Lyapunov theory for 2D Roesser models: application to the asymptotic and exponential stability

Nima Yeganefar, Nader Yeganefar, Mariem Ghamgui and Emmanuel Moulay

Abstract

This paper deals with a general class of discrete 2D systems based on the Roesser model. We first discuss the existing definitions of (Lyapunov) stability in the area of multidimensional systems and provide strong incentives to adopt new definitions in order to be coherent with the ones usually adopted in the 1D case. Once this background has been carefully designed, we develop different Lyapunov theorems in order to check asymptotic and exponential stability. Note that the theory can be applied both for linear and non linear systems. Finally we propose the first converse Lyapunov theorem in the case of exponential stability for 2D systems.

keywords: nD systems, Lyapunov stability, discrete systems.

I. Introduction

A. Context

Consider the following system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{1}$$

where $0 < q < 1$ and x^h, x^v are real scalars depending on 2 independent variables i and j representing 2 different dimensions. Here, the initial conditions are given by two sequences,

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namely $x^h(0,.)$ and $x^v(.,0)$. Ultimately this paper wants to ask and develop criteria to evaluate the stability of such systems and by "such" we mean multidimensional systems (for a general approach see [1], [2], [3], [4]). The system introduced here is indeed part of a much wider area that has been studied long since the 70ies ([5]) and is usually referred to Roesser model[1] [7]. So one may ask why to question stability now? This is the reason we start with this example. The solutions can be easily computed $x^h(i,j) = q^i x^h(0,j)$ and $x^v(i,j) = q^j x^v(i,0)$ and with $0 < q < 1$ we would like to say that the origin is asymptotically stable. The existing definition of asymptotic stability given by Fornasini ([8]) is $\lim_{i+j\to\infty} x^h(i,j) = \lim_{i+j\to\infty} x^v(i,j) = 0$. The problem we face here is that given a particular initial sequence e.g. $x^h(0,j) = 1\ \forall j$, then for a fixed $i = 0$, $\lim_{i+j\to\infty} x^h(i,j) \neq 0$. This is one of the reasons we propose a new definition for stability of 2D systems. The other reason is that in the 1D case, asymptotic stability encapsulates 2 properties. The first is that the system needs to be stable in the sense of Lyapunov, i.e. small initial conditions imply that the trajectories remain close to the equilibrium. The second one asks for the trajectories to go to 0 when $t \to \infty$. The first part of this definition has been forgotten in the study of multidimensional systems (to the exception of the work in [9]) mainly because the field is dealing almost exclusively with linear systems. Indeed, for 2D linear systems, the second property should imply the first one, although there does not seem to be an existing proof of this fact. This is however not the case for nonlinear systems. As Lyapunov framework is more and more developed in the multidimensional case, we believe that it is of high importance to properly define and evaluate the stability concept. Thus, if clearly and carefully designed, the Lyapunov approach could be also applied to nonlinear cases which are one of its most important interests.

Let us now give a brief overview on the evolution in the study of multidimensional systems. As previously said the first studies started in the mid 70's, when an engineering demand on digital filters, especially in the image processing field, led some authors to question how to generalize 1D filters. At that time, the focus was therefore mainly on linear systems and transfer functions. Instead of working with a transfer function which depends on one independent variable, they looked at polynomial functions of 2 and more generally n variables which was convenient when

[1]A second model is widely used for 2D systems and usually called Fornansini model [6] but it will not be considered in this paper.

dealing with image processing if you consider the dimensions as a discretization of the horizontal and vertical length of a picture (hence the x^h and x^v referred to the horizontal and vertical state). So the early focus was on digital filtering with multi variable transfer functions.

The first models were introduced a bit later with the work of Roesser [7] and Fornasini-Marchesini [6], [8]. With the introduction of state space models and the development of several tools for the nD case such as the use of LMIs (linear matricial inequalities) and Lyapunov techniques to derive stability conditions, a growing interest was raised concerning 2D systems ([9], [10], [11], [3], [12], [13], [14], [15], [16], [17], [18], [19] etc.). Numerous applications have been studied particularly in the image and signal processing, coding/decoding, filtering (see [1]), the study of PDEs via a discretization ([20]) or with a continuous approach (see the recent work [21] where a control of a sorption process is proposed), the analysis of time-delay systems with an algebraic approach ([22, chapter 4]), etc. More recently repetitive systems (such as long-wall cutting or metal rolling operations) and iterative learning control theory have shown to have a natural nD structure ([2]).

