{nc,d+1}(\mathcal{L}(\mathcal{W}\oplus\mathcal{U}, \mathcal{Z}\oplus\mathcal{Y}))$ *with a realization*

$$\begin{bmatrix} T_1(\boldsymbol{\delta}) & T_2(\boldsymbol{\delta}) \\ T_3(\boldsymbol{\delta}) & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} (I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta}) \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}$$

(so $\mathbf{C}_2(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B} = 0$ *for all* $\boldsymbol{\delta}$*) where*

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & 0 \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \otimes I_{\ell^2}$$

as usual. Then there exists a $K \in H^{\infty,o}_{nc,d+1}$ *so that* $T_1 + T_2KT_3$ *is in the strict noncommutative Schur-Agler class* $\mathcal{SA}^o_{nc,d+1}$ *if and only if there exist* $X, Y \in \mathcal{D}$*, with* $\mathcal{D}$ *as in* (4.11)*, satisfying LMIs:*

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AYA^* - Y & AYC_1^* & B_1 \\ C_1YA^* & C_1YC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0, \quad Y > 0,$$

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^*XA - X & A^*XB_1 & C_1^* \\ B_1^*XA & B_1^*XB_1 - I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0, \quad X > 0,$$

and the coupling condition

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0.$$

Here N_c and N_o are matrices chosen so that

$$N_c \text{ is injective and } \operatorname{Im} N_c = \operatorname{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \text{ and}$$
$$N_o \text{ is injective and } \operatorname{Im} N_o = \operatorname{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}.$$

6.4. Notes

1. The equality of $\mu_{\boldsymbol{\Delta}}(\mathbf{A})$ with $\widehat{\mu}_{\Delta}(A)$ where $\boldsymbol{\Delta}$ is as in (6.8) appears in Paganini's thesis [108]; as mentioned in the Introduction, results of the same flavor have been given in [37, 42, 60, 99, 129]. Ball-Groenewald-Malakorn [29] show how this result is closely related to the realization theory for the noncommutative Schur-Agler class obtained in [28]. There it is shown that $\mu_{\boldsymbol{\Delta}}(\mathbf{A}) \leq \overline{\mu}_{\boldsymbol{\Delta}}(\mathbf{A}) = \widehat{\mu}_{\Delta}(A)$, where $\overline{\mu}_{\boldsymbol{\Delta}}(\mathbf{A})$ is a uniform version of $\mu_{\boldsymbol{\Delta}}(\mathbf{A})$. The fact that $\mu_{\boldsymbol{\Delta}}(\mathbf{A}) = \overline{\mu}_{\boldsymbol{\Delta}}(\mathbf{A})$ is the content of Theorem B.3 in [108]. Paganini's analysis is carried out in the more general form required to obtain the result of Proposition 6.1.

The thesis of Paganini also includes some alternate versions of Proposition 6.1. Specifically, rather than letting each $\boldsymbol{\delta}_j$ be an arbitrary operator on ℓ^2, one may restrict to such operators which are *causal* (i.e., lower-triangular) and/or *slowly time-varying* in a precise quantitative sense. With any combination of these refined uncertainty structures in force, all the results developed in Section 6 continue to hold. With one or more of these modifications in force, it is more plausible to argue that the assumption made in Section 6.1 that the controller K has on-line access to the uncertainties $\boldsymbol{\delta}_i$ is physically realistic.

The replacement of the condition $\mu(\Delta) < 1$ by $\widehat{\mu}(\Delta) < 1$ can be considered as a *relaxation* of the problem: while one really wants $\mu(\Delta) < 1$, one is content to analyze $\widehat{\mu}(\Delta) < 1$ since $\widehat{\mu}(\Delta)$ is easier to compute. Necessary and sufficient conditions for $\widehat{\mu}(\Delta) < 1$ then provide sufficient conditions for $\mu(\Delta) < 1$ (due to the general inequality $\mu(\Delta) \leq \widehat{\mu}(\Delta)$). In the setting of the enhanced uncertainty structure discussed in this section, by the discussion immediately preceding Proposition 6.1 we see in this case that the relaxation is *exact* in the sense that $\widehat{\mu}(\Delta) < 1$ is necessary as well as sufficient for $\mu(\Delta) < 1$. In Remark 1.2 of the paper of Megretsky-Treil [99], it is shown how the μ-singular-value approach can be put in the following general framework involving quadratic constraints (called the *S-procedure* for obscure reasons). One is given quadratic functionals $\sigma_0, \sigma_1, \dots, \sigma_\ell$ defined on some set L and one wants to know when it is the case that

$$\sigma_j(x) \geq 0 \text{ for } j = 1, \dots, \ell \Longrightarrow \sigma_0(x) \leq 0 \text{ for } x \in L. \tag{6.21}$$

A computable sufficient condition (the relaxation) is the existence of nonnegative real numbers $\tau_1, \dots, \tau_\ell$ ($\tau_j \geq 0$ for $j = 1, \dots, \ell$) so that

$$\sigma_0(x) + \sum_{j=1}^{\ell} \tau_j \sigma_j(x) \leq 0 \text{ for all } x \in L. \tag{6.22}$$

The main result of [99] is that there is a particular case of this setting (where L is a linear shift-invariant subspace of vector-valued $L^2(0,\infty)$ (or more generally $L^2_{\text{loc}}(0,\infty)$) and the quadratic constraints are shift-invariant) where the relaxation is again exact (i.e., where (6.21) and (6.22) are equivalent); this result is closely related to Proposition 6.1 and the work of [108]. A nice survey of the S-procedure and its applications to a variety of other problems is the paper of Pólik-Terlaky [112].

