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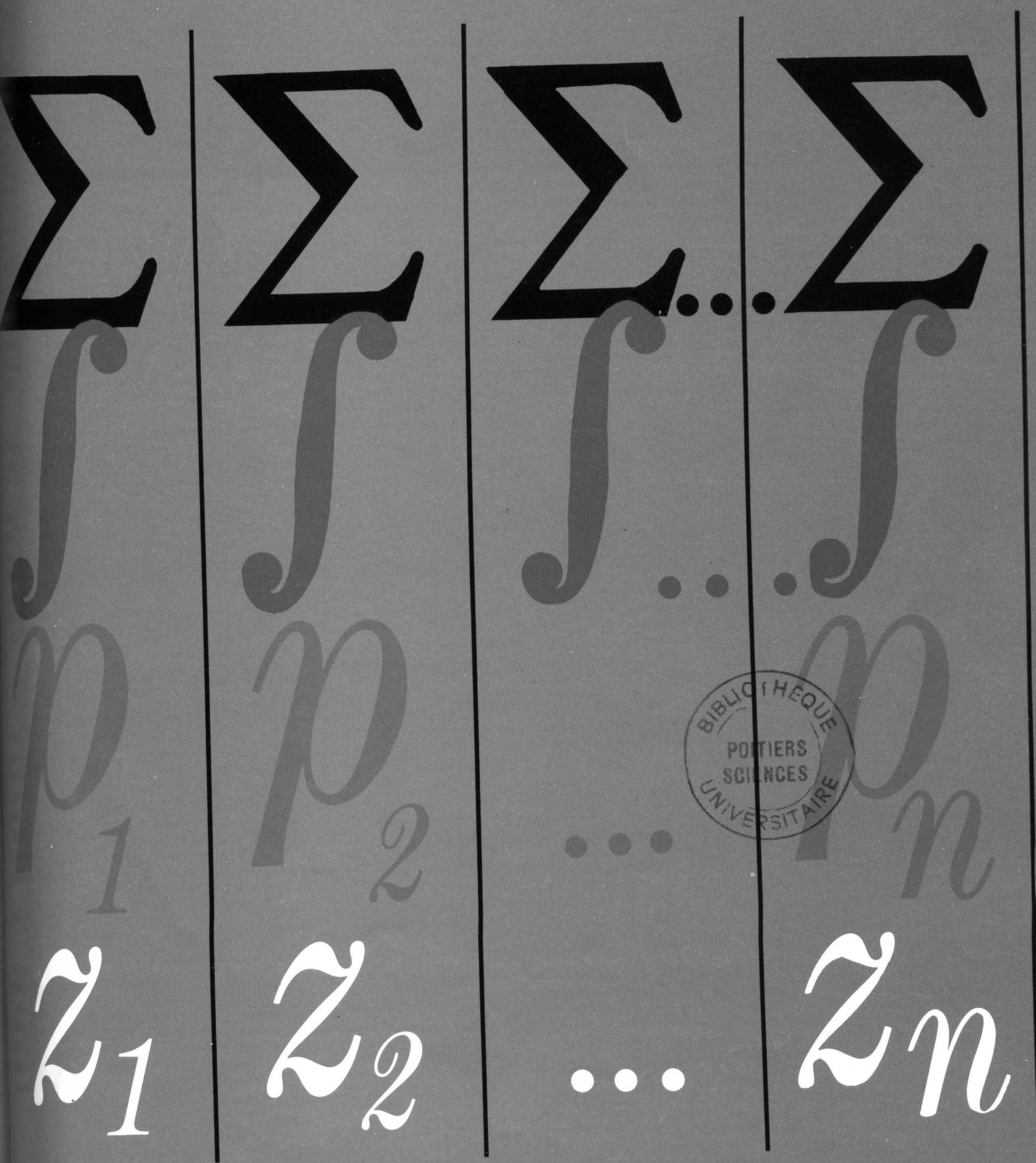
PECIAL ISSUE ON ultidimensional systems



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MULTIDIMENSIONAL SYSTEMS

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PROCEEDINGS OF THE IEEE

Scanning the Issue

SPECIAL ISSUE ON MULTIDIMENSIONAL SYSTEMS

The theories of functions and polynomials in several complex and/or real variables, along with their numerous applications in several areas of systems theory—including, but not restricted to, multidimensional digital filtering, multivariable network realizability, automatic control and communications—provide primarily the subject-matter for this special issue. Due to the wide range of topics spanned by the papers, both from the mathematical and engineering standpoint, it is hoped that the trend towards the understanding and solution of the challenging problems in this fascinating subject will continue, stimulated greatly by the avenues of discourse opened up among researchers in different but partially overlapping areas. All the papers have been reviewed, each by several reviewers, and it is hoped that in their final form they are error-free; however, since this is almost an impossibility in view of the fact that most of the material has yet to be thoroughly digested by the scientific community, the readers are urged to spare no efforts in polishing and perfecting the concepts and ideas advanced here.

The first paper, entitled "Problems and Progress in Multidimensional Systems Theory" by N. K. Bose, presents the possibilities as well as difficulties of extending established single dimensional techniques to multidimensional situations. This paper also serves as an overview of the present status and future prospects of research in the subject, with emphasis on multivariable network realizability, multidimensional recursive digital filter stability, and constructive implementation of algorithms known to exist as a consequence of elementary decision algebra. In this paper, attention is also directed to the fact that problems in different areas of multidimensional system theory often have similar mathematical characterizations requiring, then, a common mathematical solution. In connection with the computational aspects of algorithms implemented, continuation of research is necessary to determine sensitivity to parameter variation of certain properties under test; for example, in the test for global positivity of a multivariable polynomial it is useful and often necessary to know the range of values of one or more of the polynomial coefficients for which the global positivity property of the polynomial is invariant.

Recently, the potential value in systems theory of some results in algebraic geometry has been noticed. Several papers in the issue attach importance to the possible future role of algebro-geometric theorems in future multidimensional system theory. The paper entitled "Application of Algebraic Geometry to Systems Theory, Part II: Feedback and Pole Placement for Linear Hamiltonian Systems" by R. Hermann and C. Martin shows how some powerful results from algebraic geometry can be adapted to study the linear optimal regulator problem and also how the classical theory of resultants of systems of polynomials can be used to prove a version of a theorem in algebraic geometry required in mathematical systems theory. B. D. O. Anderson and R. W. Scott, in their paper entitled "Output Feedback Stabilization—Solution by Algebraic Geometry Methods," show how the difficult problem of stabilization for finite dimensional linear systems with output feedback can be reduced to a problem of calculating the solutions, finite in number, of a system of multivariable polynomial equations. The proof of finiteness is based on results from algebraic geometry, while multivariable polynomial resultants provide a method for arriving at the solution known to be finite in number. Thus, the problem finally reduces to the examination of a finite set, element by element, whose cardinality, however, may be high. The paper entitled "New Results on 2-*D* Systems Theory, Part 1: 2-*D* Polynomial Matrices, Factorization and Coprimeness" by M. Morf *et al.* extends to two dimensions the results on greatest common right or left divisor, extraction and matrix fraction descriptions. Also presented here are a criterion of relative primeness of two-dimensional polynomial matrices using concepts from algebraic geometry, and results related to existence and uniqueness questions of factorizations.

The properties of the two-variable orthogonal polynomials on the hypercircle are investigated by Y. Genin and Y. Kamp in their paper entitled "Two-Dimensional Stability and Orthogonal Polynomials on the Hypercircle," and the results developed are used to show why a well known stabilization technique for one-dimensional recursive digital filters cannot, in general, be extended to two dimensions. In the subsequent paper, entitled "A Levinson-Type Algorithm for Two-Dimensional Wiener Filtering Using Szego Polynomials," J. H. Justice shows how the two-dimensional extensions of the polynomials orthogonal on the unit circle, considered also by Genin–Kamp, can be used to derive and implement a two-dimensional analog of the Levinson algorithm occuring in the solution of normal equations in Wiener filtering. The connection between these two-dimensional orthogonal polynomials on the hypercircle and the inversion of block-Toeplitz matrices occuring in the extension of Wiener filtering to two dimensions is mentioned. Jury *et al.*, in their paper "Stabilization of Certain Two-Dimensional Recursive Digital Filters," consider, from a different viewpoint, the stabilization problem discussed in the Genin-Kamp paper by trying to identify classes of two-dimensional recursive digital filters which are stabilizable by the extension of the known one-dimensional stabilization technique.

In the paper entitled "A Stability Criterion for n-Dimensional Zero Phase Recursive Digital Filters" N. Ahmadi and R. A. King prove, using a criterion relating the stability of a recursive filter to the properties of its cepstrum, that an unstable n-dimensional recursive digital filter with a finite number of coefficients and nonzero, nonimaginary frequency response is decomposable into 2^n stable recursive filters in an infinite number of ways. In the paper entitled "Fundamentals of Digital Array Processing," D. E. Dudgeon tries to present the link between beam forming and beam spectra for sensor arrays and 2-*D* digital filtering. The relationship between beam spectra and the 2-*D* DFT is used to point out the differences in the beam forming and spectral-analysis approaches to array processing. The feasibility of using multidimensional filter design techniques to digital beamforming and array processing has been discussed. In their paper entitled "Design of Two-Dimensional Digital Filter via Spectral Transformation," S. Chakrabarti and S. K. Mitra try to present in a cohesive framework research to date on the use of spectral transformations in the design of 2-*D* digital filters. The stability-invariant property of spectral transformations often motivates its use, especially in situations when algebraic tests for stability become computationally difficult to implement.