This paper addresses the notion of stability for 2D systems. We will see that if we want to keep the same logic as the one we have for the 1D case, several issues need to be addressed. In the first section, definitions of asymptotic and exponential stability will be given and with our simple example we will try to justify why such definitions need to be taken. More specifically, we will highlight one important feature of 2D systems that we believe has been missing in the literature so far, namely the importance of the initial conditions contrary to the 1D case. In the second part of this paper, we will develop the theorems à la Lyapunov that are today used in the field but for which most of the time the stability condition is not proven (because not required in the definition!). Moreover, the first converse Lyapunov theorem is been exposed, only in the exponential case though. Keeping in mind that Lyapunov theory is a powerful tool that can also be applied very efficiently for nonlinear systems, this is also the first work on multidimensional systems that gives the necessary background to study nonlinear multidimensional systems via the Lyapunov approach [23].

But we first need to detail the notations that are going to be used in the rest of the paper.

B. Notations

A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be of class $\mathcal{K}$, if $\alpha(0) = 0$ and α is strictly increasing. α is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and if $\lim_{t\to\infty} \alpha(t) = \infty$.

For any $q \in]0, 1[$, let us denote by $E_q(\mathbb{N}, \mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued sequences which decay exponentially at rate q, that is, a sequence $u = (u(k))_{k\in\mathbb{N}}$ belongs to $E_q(\mathbb{N}, \mathbb{R}^n)$ if for some constant C, we have

$$\|u(k)\| \le Cq^k, \quad \forall k.$$

We endow $E_q(\mathbb{N}, \mathbb{R}^n)$ with two different norms. First, the sup norm $\|u\|_\infty = \sup_k \|u(k)\|$ and second $\|u\|_q = \sup_k(q^{-k} \|u(k)\|)$. Note that $\|u\|_\infty \le \|u\|_q$ but the two norms are not equivalent.

And let us call $E_q(\mathbb{N}\times\mathbb{N}, \mathbb{R}^n\times\mathbb{R}^m)$ the space of $\mathbb{R}^n\times\mathbb{R}^m$-valued sequences $x = (x(i, j))_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ for which there is a constant C such that

$$\|x(i, j)\| \le Cq^{i+j}.$$

II. Definitions

As highlighted previously, the Lyapunov approach has been extended to the 2D case in the past. A lot of work can be found in the literature using the Lyapunov approach but almost none of them – to the exception of [9] we quoted earlier – defines the concept of (Lyapunov) stability in use.

So let us first introduce the studied system, a generalization of the model introduced by Roesser:

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = f(x^h(i, j), x^v(i, j)) \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ is a given function with $f(0) = 0$, $x^h \in \mathbb{R}^n$, $x^v \in \mathbb{R}^m$ and with initial conditions the sequences $x^h(0, .)$, $x^v(., 0)$.

Definition 2.1 (stability): The point $x = 0$ is said to be stable (in the sense of Lyapunov) if for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $\|x(0, j)\| < \delta$, $\|x(i, 0)\| < \delta$ then $\|x(i, j)\| < \epsilon$ for all $i, j > 0$.

Definition 2.2 (asymptotic stability): The point $x = 0$ is said to be asymptotically stable (in the sense of Lyapunov) if:

1) $x = 0$ is stable,

2) $\lim_{i+j\to\infty} x(i,j)=0$ whenever $\lim_{j\to\infty} x^h(0,j)=0$ and $\lim_{i\to\infty} x^v(i,0)=0$.

If the initial conditions are with finite support (ie. at most a finite number of the elements of the sequences don't vanish), we will say that $x=0$ is *almost asymptotically stable*.