2. It is of interest to note that the type of noncommutative system theory developed in this section (in particular, nc-detectability/stabilizability and nc-coprime representation as in (6.17)) has been used in the work of Beck [36] and Li-Paganini [89] in connection with model reduction for linear systems with LFT-modelled structured uncertainty.

3. We note that Theorem 6.8 gives a Youla-Kučera-type parametrization for the set of stabilizing controllers for a given plant $G \in \mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{W} \oplus \mathcal{U}, \mathcal{Z} \oplus \mathcal{Y}))$ under the assumption that G_{22} has a double coprime factorization. In connection with this result, we formulate a noncommutative analogue of the conjecture of Lin: *If $G \in \mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{W} \oplus \mathcal{U}, \mathcal{Z} \oplus \mathcal{Y}))$ is stabilizable, does it follow that G_{22} has a double-coprime factorization?* If G_{22} has a realization

$$G_{22}(\boldsymbol{\delta}) = \mathbf{C}_2(I - Z(\boldsymbol{\delta})\mathbf{A})^{-1}Z(\boldsymbol{\delta})\mathbf{B}_2$$

with $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right] \otimes I_{\ell^2}$ nc-Hautus stabilizable and nc-Hautus detectable, then one can adapt the state-space formulas for the classical case (see [104, 85]) to arrive at state-space realization formulas for a double-coprime factorization of G_{22}. If it is the case that one can always find a nc-Hautus stabilizable/detectable realization for G_{22}, it follows that G_{22} in fact always has a double-coprime factorization and hence the noncommutative Lin conjecture is answered in the affirmative. However, we do not know at this time whether nc-Hautus stabilizable/detectable realizations always exist for a given $G_{22} \in \mathcal{RO}^0_{nc,d}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$. From the results of [27], it is known that *minimal* i.e., *controllable* and *observable* realizations exist for a given G_{22}. However, here controllable is in the sense that a certain finite collection of control operators be surjective and observable is in the sense that a certain finite collection of observation operators be injective. It is not known if this type of controllability is equivalent to *nc-Hautus controllability*, i.e., to the operator pencil $\begin{bmatrix} I - Z(\boldsymbol{\delta})\mathbf{A} & \mathbf{B} \end{bmatrix}$ being surjective for all $\boldsymbol{\delta} \in \mathcal{L}(\ell^2)^{d+1}$ (not just $\boldsymbol{\delta}$ in the noncommutative polydisk $\overline{\mathbb{D}}_{nc,d}$). Thus it is unknown if controllable implies nc-Hautus stabilizable in this context. Dually, we do not know if observable implies nc-Hautus detectable.

4. Theorem 6.9 can be viewed as saying that, under a stabilizability/detectability hypothesis, any stable singularity of the noncommutative function W must show up internally as a singularity in the resolvent $(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}$ of the state matrix $\mathbf{A}$. A variant on this theme is the well known fact for the classical case that, under a controllability/observability assumption, any singularity (stable or not) of the rational matrix function $W(\lambda) = D + \lambda C(I - \lambda A)^{-1}B$ necessarily must show up internally as a singularity in the resolvent $(I - \lambda A)^{-1}$ of the state matrix A. A version of this result for the noncommutative case has now appeared in the paper of Kaliuzhnyi-Verbovetskyi–Vinnikov [82]; however the notion of controllable and observable there is not quite the same as the notion of controllable and observable for non-commutative Givone-Roesser systems as given in [27].

5. Given a function $S(z) = \sum_{n \in \mathbb{Z}^d_+} S_n z^n$ (where $z = (z_1, \dots, z_d)$ is the variable in the commutative polydisk $\overline{\mathbb{D}}^d$ and we use the standard multivariable notation $z^n = z_1^{n_1} \cdots z_d^{n_d}$ if $n = (n_1, \dots, n_d) \in \mathbb{Z}^d_+$), we know from the results of [2, 3, 35] that S has a contractive realization $S(z) = D + C(I - Z(z)A)Z(z)B$. In light of the work of [28], we see that any such contractive system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : (\oplus_{k=1}^d \mathcal{X}_k \oplus \mathcal{U}) \to (\oplus_{k=1}^d \mathcal{X}_k \oplus \mathcal{Y})$ can also be used to define an element $\mathbf{S}$ of the noncommutative Schur-Agler class $\mathcal{SA}_{nc,d}(\mathcal{U}, \mathcal{Y})$:

$$\mathbf{S}(\boldsymbol{\delta}) = \mathbf{D} + \mathbf{C}(I - \mathbf{Z}(\boldsymbol{\delta})\mathbf{A})^{-1}\mathbf{Z}(\boldsymbol{\delta})\mathbf{B}$$

where $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \otimes I_{\ell^2}$. Thus a choice of contractive realization $\{A, B, C, D\}$ for the commutative Schur-Agler-class function S can be viewed as a choice of noncommutative lifting to a noncommutative Schur-Agler-class function $\mathbf{S}(\boldsymbol{\delta})$; the lifting property is that

$$\mathbf{S}(z\mathbf{I}) = S(z) \otimes I_{\ell^2} \text{ where } z\mathbf{I} = (z_1 I_{\ell^2}, \dots, z_d I_{\ell^2}) \in \overline{\mathbb{D}}_{nc,d} \text{ if } z = (z_1, \dots, z_d) \in \overline{\mathbb{D}}^d.$$

While the realization for the commutative function is highly non-unique, the realization for the noncommutative function is unique up to state-space similarity if arranged to be minimal (i.e., controllable and observable as in [27]). Philosophically one can say that evaluation of the function on the commutative polydisk $\mathbb{D}^d$ does not give enough frequencies to detect the realization; enlarging the frequency domain (or points of evaluation) to the noncommutative polydisk $\mathbb{D}^d_{nc,d}$ does give enough frequencies to detect the realization in an essentially unique way.

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