L. O. Chua and S. M. Kang, in their paper "Section-Wise Piecewise-Linear Functions: Canonical Representation, Properties and Applications," present a closed-form analytical formula for representing n-dimensional surfaces and scalar functions of n-variables which are piecewise-linear over each cross section obtained by freezing any combination of $(n - 1)$ of the n-variables. The scope for using this closed-form representation in analyzing and modeling nonlinear devices characterized by finite jump discontinuities is illustrated by examples. In the paper entitled "Nonlinear Differential Systems: A Canonic Multivariable Theory," R. W. Newcomb shows how a nonlinear polynomic differential system can be reduced to a canonic quadratic form, through the introduction of an algebra within which a power-series solution can also be found. The region of convergence of the solution is worthy of more investigation. In the paper entitled "On Nonglobal Positivity and Domains of Positivity of Multivariable Polynomials," A. R. Modarressi and N. K. Bose consider the problem of determining all real solutions—including those lying on continuous algebraic curves, closed or open, as well as isolated points—of a polynomial equation in several real variables. The results of this paper are naturally adaptable for application in diverse problems, and it is hoped that greater attention will be given to related computational problems associated especially with polynomials in greater than two variables. In their paper "New Results in 2-*D* Systems Theory Part II: 2-*D* State-Space Models-Realization and the Notions of Controllability, Observability and Minimality," S. Y. Kung *et al.* present results on a comparison between different state-space models that have been proposed in the realization theory of two-dimensional deterministic systems, after introducing the notions of (global and local) state, controllability, and observability along with their relations to minimality of 2-*D* system realizations. In the 2-*D* case, they also introduce an algebraic definition of observability (modal observability), which is shown to be equivalent to the right coprimeness of 2-*D* polynomial matrices like in the 1-*D* case where contollability and observability concepts for multiple-input multiple-output systems are known to be linked to the relative primeness properties of polynomial matrices. Subsequently, minimality of a realization is claimed, if any, only if it is controllable and observable in the modal sense. The paper entitled "Statistical Inference on Stationary Random Fields" by W. E. Larimore adapts classical identification techniques for application to multidimensional systems. The statistical techniques to model vector random processes on multidimensional Euclidean space involve parametric statistical inference via the maximum-likelihood method. Approximation of multidimensional spectra and maximization of the likelihood function under stability constraints are among topics in this paper that demand future research.

There are also several items which have been included in the *Proceeding Letters* section of this special issue. These include the letter by C. S. Koo and C. T. Chen entitled "Fadeeva's Algorithm for Spatial Dynamical Equation," the letter "On Synthesis of Class of Multivariable Positive Real Functions" by V. Ramachandran *et al.*, the letter by R. DeCarlo *et al.* entitled "A Nyquist-Like Test for the Stability of Two-Dimensional Digital Filers," and finally the letter entitled "On the Spatially Causal Estimation of Two-Dimensional Processes," by M. G. Strintzis.

This issue hopefully brings to the attention of the readers the versatile and prolific nature of research activity in this area, motivated by scopes for applications in a large number of problems of scientific and engineering interest. The natural difficulty of the topic has often forced progress to be slow, and often errors discovered in the results of earlier research have led, not only to the appreciation of the intricacies of the subject, but to a better understanding of several aspects of the subject itself. The detection of errors gives credit to a scientist's aspirations towards perfection as much as commission of errors gives credit to a scientist's belief that the only certain, but ignoble, way to avoid errors is by not doing anything. By the same token, though it may be unnatural to expect the contents of this issue to be beyond reproach, the issue will have more than served its purpose if it fulfills its objective of exposing to the readers in a coherent manner the progress made in this area and identifying the present important unsolved problems which should guide future research.

In closing, I would like to thank B. D. O. Anderson, E. I. Jury, A. R. Modarressi, R. W. Newcomb, M. G. Strintzis, and D. C. Youla for several helpful discussions from which I have greatly benefited. The cooperation given by R. W. Lucky and W. R. Crone is gratefully acknowledged.

N. K. Bose
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Preface to the Special Issue on Multidimensional Systems

ELY I. JURY, FELLOW, IEEE

IN THIS BRIEF review, I would like to recall the historical development of some problems in system theory which have been of great interest to many of the readers of this journal and which have occupied most of my professional career. I will also mention some problems which need further exploration. The historical development of this theory proceeded in the following stages.

I.

Early research in this area, including my own professional work, dealt with problems related to single variable polynomials. These problems included root clustering, special root distribution, and general root distribution. Much progress, both theoretical and computational, has been made on these problems. The theory of inners [1], [2] has shown that most of the problems connected with roots of single variable polynomials can be presented either in terms of Bezoutians (quadratic forms) and Resultant matrices (innerwise matrices). These two approaches have been developed extensively in both mathematical and engineering literature. An important feature of the Bezoutian matrices is their symmetry; an important feature of the Resultant matrices is that they possess a left triangle of zeros. Both of these patterns have been utilized effectively in calculating the various determinants and subdeterminants of these matrices.

II.

The subsequent problems to be considered were those related to multi-input multi-output (MIMO) systems. Such problems lead to polynomial matrices of a single variable. Recent results [3], motivated by Part I, indicate that most of the problems related to MIMO systems can be formulated either in terms of Generalized Bezoutian or Generalized Resultant matrices. In this case the Bezoutian matrices need not be symmetric, but the left triangle of zeros are still present in the Generalized Resultant matrices. Also, in this case the matrices are not always square as in Part I, and so the rank of these matrices plays an important role.

III.

The third set of problems to be considered were those related to multidimensional polynomials, the contents of this issue. System theory problems related to multidimensional polynomials essentially emerged in the early sixties and have increased in importance up to the present time. Much of the impetus has been generated by the technical advances made in multidimensional digital filters. Many problems related to digital filters are presented in this issue. Most of these problems can also be formulated in terms of Bezoutian or Resultant matrices, but in this case the entries of these matrices are multidimensional polynomials. Effective solution of these problems depends heavily on decision algebra [4a, b] as developed in the Tarski-Seidenberg theory. The advent of decision algebra has made many system theory problems, previously thought unsolvable, computable.

The mathematical difficulties increase considerably as one moves from single to two or multidimensional polynomials. The main source of difficulty is that single variable polynomials can be readily factored but multidimensional ones cannot be. This difficulty and others related to singular cases are elaborated upon in this issue. The computational problems related to decision algebra and algebraic geometry methods are also discussed here. Having discussed Parts II and III, it is natural to merge them and this leads us to the following fourth stage.

IV.

The fourth set of problems to be considered were related to polynomial matrices of several variables. This area of research is still in its infancy and only scattered articles are available in the literature at the present time. It is expected that as in the earlier problems most of the problems in this category can be formulated either in terms of Generalized Bezoutians or Generalized Resultants. It is expected that in the latter case the left triangle of zeros will still dominate the pattern of these matrices, thus offering a unified feature for all four parts.

The survey, presented by the guest editor in this issue, should be of much value in describing this work. Much research is needed in this area in the coming years, and indeed the developments of the theory presented in Parts II and III should advance the research on the problems described in Part IV. Many technical applications will also arise in this field of investigation, especially in image processing problems [5].

I have been quite fortunate to work on the problems mentioned in the four parts above, particularly on the single variable case. This case, and in particular the unified form of the various matrices in terms of their pattern, as well as entries, has shed much light on the other cases. The unified feature of a left triangle of zeros is of much use in the computational aspects of the various problems, and is utilized in several papers in this issue. I believe future exploitations of the inners approach to these problems will advance both the theory and computational aspects of these four categories. It should be indicated that the mathematical theory in the latter stages was developed long before its application to system theory problems.

In conclusion, I wish to thank my colleague and friend, Professor N. K. Bose for inviting me to write this preface and

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I wish to congratulate him for his many contributions to multidimensional systems and to his tireless efforts in editing this important and timely issue. It is a milestone in the progress of system theory and will motivate future work to clarify, simplify and solve most of the problems presented. I am very grateful and fortunate to participate in a minor way in this issue.

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Problems and Progress in Multidimensional Systems Theory

N. K. BOSE, SENIOR MEMBER, IEEE

Abstract—**The paper presents the troubles and trends of research in multidimensional system theory, with special emphasis in the areas of multivariable network analysis and synthesis, and multidimensional digital filters. The possibilities as well as perils of extending established single-dimensional techniques to multidimensional situations is discussed, and the nature and effect of certain fundamental problems present when applying the mathematics of several variables is given cognizance. Open problems are identified at various stages and some recommendations for future research are made.**

I. Introduction

THOUGH rational functions and matrices (whose elements are rational functions) of a single complex variable can be used to satisfactorily characterize only a limited class of systems, analysis, synthesis, and approximation techniques based on their use have been extensive, reasonably complete, and well documented. Progress in technology has been accompanied with the advent of diverse and complicated systems, many of which have been characterized by rational functions or matrices of several complex variables. Analysis and design of those wider classes of systems necessitated the use of new mathematical tools. The major new tools, especially in the context of this paper, are (though by no means not limited to) the theory of analytic functions of several complex variables, multidimensional approximation theory, abstract algebra (the growing algebraic presence in systems engineering is already known), topics in decidability theory—especially those associated with real closed fields—and the properties of algebraic curves within the more general setting of algebraic geometry. The appeal of the subject is so broad that it is impossible in the framework of a paper to present with equal emphasis most, if not all, possible applications of relevant portions of these tools in the context of those aspects of multidimensional system theory, where characterization is via rational functions or matrices of several complex variables (some or all of these variables could be specialized to be reals). Therefore, the reader is forewarned that emphasis on different areas will be nonuniform. However, more information on topics that are not adequately covered can be obtained from the references cited. It is felt that in the absence of satisfactory documentation of the expanding body of results in this area, the discussions, comments and recommendations made in this paper will be of help to future students and researchers.

In Section II, the mathematical preliminaries that are helpful in the comprehension of the concepts covered in the paper are presented. Emphasis is placed here on the similarities and differences between single and multidimensional mathematical results, on which are based the topics covered in the succeeding sections.