Remark 1: In order to verify the asymptotic stability, one needs to first check the stability condition (where the initial conditions don't need to go to 0 at infinity) and then check that $\lim_{i+j\to\infty} x(i,j)=0$ whenever $\lim_{j\to\infty} x^h(0,j)=0$ and $\lim_{i\to\infty} x^v(i,0)=0$. This is a fundamental difference with the 1D case where asymptotic stability implies that *all* the trajectories go to 0 at infinity whatever the initial conditions are. In the 2D case, one can not hope to have every trajectory approach 0 simply because it may not be the case for the initial conditions (see the example given at the end of this section).

Remark 2: Usually in the literature, the notion of stability is lost to retain only the condition $\lim_{i+j\to\infty} x(i,j)=0$ in definition 2.2 without referring to the importance of the initial conditions (see [8]).

Definition 2.3 (exponential stability): The equilibrium point $x=0$ is said to be exponentially stable if there exist $q\in]0,1[$ and a constant M such that for any initial sequences $x^h(0,.)$ and $x^v(.,0)$

$$\|x(i,j)\| \leq M\left(\left\|x^h(0,j)\right\| q^i + \|x^v(i,0)\| q^j\right)$$

Remark 3: As in the 1D case, one can see that exponential stability implies asymptotic stability. In [24], Pandolfi gives a definition of exponential stability for a linear system. Namely, the equilibrium point $x=0$ is exponentially stable if there exists a constant M such that for all trajectories $x(i,j)$:

$$\|x(i,j)\| \leq M\left(q^i \max_{0\leq j'\leq j}\left\|x^h(0,j')\right\| + q^j \max_{0\leq i'\leq i}\|x^v(i',0)\|\right)$$

With this definition, however, it is not guaranteed that $\lim_{i+j\to\infty} x(i,j)=0$ but only $\lim_{(i \text{ and } j)\to\infty} x(i,j)=0$. In particular, this definition does not imply asymptotic stability in the sense of [8] nor in our given definition.

Definition 2.4 (weak exponential stability): The equilibrium point $x=0$ is said to be weakly exponentially stable if:

1) $x=0$ is stable,

2) there exists $q \in]0,1[$ such that for all initial conditions $x^h(0,.)$ and $x^v(.,0)$ in $E_q(\mathbb{N},\mathbb{R}^n)$ and $E_q(\mathbb{N},\mathbb{R}^m)$ respectively, there is a constant M satisfying

$$\|x(i,j)\| \leq Mq^{i+j}$$

If the above criteria are verified only for initial conditions $x^h(0,.)$ and $x^v(.,0)$ with finite support, then we will call the equilibrium an *almost weakly exponentially stable point*.

Remark 4: It is easy to verify that if $x=0$ is exponentially stable then it is weakly exponentially stable. We believe that the converse is also true but have not been able to prove it yet.

Remark 5: One can not hope to have the following definition for exponential stability of 2D systems:

$$\|x(i,j)\| \leq M \max(\left\|x^h(0,j)\right\|, \|x^v(i,0)\|)q^{i+j}$$

Indeed, if it was the case, taking $x^v(.,0)=0$ and considering the inequality for $i=0$ would lead to

$$\left\|x^h(0,j)\right\| \leq \|x(0,j)\| \leq M\left\|x^h(0,j)\right\| q^j.$$

This would force $x^h(0,j)=0$ for j large enough which is not necessarily the case.

Now that everything is carefully defined let us go back to our primary example which is the linear case of system (2), with f a diagonal matrix:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{3}$$

Remember the solutions of the system: $x^h(i,j) = q^i x^h(0,j)$ and $x^v(i,j) = q^j x^v(i,0)$. As already pointed out, this system is not asymptotically stable if one takes the definition used in the literature ($\lim_{i+j\to\infty} x(i,j) = 0$) as if we choose $x^h(0,j) = 1$, then for a fixed $i = 0$, $\lim_{i+j\to\infty} x(i,j) \neq 0$ and furthermore the criterion which is known to be a sufficient and necessary condition for asymptotic stability of nD systems (roughly speaking the eigenvalues of the studied matrix needs to be inside the unit circle, see [25] or more recently [26] for a generalization) is here verified. But with our given definitions 2.2 and 2.3, the equilibrium point of system (3) is asymptotically/exponentially stable whether you have appropriate initial conditions or not. It also should be possible to prove that the quoted criterion on eigenvalues is indeed a necessary and sufficient condition for asymptotic stability if one choses our definition but this is not proven in this paper.

III. Main results

A. Direct theorems

The first two theorems introduced in this section have been applied in the literature but a complete proof is missing if one chooses the above definitions. Although the first theorem has been proved by Liu in [9], here we provide a different proof (but with shared philosophy), easier to follow and in coherence with our definitions.

Theorem 3.1: Let V_h, V_v be continuous functions from $\mathbb{R}^n \to \mathbb{R}$, $\mathbb{R}^m \to \mathbb{R}$ respectively such that for all $(x^h, x^v) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$\begin{aligned} \alpha_h(\|x^h\|) \leq V_h(x^h) \leq \beta_h(\|x^h\|) \\ \alpha_v(\|x^v\|) \leq V_v(x^v) \leq \beta_v(\|x^v\|) \end{aligned} \tag{4}$$

where α_h, α_v, β_h, β_v are functions of class $\mathcal{K}$. Define

$$V(x^h, x^v) = V_h(x^h) + V_v(x^v)$$

and ΔV as the increment of V along the trajectories of (2) by:

$$\begin{aligned} \Delta V \triangleq\ & V_h(x^h(i+1,j)) - V_h(x^h(i,j)) \\ & + V_v(x^v(i,j+1)) - V_v((x^v(i,j)) \end{aligned}$$

If $\Delta V \leq -\gamma(\|x\|)$, where γ is a function of class $\mathcal{K}$ then the equilibrium $x = 0$ is almost asymptotically stable. Similarly to the 1D case we will say that V is a Lyapunov function of system (2).

Remark 6: Conditions (4) also implies that there exist class $\mathcal{K}$ functions α, β, such that

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|) \tag{5}$$

Proof: Suppose that $\Delta V < -\gamma(\|x\|)$ and (4) holds, by hypothesis, we can write:

$$\begin{aligned} & V_h(x^h(i+1,j)) + V_v(x^v(i,j+1)) \leq \\ & V_h(x^h(i,j)) + V_v(x^v(i,j)) - \gamma(\|x\|) \end{aligned} \tag{6}$$

Let $E(r) = \sum_{(i+j=r)} V(x^h, x^v)$ denotes the 'energy' stored on a diagonal $i+j$. It follows from (6) and the positivity of γ that

$$E(r) \geq E(r+1) - V_h(x^h(0, r+1)) - V_v(x^v(r+1, 0))$$

Now consider the case when $r = i + j > L$. Taking initial conditions on a finite support implies that $E(r) - E(r+1) \geq 0$ which means that the energy on a diagonal $E(r)$ is decreasing for all $r > L$.

If $r < L$, $E(r+1) - E(r) \leq V_h(x^h(0, r+1)) + V_v(x^v(r+1, 0))$. By definition and using the positive definiteness of V, $E(0) = 0$. Adding the last inequalities for $r < L$ gives us:

$$E(r) \leq \sum_{i=1}^{r} V_h(x^h(0, i)) + V_v(x^v(i, 0))$$

V^h and V^v are continuous functions with $V_h(0) = V_v(0) = 0$, so by continuity, $V^h(x^h(0, i))$ and $V^v(x^v(0, i))$ can be as small as desired considering small initial conditions $x^h(0, i)$ and $x^v(i, 0)$. It means that for all $\epsilon > 0$ there exists a δ such that if $\left\|x^h(0, j)\right\| < \delta$ and $\|x^v(i, 0)\| < \delta$,

$$\max_{r<L}(E(r)) \leq \sum_{i=1}^{r} V_h(x^h(0, i)) + V_v(x^v(i, 0)) \leq \epsilon.$$

For $r > L$ as $E(r)$ is decreasing, $\max_{r \geq 0} E(r) \leq \epsilon$ stands, therefore:

$$\alpha(\left\|(x^h, x^v)\right\|) \leq V(x^h, x^v) \leq \max_{r \geq 0} E(r) \leq \epsilon$$

We conclude from definition 2.2 that the system (2) is stable.