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TABLE I

Summary of Certain Distinguishing Features of Single and Multidimensional Problems

Topic	Single Dimensional	Multidimensional
1. Factorization of polynomial.	Uniquely factorable as a product of irreducible linear factors.	Uniquely factorable as a product of irreducible factors, which may not be linear [183].
2. Singularities of rational functions.	Isolated poles.	Nonessential singularities of first and second kinds which lie on continuous algebraic curves.
3. Real or imaginary parts parts of analytic functions.	Harmonic functions.	Pluriharmonic functions.
4. Relationship between real and imaginary parts of holomorphic functions.	One computable from the other via Hilbert transform.	One may not actually be computable from the other.
5. Stability (positivity) algorithms.	No degree reduction of polynomial.	Degree reduction of polynomial, written in recursive cenonical form.
6. Approximation technique.	Haar condition holds.	Haar condition does not hold (lack of uniqueness of best approximation).
7. Stabilization.	Least-square inverse is stable.	Least-square inverse not stable in general.
8. Multiplicative positivity.	Necessary and sufficient for the solvability of moment problem.	Not sufficient for solvability of moment problem.
9. Synthesis.	Positive realness necessary and sufficient for LLFPB synthesis.	Sufficiency of positive realness not yet known.
10. Recursive filter BIBO stability.	Numerator polynomial does not play any role.	Numerator polynomial might play a role.

In Section III, the progress made and the problems present in the subject of multivariable network synthesis are discussed. Section IV is concerned with multidimensional signal processing where special emphasis is given to stability problems in multidimensional digital recursive filters. Section V is concerned with several areas where the results and concepts of interest in this paper become applicable. Section VI presents certain conclusions, and recommendations are made towards future research, particularly along the direction leading to development and implementation of algorithms relevant to multidimensional systems via modular methods. Though most of the notations used are standard and self-explanatory, a list of nomenclature is contained at the end of the paper for added convenience. Table I briefly summarizes some of the distinguishing features that differentiate between single and multidimensional problems discussed in this paper.

II. Mathematical Preliminaries

A. Polynomial and Rational Functions of Several Variables

Broadly speaking, the subject matter of this paper will be primarily concerned with the role of real rational functions of n complex variables (some or all of the variables could be real) and matrices whose elements are real rational functions of such variables, in the analysis or synthesis of systems which are characterizable by these classes of functions or matrices. As in the case $n = 1$, a rational function, $H(p_1, p_2, \cdots, p_n) = H(p)$ in the complex variables $p_1, p_2, \cdots, p_n$ is defined to be a quotient, of two polynomials $P(p_1, p_2, \cdots, p_n)$ and $Q(p_1, p_2, \cdots, p_n)$, i.e.

$$H(p_1, p_2, \cdots, p_n) = \frac{P(p_1, p_2, \cdots, p_n)}{Q(p_1, p_2, \cdots, p_n)} \tag{2.1}$$

where in (2.1), $Q(p_1, p_2, \cdots, p_n)$ is not identically zero, i.e., $Q(p_1, p_2, \cdots, p_n) \not\equiv 0$. When the coefficients of $P(p)$ and $Q(p)$ are real, $H(p)$ is called real rational. The case $n > 1$ differs from the $n = 1$ case in several significant respects and those most relevant to our study here are outlined. For further details, the readers can consult the several references available but are also forewarned that only a small fraction of the details which are in general an order of magnitude difficult than the $n = 1$ case, are really required for our present purpose.

In (2.1), every polynomial $P(p)$ (or $Q(p)$) in the n indeterminates can be uniquely written in the form

$$P(p_1, p_2, \cdots, p_n) = \sum a_{k_1 \cdots k_n} p_1^{k_1} \cdots p_n^{k_n} \quad \cdot k_1 + k_2 + \cdots + k_n \leqslant d \tag{2.2}$$

where $a_{k_1 \cdots k_n} \in R$ or C and d is the degree of $P(p)$. A polynomial in n variables of degree d as in (2.2) has not more than $(n+d)!/n!d!$ terms. Polynomials of the form $p_1^{k_1} \cdots p_n^{k_n}$ are called monomials. The exponent k_i is called the degree of the monomial $p_1^{k_1} \cdots p_n^{k_n}$ with respect to indeterminate p_i and $k = k_1 + k_2 + \cdots + k_n$ is called the total degree of the monomial. A polynomial which can be expressed as a sum of monomials all of the same degree is called a *homogeneous polynomial* or a *form*. The monomials of a given degree k generate a subspace of the polynomial algebra in n indeterminates, and after assignment of degree k, for $k = 1, 2, \cdots$ to the elements of the respective subspaces, the polynomial algebra in the n-indeterminates is termed a graded algebra. For any element $P(p_1, p_2, \cdots, p_n)$ of this graded algebra, the following Assertion is valid. It is understood that the coefficients $a_{k_1 \cdots k_n}$ of $P(p_1, p_2, \cdots, p_n)$ in (2.2) belong to a number field K and $K[p_1, p_2, \cdots, p_n]$ denotes the ring set of all polynomials over K.

Assertion 2.1: Every polynomial of degree $\geqslant 1$ belonging to $K[p_1, p_2, \cdots, p_n]$ can be expressed, within units, as a product of factors irreducible in K in an unique manner. When $n > 1$, these irreducible factors may not be linear (i.e., of degree 1) even when K is algebraically closed.

Further details concerning properties of polynomials in several variables can be found in [1], and standard definitions are not repeated here for the sake of brevity. It may be noted that a polynomial, irreducible in one number field may be reducible in another. However, a polynomial irreducible in every number field is called absolutely reducible. A polynomial like $p_1^2 + p_2^2 + p_3^2$ is absolutely irreducible while $p_1^2 + p_2^2$, though irreducible over the real number field becomes reducible when the number field is chosen to be complex. An important item linked to the problem of factorization is the zero-set or solution-set of a n-variable polynomial equation. The problem becomes more challenging when the number field in which solution is sought is not arbitrary but specified. For example, over the field of real numbers there are many homogeneous equations in the real variables $x_1, x_2, \cdots, x_n$, such as

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \tag{2.3}$$

which have only the trivial solution $(0, 0, \cdots, 0)$ for any positive integer-valued n. However, other fields exist in which a form $F(x_1, x_2, \cdots, x_n)$ in n-variables has a nontrivial zero, provided n is sufficiently large compared to the degree of $F(x_1, x_2, \cdots, x_n)$. The study of such coefficient fields has led to the development of an exciting mathematical discipline of relatively recent origin [8], while the existence or not of nontrivial solutions in the real number field of a polynomial or form is directly linked to the question of multivariable polynomial positivity or nonnegativity which has several engineering applications already [28], [54]. Most of the results of this paper will be based on the hypothesis that the field K is either real or complex. For the sake of compactness $K[x_1, x_2, \cdots, x_n]$ will be used to denote the set of all polynomials in the real variables $x_1, x_2, \cdots, x_n$ over a real field K, while $K[p_1, p_2, \cdots, p_n]$ will be used to denote the set of all polynomials in the complex variables $p_1, p_2, \cdots, p_n$ over a complex field K, which, however, for most of the results of our paper will be specialized to the field of reals.

In (2.1), whether or not the polynomials $Q(p_1, p_2, \cdots, p_n) \in K[p_1, p_2, \cdots, p_n]$ and $P(p_1, p_2, \cdots, p_n) \in K[p_1, p_2, \cdots, p_n]$ are relatively prime can be determined, and if not relatively prime, there common factors can be conveniently extracted [113]. Therefore, it is no restriction to assume that the numerator and denominator of a rational function are relatively prime polynomials in the following discussion. When $n > 1$, even if $P(p)$ and $Q(p)$ are relatively prime, their zero-sets might intersect, resulting in a bad type of singularity referred to as the nonessential singularity of the second kind. The effect of this type of singularity in various multidimensional problems will become evident in subsequent sections. A zero of $Q(p)$ which is not simultaneously a zero of $P(p)$ is referred to as a nonessential singularity of the first kind.

The theory of analytic functions of several complex variables [3]–[5] which is useful in the study of special classes of functions like those which have a rational characterization as in (2.1), have several similarities and dissimilarities with the corresponding single variable theory. In the present context, it is only necessary to point out some of these distinguishing features. First, the term holomorphic will be defined.

Definition 2.1: A complex-valued analytic function $F(p_1, p_2, \cdots, p_n)$ defined in some open set $\{s\} \in C^n$, where C^n is the Cartesian product of n copies of the complex field, is said to be holomorphic in $\{s\}$, provided: i) $F(p)$ is continuous in $\{s\}$; and ii) $F(p)$ is holomorphic in each variable separately. Also, the fact that ii) implies i) is a remarkable consequence of a deep theorem due to Hartog [3, pp. 1–2].[1]

The continuity of a holomorphic function in the set of its variables can be used to obtain its representation in the form of an n-dimensional Cauchy integral. From this integral representation of Cauchy, the representation of a holomorphic function as a multiple power series [4, p. 39] or Laurent series [4, p. 88] can be obtained. Similar to the $n = 1$ case, the existence and continuity of all partial derivatives of a holomorphic function $F(p)$ follow from the Cauchy integral formula. Furthermore as in the one-dimensional case the multidimensional counterpart of the Cauchy–Riemann equations are satisfied by a holomorphic function $F(p)$ as summarized in the Assertion below.

Assertion 2.2: $F(p)$ is holomorphic in a open set $\{s\} \in C^n$,

[1] It is noted, however, that for a function of several real variables to be analytic, it is not sufficient that the function be analytic in each variable separately when the others are held fast [168, p. 142].

if and only if, [5, p. 3]

$$\frac{\partial U}{\partial x_k} = \frac{\partial V}{\partial y_k}, \quad \frac{\partial U}{\partial y_k} = -\frac{\partial V}{\partial x_k}, \quad k = 1, 2, \cdots, n \tag{2.4}$$

or equivalently [4, pp. 20–21]

$$\frac{\partial F}{\partial p_k^*} = 0, \quad k = 1, 2, \cdots, n \tag{2.5}$$

where $F(p) = U(x, y) + jV(x, y)$ and $p_k = x_k + jy_k$, $p_k^* = x_k - jy_k$, $k = 1, 2, \cdots, n$.

The conditions in (2.4) and the fact that successive derivatives of $F(p)$ exist lead to the pluriharmonic conditions in (2.6) where $k = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, n$.

$$\frac{\partial^2 U}{\partial x_k \partial x_i} + \frac{\partial^2 U}{\partial y_k \partial y_i} = 0, \quad \frac{\partial^2 U}{\partial x_k \partial y_i} - \frac{\partial^2 U}{\partial x_i \partial y_k} = 0. \tag{2.6}$$

Similar conditions as in (2.6) are valid for $V(x, y)$. Thus the real and imaginary parts of a holomorphic function in n complex variables are pluriharmonic functions, which constitute evidently a subclass of the harmonic functions. For the $n = 2$ case the pluriharmonic property is called the biharmonic property, and a biharmonic function $U(x_1, x_2, y_1, y_2)$, the real part of a holomorphic function, $F(x_1 + jy_1, x_2 + y_2) = U(x_1, x_2, y_1, y_2) + jV(x_1, x_2, y_1, y_2)$ satisfies the set of conditions in (2.7), as a consequence of the results in (2.5) and (2.6) when specialized for the $n = 2$ case.

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial y_1^2} = 0 \quad \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial y_2^2} = 0 \tag{2.7a}$$

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} + \frac{\partial^2 U}{\partial y_1 \partial y_2} = 0 \quad \frac{\partial^2 U}{\partial x_1 \partial y_2} - \frac{\partial^2 U}{\partial y_1 \partial x_2} = 0. \tag{2.7b}$$

Thus condition (2.7b) is the restriction added on to the harmonic property of the real (or imaginary) part of a holomorphic function in two complex variables ($n = 2$), in contrast to the $n = 1$ case. Due to this added restriction in the $n > 1$ case it is not possible, in general, to construct a pluriharmonic function in a closed domain which shall taken on values preassigned over some portion of the boundary of the domain under construction, as in the $n = 1$ case. Actually, in the $n > 1$ case multifold application of the Poisson's formula gives a function which satisfies the harmonicity condition as in (2.7a) but not necessarily the pluriharmonicity condition of (2.6) or (2.7b) [4, pp. 276–277]. A continuous complex function in an open set in C^n will be called n-harmonic if the function is harmonic in each variable separately. The results discussed in this paragraph can now be briefly summarized. The class of all functions which are real parts of holomorphic functions form a subclass of the real n-harmonic functions, and particularly when $n > 1$ a n-harmonic function need not correspond to the real part of a holomorphic function. For example, $x_1x_2 + y_1y_2$ is 2-harmonic but cannot be the real part of a holomorphic function, while $x_1x_2 - y_1y_2$ is 2-harmonic and is the real part of $F(p_1, p_2) = p_1p_2$. This fact, for example, affects the construction of the real part of a network function, which is holomorphic in a polydomain Re $p_i > 0$ as will be seen in Section III, from prescribed values on Re $p_i = 0$. The real and imaginary parts, $U(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ and $V(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$, of a n-variable holomorphic function are, as expected, related to each other as in (2.8).

$$V(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n) = \int_{p_0}^{p} \sum_{k=1}^{n} \left\{ \frac{\partial U}{\partial y_k} dx_k + \frac{\partial U}{\partial x_k} dy_k \right\} \tag{2.8}$$

where the line integral is taken over a path extending from some fixed point p_0 to the point p. However, there are pitfalls in direct extension of one variable results like Hilbert transforms (which relate explicitly the real and imaginary parts of a single variable analytic function as opposed to the implicit relationship provided by the Cauchy–Reimann equations [114, pp. 433–439]) to the multivariable case [115].

A result in several complex variable theory which has a parallel in the single variable case is the maximum modulus theorem, summarized in Assertion 2.3. This has direct application in the results to be discussed in the next section.

Assertion 2.3: If the function $F(p)$) is holomorphic in the open region $D \subset C^n$ and is not constant there, then $|F(p)|$ cannot take on its maximum value inside the region. If the function is also continuous in the closed region composed of D and its boundary, then $|F(p)|$ takes in its maximum value on the boundary of D.

An implication of the above Assertion in relation to the topic of the succeeding section is that for a real rational function $Z(p)$, holomorphic in Re $p > 0$, the test to determine whether Re $Z(p) \geqslant 0$ in Re $p > 0$ cannot be replaced by the simpler test to determine whether Re $Z(j\boldsymbol{\omega}) \geqslant 0$ in Re $p = 0$, when nonessential singularities of the first or second kind are present on the boundary Re $p = 0$ of the open polydomain Re $p > 0$, as will also be substantiated by examples later. For the $n \geqslant 2$ case, any singularity of $Z(p)$ in Re $p \geqslant 0$ present in the set of points which are contained totally neither in Re $p > 0$ nor Re $p = 0$, may induce a singularity in Re $p = 0$ [3, pp. 97–99]. This type of behavior of the singularities on Re $p = 0$, complicates, in general, their extraction, unlike in the $n = 1$ case. Various other details concerning the distinguishing features of analytic functions of several complex variables, which are not of direct interest here can be found in [3] and [4]. An interesting discussion of the basic similarities and dissimilarities between analytic functions of complex variables and analytic functions of real variables can be found in [116, pp. 30–44]. A common feature in the development of the theory of analytic functions is the property of single-valuedness [4, p. 17]. On the other hand, algebraic functions are multiple-valued and some useful material concerning these as well as analytic functions can be found in [6]. The definition of an algebraic function is generated from a several variable polynomial equation and is illustrated by considering a two-variable polynomial $Q(p_1, p_2) \in K[p_1, p_2]$ in (2.9)

$$Q(p_1, p_2) = a_0(p_1)p_2^m + a_1(p_1)p_2^{m-1} + \cdots + a_{m-1}(p_1)p_2 + a_m(p_1). \tag{2.9}$$

In (2.9), $a_j(p_1) \in K[p_1]$, for $j = 0, 1, 2, \cdots, m$. An algebraic function is then a function, $p_2 = G(p_1)$ defined for values of p_1 in C by an equation of the form $Q(p_1, p_2) = 0$. With $Q(p_1, p_2)$ written as in (2.9) as a polynomial in p_2 with coefficients which are polynomials in p_1, primitivity is assumed, i.e., there is no factor common to all the polynomial coefficients $a_0(p_1)$, $a_1(p_1)$, $\cdots$, $a_{m-1}(p_1)$, $a_m(p_1)$. This, incidentally, does not imply that there is no common factor to the polynomial coefficients in p_2 when $Q(p_1, p_2)$ is written as a polynomial in p_1 with coefficients which are polynomials in p_2, as is evident from the example given next.

$$Q(p_1, p_2) = p_1 p_2^3 + p_2^2 + p_1 p_2 + 1$$
$$= (p_2^3 + p_2)p_1 + (p_2^2 + 1).$$

The zeros of $a_0(p_1)$, corresponding to values of p_1 for which degree reduction of $Q(p_1, p_2)$ takes place alongwith other values of p_1 for which the discriminant $D(p_1)$ of $Q(p_1, p_2)$, considered as a polynomial in p_2 [$D(p_1)$ equals the resultant of $Q(p_1, p_2)$ and $\partial Q(p_1, p_2)/\partial p_2$] is zero constitute the singular points of the algebraic function $p_2 = G(p_1)$. Since, the concept of "degree reduction" plays an important role in various types of tests performed on multivariable polynomials [72], [73], [54], [117], its relation to the theory of singular points of algebraic functions, a topic well explored in [6], is worthy of attention.

Due to the fact that this paper is concerned with problems which are characterizable by a special class of multivariable functions namely those which are meromorphic at every point in the space of analysis (i.e., rational functions), the properties related to these types of functions are underscored. For example, only nonessential singularities are mentioned as rational functions in several complex variables do not have other types of singularities. Readers interested in distinguishing features of other types of singularities for functions of several variables are referred to the very readable book by Osgood [168]. It is also mentioned that another branch of mathematics, namely algebraic geometry, studies rational functions systematically. As several papers in this issue, including the ones by Martin and Hermann, Anderson and Scott, and Morf *et al.*, deal with various aspects of applications of algebrogeometric theorems in system theory, the readers are referred to those papers as well as the excellent treatise by Shafarevich [10] for information.

B. Multidimensional Approximation

The problem of multidimensional approximation is very important in the context of this paper, and as reference will occasionally be made to this problem in the succeeding sections, it is considered pertinent to present a brief state-of-the-art summary of aspects of approximation theory in several variables which have relevance here. Though several textbooks [118]-[121] have appeared which deal extensively with single variable approximation theory, the topic of multidimensional approximation, though mentioned briefly in some books [119], is mostly found scattered in various technical journals. A very fundamental and important result in approximation theory is that of Weierstrass, who essentially proved that the set of all polynomials is dense in the space of all continuous complex-valued (not only real-valued) functions defined on a bounded closed interval of real numbers, implying that such functions can be uniformly approximated by polynomials. The generalization of this result to functions of several variables has been proved [119, p. 8]. This generalized results of Weierstrass, because of its fundamental importance, is summarized next.

Assertion 2.4: Let $\{s\}$ be a closed bounded subset of the n-dimensional Euclidean space. Then, the set of polynomials

$$\sum_{0 \leq k_1 + k_2 + \cdots + k_n \leq d} a_{k_1 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}$$

is dense in the space of all continuous functions $G(x_1, x_2, \cdots, x_n)$ on $\{s\}$.

A nice proof of the result in the above Assertion along with other interesting discussions can be found in [122, especially pp. 244-245], though the first extension of the Weierstrass' theorem to several variables was made by Picard [123]. The role of Bernstein polynomials in the proof of Weierstrass' celebrated theorem is well known [118, pp. 66-69] and modifications of these polynomials for purposes of approximation on infinite intervals have been considered along with their use to approximate discontinuous functions [124]. For this reason, it is interesting to note that Bernstein polynomials for functions of two variables have been introduced [125] and a study of some of their properties has been made [126], [119, pp. 69-72]. Extensions to several variables of the well established classical theory of interpolation has also been made [127], [106] and some added comments on this are reserved for a later section. In [127], attention is given to the adaptation of results for numerical computation, and the key concepts are nicely illustrated by an example.