To conclude the proof we need to show that $\lim_{i+j \to \infty} \|x(i, j)\| = 0$. For $r > L$, $E(r)$ is a decreasing positive series that converges to a given limit, hence

$$\lim_{r \to \infty} E(r) - E(r+1) = 0.$$

But going back to inequality (6), observe that for $r > L$,

$$E(r) - E(r+1) \geq (r+1)\gamma(\|x(i, j)\|) \geq \gamma(\|x(i, j)\|) \geq 0.$$

This proves that $\lim_{i+j \to \infty} \gamma(\|x(i, j)\|) = 0$, hence as γ is of class $\mathcal{K}$, leads to $\lim_{i+j \to \infty} \|x(i, j)\| = 0$ which concludes the proof. ■

Theorem 3.2: Let V_h, V_v be continuous functions from $\mathbb{R}^n \to \mathbb{R}$, $\mathbb{R}^m \to \mathbb{R}$ respectively such that for all $(x^h, x^v) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$a\left\|x^h\right\|^p \leq V_h(x^h) \leq b\left\|x^h\right\|^p$$
$$a\left\|x^v\right\|^p \leq V_v(x^v) \leq b\left\|x^v\right\|^p \tag{7}$$

where a, b and p are positive constants. Define

$$V(x^h, x^v) = V_h(x^h) + V_v(x^v)$$

If there exists a constant $0 < \alpha < 1$ such that $\Delta V \leq -\alpha V$, where for any trajectory $(x(i,j))$ solution of system (2)

$$\begin{aligned}\Delta V =& V^h(x^h(i+1,j)) - V^h(x^h(i,j)) \\ &+ V^v(x^v(i,j+1)) - V^v(x^v(i,j))\end{aligned}$$

then, $x = 0$ is almost weakly exponentially stable.

Proof: For simplicity, we will only consider $p = 1$ during this proof but the general case is similar. Let $x(i,j)$ be a solution of system (2) with initial conditions $x^h(0,.)$ and $x^v(.,0)$ with finite support. Consider the quantity $E(d) = \sum_{i+j=d} V(x(i,j))$ which represents the energy stored on a diagonal of the system and let $q = 1 - \alpha$. Recall that

$$\begin{aligned}&V^h(x^h(i+1,j)) - V^h(x^h(i,j)) \\ &+ V^v(x^v(i,j+1)) - V^v(x^v(i,j)) \leq -\alpha V\end{aligned}$$

Then

$$\begin{aligned}&E(d+1) - E(d) = \sum_{i+j=d+1} V^h(x^h(i,j)) - \sum_{i+j=d} V^h(x^h(i,j)) \\ &+ \sum_{i+j=d+1} V^v(x^v(i,j)) - \sum_{i+j=d} V^v(x^v(i,j)) \\ &\leq -\alpha E(d) + V^h(x^h(0,d+1)) + V^v(x^v(d+1,0))\end{aligned} \tag{8}$$

By induction on d, it follows that

$$\begin{aligned}E(d) &\leq q^d E(0) + \sum_{j=1}^{d} q^{d-j} V^h(x^h(0,j)) + \sum_{i=1}^{d} q^{d-i} V^v(x^v(i,0)) \\ &= \sum_{j=0}^{d} q^{d-j} V^h(x^h(0,j)) + \sum_{i=0}^{d} q^{d-i} V^v(x^v(i,0))\end{aligned}$$

The use of inequality (7) leads to:

$$\begin{aligned}E(d) \leq b \sum_{j=0}^{d} q^{d-j} \left\|x^h(0,j)\right\| + \\ b \sum_{i=0}^{d} q^{d-i} \left\|x^v(i,0)\right\|\end{aligned}$$

As the initial conditions are with finite support, there is a d_0 such that for $d \geq d_0$:

$$E(d) \leq q^d \left(b \sum_{j=0}^{d_0} q^{-j} \sup_{j'} \left\| x^h(0, j') \right\| + \right.$$
$$\left. b \sum_{i=0}^{d_0} q^{-i} \sup_{i'} \left\| x^v(i', 0) \right\| \right)$$