From the practical standpoint, the theory of uniform approximation based on Chebychev's result is highly important and the literature available on the topic is vast [119]. A. Haar presented a set of necessary and sufficient conditions that an unique solution exists to the approximation problem of a real continuous function on a compact set in Euclidean space in a Chebyshev sense [118], [119], [128]. An extension of Haar's condition to the complex case was proved by Kolmogorov in 1947 and a different proof based on a technique (mainly based on the Hahn-Banach extension theorem and the Reisz representation theorem for linear functionals), using functional analysis was given in 1960 [129]. Attempts to extend the Chebyshev approximation theory to several variables suffer because of a serious obstacle—the lack of uniqueness of best approximations to functions of more than one variable [130]. This is because the multidimensional counterpart of the Haar condition, referred to above, is not generally satisfied. In particular, it has been shown [128] that best approximations cannot be unique unless the functions are defined on a space homeomorphic to a subset of the unit circle. In spite of this, some workable theories for the computation of best approximations in several variables have been obtained [131]. An intrinsic feature of Chebychev approximations is that of alternation or oscillation [119, pp. 16-36]—a feature for which a simple geometric interpretation is difficult to give for functions of several variables. Therefore, though it appears that the famous Remez exchange algorithm [118, pp. 96-100] cannot be generalized to functions of more than two variables, suitable modifications of the "one for one exchange" algorithms used for actual computation of best approximations of functions of several variables were first given by Rice [131, pp. 461-465]. For some special results on unicity of approximation for classes of functions in several variables the reader is referred to [132], [133], [119, pp. 103-104, p. 126], [134]. Approximation by rational functions is the subject of a treatise by Walsh [135], among others and various aspects of this topic from a different viewpoint are also discussed in [118, ch. 5], where an extensive bibliography on the subject is also present. A nice but concise discussion on rational Chebychev approximation in several variables occur in [119, pp. 153-158]. The preceding discussion summarizes some of the limited progress that has been made for approximating functions of several variables. These limitations provide some motivation for the use of piecewise-linear approximation of nonlinear multivariable required in his realization. The solution to the problem of the minimum number of ideal gyrators required for synthesizing an arbitrary nonsymmetrical two variable reactance matrix is favorably influenced by the recent solution to the important problem of minimum-gyrator synthesis of an one variable nonbilateral dissipative m-port by Oono [26]. Another paper [27], similar to Youla's in several respects, proved that a minimal realization of a $(m \times m)$ two-variable reactance matrix as a lossless $(m + k)$-port in one variable terminated at its k-port by unit reactances in the remaining variable, is possible. The heart of the synthesis scheme in this case dwells on the feasibility of factorization of a $mr \times mr$ real polynomial parahermitian matrix $T(p_1)$, nonnegative on the $p_1 = j\omega_1$ axis in the form

$$T(p_1) = M(p_1) M_*(p_1) \tag{3.7}$$

where $M(p_1)$ is a $mr \times k$ real polynomial matrix with a left inverse which is analytic in Re $p_1 > 0$, r and k being, respectively, the p_1 and p_2 degrees of the specified reactance matrix, $Z(p_1, p_2)$ of order $(m \times m)$. It is also possible in this case to implement the synthesis via factorization as in (3.5) of a real rational parahermitian matrix, positive definite on the imaginary axis. The absence of the multivariable counterpart to the factorization in (3.7) as well as the problems encountered in the multivariable factorization of the type in (3.6), discussed earlier, prevents extension of the synthesis scheme to reactance matrices in greater than two variables. Koga [20] made a valiant attempt to prove that the positive realness condition is also sufficient for synthesis of an arbitrary passive multiport when the number of variables in the prescribed matrix is greater than one. The shortcomings and error in the general applicability of the procedure has been recently discussed [28]. Consequently, a significant open problem remains as to whether a synthesis procedure can be given for a prescribed positive real matrix in several variables. It is recommended that new results towards this goal be sought by considering first the synthesis problem of an arbitrary two-variable positive real matrix. In case, positive realness is proved to be not a sufficient condition for multivariable multiport synthesis, then the derivation of a complete set of conditions (including positive realness) that will serve as necessary and sufficient conditions in the context under discussion, is of interest.

C. Synthesis with Constrained Topology

In practice, constraints are often imposed upon the topology of the network to be synthesized. A very important practical class of lumped-distributed networks, especially at microwave frequencies is a cascaded structure of uniform commensurate or rationally related transmission lines separated in general, by lumped 2-port lossless networks terminated in a load that is passive but otherwise arbitrary (when there are m lines with same one way delay $\tau > 0$, the network is called m-lines, lumped-terminated reactive, τ-cascade). It is well known that these networks are often realized by the use of coaxial cables, striplines, or waveguides with lumped elements in the form of step discontinuities, dielectric beads, irises, or posts and are useful as filters especially with the growing interest in microwave integrated circuits. It has long been appreciated that over and above the two variable positive real constraint on the driving point condition, an "even part" constraint must be satisfied before realization can be implemented. References to related synthesis considerations along with discussions of conditions on the driving-point admittance when each of the lumped reactive two-ports consists of only a single shunt capacitor or a series inductor, are given in [29]. A correct, complete, and compact solution was recently advanced and because of the importance of this result the main result in [30] is briefly stated below as a theorem.

Theorem [30]: The driving-point impedance of an m_1-line, lumped terminated reactive τ-cascade can always be expressed in the irreducible form

$$Z(p_1, z_2) = \frac{b_0(p_1) + b_1(p_1) z_2 + \cdots + b_m(p_1) z_2^m}{a_0(p_1) + a_1(p_1) z_2 + \cdots + a_m(p_1) z_2^m} \equiv \frac{M(p_1, z_2)}{N(p_1, z_2)}$$

where the a_i's and b_i's for $i = 0, 1, 2, \cdots, m$ are real polynomials in p_1, $z_2 = e^{-2p_1\tau}$, $m \leqslant m_1$ and

$$b_m(p_1) a_m(p_1) \not\equiv 0.$$

Moreover

(i) $a_0(p_1) + b_0(p_1) \neq 0$, Re $p_1 \geqslant 0$

(ii) $M(p_1, z_2) N(-p_1, z_2^{-1}) + M(-p_1, z_2^{-1}) N(p_1, z_2) = \mu(p_1)$

where $\mu(p_1) = \mu(-p_1)$, $\mu(p_1)$ has real coefficients, and $\mu(j\omega_1) \geqslant 0$, $-\infty < \omega_1 < \infty$

(iii) For $m \geqslant 1$ let

$$A(p_1) = \begin{bmatrix} a_0(p_1) & & & \\ a_1(p_1) & a_0(p_1) & & \\ \vdots & \vdots & \ddots & \\ a_{m-1}(p_1) & a_{m-2}(p_1) & \cdots & a_0(p_1) \end{bmatrix}$$

$$B(p_1) = \begin{bmatrix} b_0(p_1) & & & \\ b_1(p_1) & b_0(p_1) & & \\ \vdots & \vdots & \ddots & \\ b_{m-1}(p_1) & b_{m-2}(p_1) & \cdots & b_0(p_1) \end{bmatrix}$$

and define the $m \times m$ parahermitian "resistivity matrix" $K(p_1)$ associated with $Z(p_1, z_2)$ by

$$K(p_1) = \frac{A(p_1) B_*(p_1) + B(p_1) A_*(p_1)}{2}.$$

Then $K(p_1)$ admits the factorization

$$K(p_1) = L(p_1) L_*(p_1) \tag{3.8}$$

where $L(p_1)$ is real, square, polynomial, lower triangular, and minimum-phase; i.e.,

$$\det L(p_1) \neq 0 \quad \text{Re } p_1 > 0.$$

Conversely, a $Z(p_1, z_2)$ of the form above satisfying conditions (i)–(iii) is realizable as the input impedance of an m_1-line lumped-terminated reactive τ-cascade and the synthesis may always be carried out with linear lumped reactive networks which are "all-pass free on their output side."

The remarkable fact about the above theorem is that the difficulty of testing a two variable polynomial for positive realness is replaced by relatively simple tests. Moreover, over and above the well known conditions that were known to be necessary for this type of cascade synthesis, the theorem brings out clearly an additional constraint (involving the so-called resistivity matrix) which must be met if the passive topological structure of the cascade is to be retained. The preceding theorem provides the basis for an explicit solution to the m-line, lumped-terminated reactive τ-cascade. The details of the procedure yielding expressions for the junction reflection coefficients of the lumped networks between any pair of lines as given in [31], uses properties of sequences of polynomials orthogonal on the unit circle [32, pp. 182–184]. In the one variable case this procedure yields an efficient alternative to the cascaded synthesis based on Richard's transformation and theorem [33] or the algorithm of Kinariwala [34] in which the transcendental impedance function is not converted to a rational function via a transformation. For the synthesis of lumped-distributed networks, it is noted that Weinberg [35] and Riederer [36] instead of converting the single variable transcendental function problem to a two variable rational function problem (in the case of m-lines, lumped-terminated reactive τ-cascade) followed a philosophy similar to that in [34] to present a synthesis algorithm. It appears that there are some merits in this approach and it is worthwhile to make a detailed comparison including considerations of computational complexities between the two basic approaches—one using two variable rational functions and the other using single variable transcendental functions. The n-variable characterization $n > 2$ becomes necessary when in the cascaded structure of the generic type, the electrical line lengths are incommensurate or not rationally related. As such a possibility has no practical relevance, the scopes for extension of the results in [30] and [31] will not be considered here. It may be possible, however, that more than two variables may be required in the characterization of other types of networks consisting of say, RC as well as LC uniform or nonuniform transmission lines along with lumped elements, and future research should not totally overlook n-variable synthesis, when $n > 2$. A word of caution is pertinent here. Though, as stated above, there has been some nice results in two variable theory applicable to cascaded synthesis of lumped-distributed networks, there are certain problems in the lumped-distributed area which are not benefited via transformation into multivariable problems. For example, it is not true that the stability of a given lumped-distributed network can be checked by checking the Hurwitz property of a multivariable polynomial as asserted in [37]. The multivariable Hurwitz property might serve as a sufficient but not necessary condition for the stability of a given lumped-distributed network. The catch lies in the fact that the τ_i invariant (where $\tau_i > 0$ corresponds to line delay) realizability conditions in the single complex frequency plane are in one-to-one correspondence with the multivariable realizability conditions provided *all* possible τ_i are considered [38]. This also delineates as a open problem the devising of procedures to determine the range of τ_i that will guarantee stability of a prescribed active lumped-distributed feedback network, via multidimensional techniques. The importance of this type of research is further enhanced by the fact that though there are established computationally feasible techniques to test, for example, whether a polynomial $Q(p_1, p_2)$ has zeros in Re $p_i \geqslant 0$, $i = 1, 2$ [14], [21], the techniques advanced so far to test the characteristic equations (which are entire functions of a single complex variable) obtained from the meromorphic system functions of lumped-distributed networks, for zeros in the right-half plane, are either very difficult to implement [39] or approximate at best [40].

Multivariable rational functions have also been used in the characterization and synthesis of variable-parameter networks including passive variable networks [12], [41] as well as active variable networks comprised of constant elements in association with variable active elements [42], [43]. The characterization considered for networks of this type (in combination, if necessary, with distributed elements as well) is in terms of rational functions in several complex as well as real variables. There is considerable scope for developing applicable synthesis techniques for these types of functions, using as basic building blocks not only passive but active elements as well [44], [45]. Another important problem which demands considerable attention in the future is that of multivariable approximation, which, really, in practice serves as a prelude to synthesis of desired amplifier or filter characteristics. Though, the importance of the approximation problem was mentioned on several occasions [12], [46], most of the work to date either does not undertake or side-steps this approximation problem. In [47], attention has been given to the setting up of conditions for maximally flat approximation for a cascaded connection of noncommensurate transmission lines, while in [48] the approximation problem for a class of two-variable resonant ladder networks has been considered. As the multidimensional approximation problem is also germane in other areas outside network theory, some additional comments on it will be made later. Finally, it may be worthwhile to point out that several dissertations have been written in the area under discussion in this section and some of these dissertations which have not so far been directly cited are listed for further reference, [49]–[55].

IV. Multidimensional Signal Processing

The range and depth of topics that could be covered in this area is tremendous as is evident from the volume of publications in the form of books, reports, and research papers over the last decade or so. The purpose here is to discuss in what way the mathematical tools or principles discussed, for example, in the previous sections are adaptable in other areas requiring processing of multidimensional data. The characterization by rational transfer functions of the input-output behavior of space-invariant multidimensional optical processing systems has been considered in [56], [57], and the literature on multidimensional digital filters used for a variety of signal processing applications can to a great extent be found in [58]–[50], [61, ch. 7], and [62], though the literature in all these areas is expanding so fast that continuous updating of references become necessary.

A. Multidimensional Digital Filter Stability Problems

In order to emphasize upon the breadth of applications, the emphasis in this section, as opposed to the previous one, will be on multidimensional linear descrete-time systems, whose input-output characterization is via a rational function in the variables $(z_1, z_2, \cdots, z_n) = z$. A multidimensional digital recursive filter, for example is characterized by the multidimensional z-transform $H(z)$ in (4.1), obtained from the spatial-domain difference equation relating the input and output multidimension sequences

$$H(z) = \frac{P(z)}{Q(z)} \tag{4.1}$$

where $P(z)$ and $Q(z)$ will be assumed throughout to be relatively prime polynomials. One of the major problems in the design of a recursive filter is the problem of stability. For quite some time the stability of a multidimensional filter in the bounded-input, bounded-output sense has been related to the absolute summability of the impulse response and subsequently to the condition

$$Q(z) \neq 0, |z| \leqslant 1 \tag{4.2}$$

where $|z| \leqslant 1$ will be interpreted as equivalent to $|z_1| \leqslant 1$, $|z_2| \leqslant 1, \cdots, |z_n| \leqslant 1$ simultaneously. Very recently, Goodman [63] has shown with clever counterexamples that the nonessential singularities of the second kind on the boundary of the unit polydisc in the z-plane, can cause problems and that though the condition in (4.2) is sufficient for stability of the system characterized by (4.1), it is by no means necessary. Therefore, interestingly enough the stability problem for $n > 1$ is influenced not only by the denominator polynomial of $H(z)$ but also by its numerator polynomial. It is intuitively felt by several digital filter designers that it is not worth implementing filters when the condition in (4.2) is not satisfied, i.e., nonessential singularities of the second kind on the unit polydisc must at all cost be avoided even if those do not lead to stability problems on paper. It is the author's opinion, nevertheless, that a more thorough study of transfer functions that lead to such singularities is necessary. When the multidimensional transfer function $H(z_1 \cdot z_2, \cdots, z_n)$ is separable, i.e.,

$$H(z_1, z_2, \cdots, z_n) = \prod_{i=1}^{n} H_i(z_i) \tag{4.3}$$

where $H_i(z_i), i = 1, 2, \cdots, n$ is a transfer function in one complex variable, this problem can be satisfactorily addressed to. However, when the separability condition of (4.3) is not satisfied, or when the transfer function $H(z)$ in (4.1) is not even upper (lower) semi-1-reducible (i.e., when either the numerator $P(z)$ or the denominator $Q(z)$, but not both, can be expressed as a product of single variable polynomials, thus one or the other, but not both, is 1-reducible) [64], the problem of checking (4.1) for absence of the singularities referred to on the unit polydisc appears to be a difficult one. The question then arises as to what the precise limitations of separable (1-reducible) or upper (lower) semi-1-reducible filters are, because in the former case the nonessential singularities of the second kind on the unit polydisc do not occur (when the numerator and denominator are relatively prime polynomials) and in the latter case, it may be feasible without too much difficulty to check into the presence or absence of such singularities on the unit polydisc. This is because in the event a filter designed to meet certain specifications is found to be not separable or upper (lower) semi-1-reducible, it may become necessary to test for the presence or absence of nonessential singularities of the second kind on the unit polydisc, which problem is equivalent to that of ascertaining whether or not there exists a $z = z_0$ satisfying simultaneously

$$P(z_0) = 0, Q(z_0) = 0 \text{ and } |z_0| = 1 \tag{4.4}$$

where $P(z)$ and $Q(z)$ are polynomials as in (4.1) and $|z_0| = 1$ is interpreted to be equivalent to $|z_{10}| = |z_{20}| = \cdots = |z_{n0}| = 1$, when $z_0 = (z_{10}, z_{20}, \cdots, z_{n0})$. Though it is possible to solve the above problem as implied by the results from elementary decision algebra [15], the computational complexities are very high.

The test for the condition in (4.2) has been considered by several researchers. For the $n = 2$ cases, several solutions have been offered in [14] and [65]–[68]. Some of these more recent results along with previous contributions on the subject have been treated in a recent review paper [69]. While at present, two-dimensional recursive filtering is finding wide use in a variety of technological problems, false scepticism or doubts regarding the applications of higher dimensional filtering should not set the boundaries of present research activity. In fact, the importance and need for multidimensional filtering in certain areas like seismology have already been discussed over several years [70], [71]. In [72], the test for stability of three-dimensional filters was explicitly presented, the problem of degree reduction of multivariable polynomials (when written as a polynomial in one variable with coefficients in the remaining variables) mentioned, and the need for generating a constructive algorithm for stability tests for higher than three-dimensional digital filters using Tarski's generalization of Sturm's theorem was discussed [7, vol. 3, pp. 312–316]. Before this, there were some doubts regarding even the existence of such procedures. In [73], a constructive tabular approach to implement the simplified test conditions for multidimensional digital filters [74] has been suggested. Preliminary investigations have revealed that the computational complexities in the tabular approach to multidimensional problems tend to be high. This is especially true when a self-inversive polynomial like $C(z_1, z_1^{-1}, z_2, z_2^{-1}, \cdots, z_n, z_n^{-1})$ cannot be expressed as a polynomial in $(z_1 + z_1^{-1}), (z_2 + z_2^{-1}), \cdots, (z_n + z_n^{-1})$. This is always possible when $n = 1$, where

$$C(z_1, z_1^{-1}) = \sum_{j=0}^{m} c_j(z_1^j + z_1^{-j})$$

can always be expressed as [14], [65]

$$C(z_1, z_1^{-1}) = \sum_{j=0}^{m} d_j(z_1 + z_1^{-1})^j$$

where d_j's are constants expressed in terms of constant c_j's. A case in point is the polynomial $z_1 z_2^{-1} + z_2 z_1^{-1}$ which cannot be expressed as polynomial in $(z_1 + z_1^{-1})$ and $(z_2 + z_2^{-1})$. This fact also complicates the multidimensional stability test problem if other methods are used [75]. Though multidimensional digital filter stability test using schemes like inners [76, pp. 28–29] should also be possible, the efficiency of each type of implementation has to be assessed. A comprehensive study of these problems is being carried out and will be reported in due course. It is pointed out that an alternate set of conditions, besides those already given in [74], to test for the condition in (4.2) has been presented [77]. Here it is claimed that $Q(z_1, z_2, \cdots, z_n) \neq 0$, in $|z_1| \leqslant 1, |z_2| \leqslant 1, \cdots, |z_n| \leqslant 1$ if and only if the following conditions are met:

$$Q(1, 1, \cdots, 1, z_i, 1, \cdots, 1) \neq 0 \text{ in } |z_i| \leqslant 1 \text{ for } i = 1, 2, \cdots, n \tag{4.5a}$$

and

$$Q(z_1, z_2, \cdots, z_n) \neq 0 \text{ in } |z_1| = |z_2| = \cdots = |z_n| = 1. \tag{4.5b}$$

The degree of difficulty in the implementation of the test for condition (4.5b) remains to be assessed.

In [63] also are considered the problems that occur in trying to extend to the two-dimensional case some of the results in the one dimensional case where, for example, either of the conditions in (4.6a) and (4.6b) serve as necessary and sufficient for the filter with a rational transfer function $H(z_1)$ and impulse response $h_n(H(z_1) = \sum_{n=0}^{\infty} h_n z^n)$ to be bounded-input bounded-output stable

$$\lim_{n \to \infty} h_n = 0 \tag{4.6a}$$

$$\sum_{n=0}^{\infty} |h_n|^k < \infty \quad \text{for some } k \geqslant 1. \tag{4.6b}$$

The validity of the extensions is, however, claimed when some restrictions are imposed on the two-dimensional filter transfer function [78]. Also, here and in [79] investigations into stability conditions are made in a more general setting.