The last inequality also holds for $d < d_0$. We may rewrite this as

$$E(d) \leq M \max \left(\sup_{j'} \left\| x^h(0, j') \right\|, \sup_{i'} \left\| x^v(i', 0) \right\| \right) q^d,$$

where $M = b \sum_{j=0}^{d_0} q^{-j}$ is a constant which depends on d_0. To conclude the first part of the proof, let use the first part of inequality (7) which gives:

$$\begin{aligned} \|x(i,j)\| &\leq a^{-1} V(x(i,j) \leq E(i+j) \\ &\leq M \max \left(\sup_{j'} \left\| x^h(0, j') \right\|, \sup_{i'} \left\| x^v(i', 0) \right\| \right) q^{i+j}. \end{aligned}$$

The stability is a direct consequence of theorem 3.1. ■

B. Converse theorem

The next theorem is the first converse Lyapunov theorem introduced for 2D systems. We assume that system (2) is exponentially stable. We can view $E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$ as the space of initial conditions of system (2) which decay exponentially at rate q: if $(u^h, u^v) \in E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$, then there is a unique trajectory x of system (2) such that $x^h(0, .) = u^h$ and $x^v(., 0) = u^v$. Conversely, if $x = (x^h, x^v)$ is an exponentially decaying trajectory of system (2), then for all (i, j), x defines an element (u^h, u^v) in $E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$, namely $u^h = x^h(i, . + j)$ and $u^v = x^v(. + i, j)$. Note that for fixed (i, j), the trajectory $\tilde{x}$ corresponding to these initial conditions is simply given by $\tilde{x}(., .) = x(. + i, . + j)$.

Now, let $V : E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m) \to \mathbb{R}$ be a function defined on the space of initials conditions. If x is an exponentially decaying trajectory of system (2), we define V along x to be the the (double) sequence $(i, j) \mapsto V(x^h(i, . + j, x^v(. + i, j))$. We would like also to define the variation ΔV of V along x. To this end, we assume that V can be written in the following form: $V = V^h + V^v$, where $V^h, V^v : E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m) \to \mathbb{R}$ are functions such that if $(u^h, u^v) \in E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$ and x is the corresponding trajectory, then $V^h(u^h, u^v)$

(respectively $V^v(u^h, u^v)$) depends only on x^h (respectively on x^v). For such a function V, its variation ΔV along a trajectory x is the sequence defined for all (i, j) by

$$\Delta V = V^h(x^h(i+1, .+j), x^v(.+i+1, j)) + \\ V^v(x^h(i, .+j+1), x^v(.+i, j+1)) - V(x^h(i, .+j), x^v(.+i, j)),$$

which is the same as

$$\Delta V = V^h(x^h(i+1, .+j), x^v(.+i+1, j)) - V^h(x^h(i, .+j), x^v(.+i, j)) \\ + V^v(x^h(i, .+j+1), x^v(.+i, j+1)) - V^v(x^h(i, .+j), x^v(.+i, j)).$$

With these preliminaries in mind, we can state our result

Theorem 3.3: Assume that system (2) is exponentially stable. Then there exists a function $V : E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m) \to \mathbb{R}$ having the following properties:

1) There are two positive constants C_1 and C_2 such that for all $(u^h, u^v) \in E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$, we have
$$C_1 \left(\|u^h\|_\infty + \|u^v\|_\infty\right) \leq V(u^h, u^v) \leq C_2 \left(\|u^h\|_q + \|u^v\|_q\right).$$
2) There are two functions $V^h, V^v : E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m) \to \mathbb{R}$ such that $V = V^h + V^v$ and for any $(u^h, u^v) \in E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$ with corresponding trajectory x, $V^h(u^h, u^v)$ (respectively $V^v(u^h, u^v)$) depends only on x^h (respectively on x^v).
3) There is a positive constant α such along any exponential decaying trajectory of system (2), we have $\Delta V \leq -\alpha V$.

Proof: For $(u^h, u^v) \in E_q(\mathbb{N}, \mathbb{R}^n) \times E_q(\mathbb{N}, \mathbb{R}^m)$, we let x be the corresponding trajectory which is also exponentially decaying at rate q. We define V^h by

$$V^h(u^h, u^v) = \sum_{i,j} \|x^h(i, j)\|.$$

As x decays exponentially, V^h is well defined. We can also define V^v in a similar way, and setting $V = V^h + V^v$, we see that V satisfies the second property of our theorem.