B. Realization Problem

Several approaches to the realization problem of two-dimensional recursive digital filters have been proposed since the first paper on the subject [80] introduced a scheme of direct implementation of the two-dimensional difference equation represented by the rational transfer function in two complex variables z_1, z_2. Other direct form realizations of arbitrary two-dimensional transfer functions have been presented [81]. Nevertheless, because of the fundamental problem of factorizing multivariable polynomials (as in Assertion 2.1, when $n > 1$), some special types of structures which do not, in general, lead to multidimensional realizations are the parallel, cascade, continued fraction or ladder and lattice structures. The role of the continued fraction expansion in synthesis of classes of two variable reactance functions [82] as well as in infinite impulse response (IIR) two-dimensional (2-*D*) filter realizations [81], has been considered. The fundamental hindrance to generality is associated with the nature of generalization of Euclid's division algorithm [2]. Certain state-space models of 2-*D* systems have also been introduced [83], [84]. Since a unified treatment of these and other approaches along with results concerning existence or not of minimal realization plus related results form the subject of another paper by Kung, Levy, Morf, and Kailath, duplication will be avoided here. In contrast to the recursive deterministic filter realization techniques mentioned so far, some progress in filter synthesis in a stochastic setting has also been made [85], [180]–[182] where the design of an optimal steady-state filter that avoids phase error problems (which, in general, demand serious attention in 2-*D* image data, unlike the 1-*D* case [86]) has been considered. Extensive references to existing literature in the topics mentioned in this paragraph can be found in recent dissertations like [87], [88]. In [87], the theory of spectral transformation has been extended to two-dimensional filter design using the stability preserving property of the spectral transformation operator. Further details concerning this approach can be found in [89] and in a paper in this issue by Chakrabarti and Mitra.

C. Stabilization Problem, Approximation, and Miscellaneous

The tests for stability discussed solve only part of the overall stability problem. The problems of designing filters that are guaranteed to be stable or that of stabilisation of an unstable filter without significant change in its frequency response are, in practice, very important. With respect to the first problem, the role of spectral transformation mentioned above can be useful as it does yield some information on what types of frequency responses one can get from classes of simple filters whose stability property is conveniently controllable. Thus a stable filter design can be altered quickly to produce other stable designs. With respect to the second question, the proof of a conjecture by Shanks [90] had been long outstanding. The invalidity in general of the conjecture has been demonstrated by a neat counter example [91], and the detailed theoretical developments based on the properties of two variable orthogonal polynomials on the hypercircle that lead to the construction of the counterexample to Shanks' conjecture, are contained in the paper by Genin and Kamp in this issue of PROCEEDINGS. Recently, after observing that the counterexample of Genin and Kamp involves a polynomial of third degree in two variables which allows an inverse polynomial of lower degree in each variable (linear in the case under discussion) violating the stability conditions, Jury conjectured that the double planar least square inverse polynomial of the same degree as the original unstable polynomial is stable. Even if this conjecture is true, the proof is expected to be difficult [92]. Jury *et al.* explore a class of polynomials in a paper (this issue of PROCEEDINGS) that satisfy Shank's conjecture, and research directed towards determining precisely the scope for broadening this class is of interest. In the realization of a goal to obtain stable, 1-*D* recursive filters, factorization of polynomials into their causal and anticausal components is necessary. This is done after a prescribed magnitude characteristic is approximated by a ratio of two cosine polynomials. In [68] a theory of 2-*D* spectral factorization is presented with the objective of designing half-plane recursive filters conveniently. In the 2-*D* case, though spectral factorization retains analyticity properties, the factors may not be of finite order as in the 1-*D* case. Therefore, imposition of a finite order constraint on the spectral factors becomes necessary in a practical design algorithm. Ahmadi and King consider the problem of generalization of some of the ideas in [68] to *n*-dimensional recursive filters, in a paper appearing in this issue of PROCEEDINGS. It becomes evident from the above discussion that useful progress has been made towards the resolution of the important problems related to overall filter stability considerations of multidimensional recursive filters since 1972 [93, p. 163].

Considerable work has also been done in the design of 2-*D* nonrecursive filters which have the advantage of linear phase, are less sensitive to quantization errors, and do not suffer from stability problems. A large volume of result on the design and implementation of such filters is available and the reader is referred to some recent publications which present new results and refer to previous contributions on the subject as well [87], [94]–[96]. In addition to implementation via direct convolution or transform techniques like FFT and number theoretic transforms, an interesting approach involving the use of residue arithmetic and computations in finite fields with subsequent advantages of parallel processing has been used in the implementation of 1-*D* nonrecursive filters [97]. Research into the possibility of extension of this type of result in the design of multidimensional filters might be helpful. Space limitations do not permit detailed survey of the various results, especially those which have already been adequately exposed in numerous articles including thesis and dissertations. In a nutshell, it has been seen that in spite of the difficulties encountered in higher dimensional filtering due to the absence of the fundamental factorization theorem of algebra ($n > 1$ case in Assertion 2.1) and the Haar condition (Section II-B) considerable progress has been made, especially in two-dimensional recursive as well as nonrecursive filtering. For additional detailed reference, the readers are alerted to some of the pertinent dissertations in this area, which often contain valuable information not available in papers publishing considerably condensed versions [98]–[103].

This section will be terminated after additional discussion pertaining especially to the problem of multidimensional approximation, which actually is a prelude to design and synthesis. The design of a multidimensional digital or analog filter from a prescribed magnitude response specification reduces to the construction of a rational function in several complex variables whose magnitude function approximates within suitable error specifications the prescribed magnitude characteristic. In the case of a nonrecursive digital filter, the rational function is a polynomial. The theory of interpolation is well established [105], and this theory even with several variables is years old [106]. In the case of two variables, for example, let $F(x_1, x_2)$ be the function required whose values are defined at the points $x = a_i, y = b_j, i = 0, 1, 2, \cdots, j = 0, 1, 2, \cdots$. A polynomial $P(x_1, x_2)$ such that

$$P(a_i, b_j) = F(a_i, b_j) \tag{4.7}$$

and its degree is not greater than m and n, respectively, can be represented in the form

$$P(x_1, x_2) = \sum_{k_1=0}^{m} \sum_{k_2=0}^{n} c_{k_1,k_2} \prod_{\substack{i=0\\ i\neq k_1}}^{m} (x_1 - a_i) \prod_{\substack{j=0\\ j\neq k_2}}^{m} (x_2 - b_j). \tag{4.8}$$

The coefficients c_{k_1, k_2} in (4.8) can be successively found. The most remarkable property of the coefficients is that those need not be reestimated even if the number of points at which the value of the function is given is increased. This type of basic 2-*D* Lagrange interpolation formula and its ramifications have been used in the approximation (in a Chebyshev or equiripple sense over closed compact sets) of frequency sampling and optimal nonrecursive filters [107] as well as in realization [87], [108]. An authoritative account of the role of Chebyshev approximation in the design of 1-*D* or 2-*D* nonrecursive digital filters, the possibility and problems in extending 1-*D* theory to the 2-*D* case, and open questions in multidimensional approximation can be found [112]. In another instance, similar to that in the 1-*D* case, an optimization algorithm to minimize the l_p-error criterion subject to stability constraints has been proposed for the approximation and design of 2-*D* recursive filters [109]. However it has been shown in [110], that examples can be found in two dimensions for which the algorithm in [109] will attempt to converge to an unstable solution—the source of the problem in the counterexample being the presence of nonessential singularities of the second kind (see Section II) on the boundary of the unit bidisc in the z_1, z_2 planes. Another technique has been presented for designing stable 2-*D* recursive filters whose magnitude response is approximately circularly symmetric [111]. The problems that occur here and in [80] due to nonessential singularities of the second kind on the distinguished boundary (defined by $|z_1| = |z_2| = 1$, which in this paper has been also referred to as boundary, without scopes for confusion because of context) in the 2-*D* rotated filter has been considered in [142]. Though the problems of stability, design and synthesis, and approximation problems have been discussed with attention to the role of nonessential singularities of the second kind, recent results on topics like error analysis of multidimensional digital filters [169], [170] are not considered to limit the size of the paper. Also, the emphasis has been on multidimensional digital filtering though there is considerable overlap in the areas of digital filtering and image processing. Image enhancement, for example, is directly related to digital filtering, especially when the classical procedures of Wiener filtering and regression techniques are applied. The special issue [171], provides considerable information about research in the area till 1975, along with a listing in its editorial of other previous special issues on the subject.

The two separate disciplines of digital image processing and numerical analysis often merge when one models imaging techniques in linear or matrix formalism. The problem of computer storage in digital form of a 2-*D* array of numbers characterizing individual brightness values taken from an original photograph, is inherent in the representation and restoration schemes of images in digital image processing. However, a sampled and quantized image is merely a matrix of nonnegative numbers, which is open to manipulations by a large class of linear or nonlinear operations on a digital computer. The efficient implementation of these operations is almost a subject by itself and the readers are referred to [172, pp. 51–54] for more information and additional references on this subject, in the context of multidimensional problems.

V. Other Areas of Multidimensional System Theory

This section presents other areas of research, besides those covered in the preceding sections, that fall within the scope of multidimensional system theory from the standpoint of this issue. In order to keep the size of the paper within reasonable bounds, only brief comments on recent progress in the areas are made and the readers are referred to the original sources for specific technical details.