The first property of the theorem is also easily verified with the given norms. First $V = V^h + V^v$ verifies the first part of the inequality with $C_1 = 1$. To see why, observe that

$$\sum_{i,j} \left\|x^h(i, j)\right\| \geq \sum_i \left\|u^h(i)\right\| \geq \left\|u^h\right\|_\infty.$$

The second part of inequality comes from the following observation: for any sequence u, $\|u(k)\| \leq \|u\|_q q^k$. As x is the corresponding trajectory of system (2) also decaying exponentially at rate q, one can derive the following inequality:

$$\left\|x^h(i,j)\right\| + \|x^v(i,j)\| \leq 2M\left(q^i \left\|x^h(0,j)\right\| + q^j \|x^v(i,0)\|\right) = 2M\left(q^i \left\|u^h\right\| + q^j \|u^v\|\right)$$

which leads to:

$$V(x^h, x^v) \leq 2M \sum_{i,j} q^{i+j} (\left\|u^h\right\|_q + \|u^v\|_q)$$

V verifies the second part of inequality with $C_2 = 2M/(1-q)^2$.

To prove the last property, we have to compute, for fixed (i,j), terms such as $V^h(x^h(i,.+j, x^v(.+i,j))$ for a trajectory x of system (2). To achieve this, we just notice that the trajectory $\tilde{x}$ corresponding to the initial conditions $(x^h(i,.+j, x^v(.+i,j))$ is given by $\tilde{x}(.,.) = x(.+i,.+j)$. This implies that

$$V^h(x^h(i,.+j, x^v(.+i,j)) = \sum_{k,l} \|x^h(k+i, l+j)\|,$$

and similar expressions can be found to compute the other terms involved in ΔV and V along the trajectory x. Thus, we have

$$\begin{aligned} \Delta V &= \sum_{k,l} \left[\|x^h(k+i+1, l+j)\| - \|x^h(k+i, l+j)\|\right] \\ &\quad + \sum_{k,l} \left[\|x^v(k+i, l+j+1)\| - \|x^v(k+i, l+j)\|\right] \\ &= -\sum_l \|x^h(i, l+j)\| - \sum_k \|x^h(k+i, j)\|. \end{aligned}$$

It follows that we have to find some $\alpha > 0$ satisfying

$$\alpha \left(\sum_{k,l} \|x^h(k,l)\| + \sum_{k,l} \|x^v(k,l)\|\right) \leq \sum_l \|x^h(i, l+j)\| + \sum_k \|x^h(k+i, j)\|.$$

Now, the exponential stability condition of Definition 2.3 applied to $\tilde{x}$ yields

$$\|x^h(k+i, l+j)\| \leq M\left[\|x^h(i, l+j)\| q^k + \|x^v(k+i, j)\| q^l\right],$$

and a similar estimate holds for $x^v(k+i, l+j)$. Summing over k and l, we get

$$\sum_{k,l} \|x^h(k,l)\| + \sum_{k,l} \|x^v(k,l)\| \leq \frac{2M}{1-q}\left(\sum_l \|x^h(i, l+j)\| + \sum_k \|x^h(k+i, j)\|\right).$$

In other words, the third property in the theorem is true with $\alpha = (1-q)/(2M)$. ■

IV. Conclusion

Let us highlight the contributions of this paper. We believe this work is the first necessary step in order to apply Lyapunov theories to nonlinear multidimensional systems. Indeed definitions on stability, asymptotic stability and exponential stability have been provided and the coherences/differences with the ones in the 1D case have been discussed. We have also carefully explained in the introduction and first section what problems we face if we keep the usual definitions. In the second part of the paper we provided 2 different Lyapunov theorems which give sufficient conditions to check the quoted definitions. Finally the first converse theorem has been provided in the exponential case. This opens new possibilities for multidimensional systems and solid basis in order to extend some well-known techniques from the 1D to the 2D case. Furthermore as said earlier, this opens the doors for the study of systems with nonlinearities.

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