A. Multidimensional Polynomial Positivity, Nonnegativity, and Solution Regions

The concepts of positivity and positive definiteness in relation to the theory of generalized functions has been the subject of intensive research by mathematicians and an excellent account of this topic can be found in [143, ch. 2]. The discussions of positive definite generalized functions and conditionally positive generalized functions of one and several variables are useful in the theory of random processes and random fields. The basic concepts there of positivity [143, p. 142], positive definiteness [143, p. 151], and multiplicative positivity [143, p. 230] pertain to the theory of topological algebras with involutions [143, p. 229]. One example of a topological algebra with involution is the algebra of polynomials of several variables discussed in Section II. The study of positivity of these multivariable polynomials is our major concern here. In the algebra of polynomials of two or more variables, some multiplicatively positive linear functionals are not positive, unlike in the algebra of polynomials of a single variable. This fact follows from Hilbert's example of a positive two variable polynomial (not form) with real coefficients which cannot be expressed as a sum of squares of two variable polynomials with real coefficients (the independent variables are also real in the polynomials). The relevance of the sum of squares representation problem in network synthesis has been discussed in [28]. Interestingly enough, Hilbert's two variable polynomial example is also connected with the moment problem for functions of two variables. Unlike in the case of one variable, the weaker requirement of multiplicative positivity of the linear functional F in the space of polynomials in two real variables is not sufficient for the moment problem to be solvable [143, pp. 235–236]. However, the requirement of positivity is both necessary and sufficient for the moment problem in two variables to be solvable.

A large number of problems in system theory are essentially reducible to determination of global or nonglobal (including local and semilocal) positivity (nonnegativity) of a polynomial in several real variables. Some of these problems are global asymptotic stability investigation by the direct method of Lyapunov, determination of the existence or not of limit cycles using the Poincare–Bendixon theorem, determination of positive realness in multiport network analysis and synthesis, constrained as well as unconstrained optimization, the classification of singularities of nonlinear systems, and stability problems associated with recursive digital filters used in areas like image processing and seismic or geophysical data exploration. In view of such a broad scope for applications, it seems appropriate to develop algorithms which are suitable for implementation with storage, time, and cost constraints. Very recently, a general procedure [117] to test a multivariable polynomial for global positivity has been given. The n-variable polynomial global nonnegativity test has also been formulated in terms of a class of $(n + 1)$ - variable global positivity test [144]. Also, a procedure to test a two variable polynomial for local positivity test has been given [75] and nonglobal multivariable polynomial positivity test considerations have been included in [145], [146]. The polynomial global positivity test algorithm has been programmed and its complete listing can be found in [54]. The paper in this issue of PROCEEDINGS by Modarressi and Bose takes steps towards the getting of exact solution regions in the real number field when a given polynomial in several real variables is found to be not sign definite. The restriction of solutions to a real field results in the possible presence of isolated points along with continuous algebraic curves (closed or open) in the solution space. It is mentioned that scopes for implementing the multivariable polynomial global positivity test using other procedures—in particular via use of Routh type of array—has been investigated. However, the computational shortcomings of such approaches appear to be more than the method used in [117]. The use of a Routh type of array in the extraction of the greatest common factor from two multivariable polynomials also appears to lead to worse computational problems than the method suggested in [113].

In many applications requiring positivity test on a polynomial $V(x_1, x_2, \cdots, x_n)$ in n real variables, the magnitude of n is rather high. For example, consider the important problem of determining the portion x_i of power which is to be supplied by each of the generators G_i $i = 1, 2, \cdots, n$ in a generating system to a fixed load h. The generators have costs C_i which are functions of the power output x_i, i.e.,

$$C_i = P_i(x_1, x_2, \cdots, x_n) \tag{5.1}$$

where $P_i(x_1, x_2, \cdots, x_n)$ is usually a polynomial of low degree in the variables x_i, $i = 1, 2, \cdots, n$. The transmission line looses L_i of each of the generators are also usually given by

$$L_i = Q_i(x_1, x_2, \cdots, x_n) \tag{5.2}$$

where the Q_i's are also polynomials, usually of degree not greater than 2 in each of the variables x_i, $i = 1, 2, \cdots, n$. The problem reduces to minimizing the cost function

$$F(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} P_i(x_1, x_2, \cdots, x_n) \tag{5.3}$$

subject to constraint

$$G(x_1, x_2, \cdots, x_n) = \sum_{x=1}^{n} (x_i + Q_i) - h = 0. \tag{5.4}$$

The Lagrangian multiplier $\lambda = x_{n+1}$ is introduced in the new function

$$F_1(x_1, x_2, \cdots, x_n, x_{n+1}) = F + x_{n+1} G$$

and a real solution provided one exists, to the system of equations in (5.5) is sought.

$$\frac{\delta F_1}{\delta x_i} = 0, i = 1, 2, \cdots, n+1. \tag{5.5}$$

This is equivalent to determining a real solution, provided one exists, of the multinomial in (5.6)

$$V(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n+1} \left\{ \frac{\delta F_1(x_1, x_2, \cdots, x_{n+1})}{\delta x_i} \right\}^2 = 0. \tag{5.6}$$

The question of existence of a real solution can be settled by investigating into whether or not $V(x_1, x_2, \cdots, x_{n+1})$ or its negative is globally positive. In case $V(x_1, x_2, \cdots, x_{n+1})$ is found to be not sign definite, the question of constructing a real solution arises. It may be noted that physical problems of the type just considered involve in the mathematical formulation a multinomial having quite a few variables each of relatively low degree m_i, where m_i is the degree in x_i. In problems of different physical origin like those encountered in stability studies of bidimensional recursive filters in image processing or in realizability theory of networks composed of commensurate transmission lines and lumped reactances, the number of variables n is usually small while the magnitude of m_i in the polynomials associated in the characterizations can run quite high.

B. Nonlinear System Characterization via Volterra Series

The application of multidimensional transform methods in the analysis of continuous nonlinear systems represented by Volterra functional series is several years old [147], [148]. Analysis with multidimensional z-transforms of nonlinear sampled-data systems, for which the continuous representation is a Volterra functional series or in which the Volterra kernels take a form which is easily transformable is also several years old [149], [150]. Multidimensional power series expansions of multidimensional transforms expressed as rational functions of several complex variables have been used in the identification and synthesis of classes of nonlinear systems [151, pp. 311–313]. References to a number of previous papers on the use of Volterra series in the development of distortion analysis techniques for electronic amplifiers, analytical modelling, identification, and synthesis of classes of nonlinear systems can be found in [152]. The polynomial separability results presented in [64] are applicable to the synthesis algorithm suggested in [152]. The identification problem for a class of nonlinear systems composed of certain interconnections of stable linear systems and integer power nonlinearities have been recently considered [153], [155]. A realization algorithm has been provided for a class of nonlinear systems composed of linear dynamic systems connected in parallel with outputs multiplied in the time domain, using the state variable representations for internal behaviour in conjunction with the

Volterra multidimensional transfer function representation of the input-output property [154]. It may be noted that the algorithm proposed does not in general lead to a minimal realization and that though the class of nonlinear systems, assumed to be stable, are completely identifiable from steady state measurement of responses to two-tone or two-frequency inputs, a more satisfactory upper bound on the number of such two-tone inputs required should be obtainable.

C. Stiff Differential Systems

In order to exploit the stiff property satisfied by the differential equations characterizing most practical circuit and system problems, scientists have given considerable emphasis to the study of stiff differential systems over the past several years [156]. In a program designed to solve a set of ordinary differential equations with prescribed initial conditions on a digital computer, the computation time, which is directly proportional to the number of integration steps, is considerably reduced if the integration formula implemented enjoys a particular kind of numerical stability, termed A-stability [157], a condition which is well suited to match the stiff property of certain differential equations. A comprehensive discussion of A-stability and multistep methods can be found in the article by Bickart and Rubin contained in [156]. Also the article [158] is of interest in this context. A new approach to the synthesis of stiffly stable linear multistep formulas, based on the concepts of positive real functions and maximally flat approximations at infinity was presented by Genin in an important paper [159]. In [159], a canonical fraction, which is single variable real rational function, was associated with each linear multistep formula and a necessary and sufficient condition for A-stability of this formula was expressed in terms of positive realness of the associated canonical fraction. Furthermore, the canonical fraction of an optimal linear multistep formula was claimed to be a maximally flat approximation at infinity of a logarithmic function. A significant feature of the approach just mentioned is that the canonical fraction permits a complete decoupling of the problem of accuracy from the problem of stability with subsequent simplification of both problems. In a more recent report [160] Genin extended the canonical fraction concept to that of a polynomial in two complex variables called a canonical polynomial, which can be associated with linear any multistep integration formula containing derivatives of any order [161]. In particular an algebraic criterion for A-stability is arrived at for a linear multistep-multiderivative formula in terms of the properties of a canonical polynomial $Q(p_1, p_2)$, summarized in (5.1) below

$$Q(p_1, p_2) \neq 0 \quad \text{Re } p_1 \geqslant 0 \quad \text{Re } p_2 > 0 \tag{5.1a}$$

$$Q(p_1, p_2) \neq 0 \quad \text{Re } p_1 > 0 \quad \text{Re } p_2 \geqslant 0. \tag{5.2a}$$

A polynomial in two complex variables satisfying the conditions in (5.1) is called a Hurwitz polynomial in a narrow sense and a procedure to test whether or not a polynomial belongs to this class is considered in [162]. Other equivalent formulations in terms of positivity of algebraic functions are considered in [160]. It is worthwhile to investigate into alternate tests possibly simpler from computational standpoint, for the verification of the conditions in (5.1). It may be possible to formulate alternate tests along lines resembling the tests in [14] or [21] which were given for verification of a slightly different condition for $Q(p_1, p_2)$ viz. $Q(p_1, p_2) \neq 0$, in $\text{Re } p_1 \geqslant 0$, $\text{Re } p_2 \geqslant 0$.

D. Miscellaneous Areas

In this subsection brief references will be made to other areas, where applications of the ideas in this paper have recently been made. In [163], the application of decision methods, described in Section II and in [15] have been made to minimal-order observer design. In [164], two equivalent sets of necessary and sufficient conditions for the existence of an asymptotically stable partial realization (in the determination of minimal stable realization from partially specified Markov parameters) are presented, and the conditions are in a form where methods of elementary decision algebra become applicable. In both the above papers, it appears that adequate attention has not been given to the computational problems, when those appear to be intractable. Attention to computational algorithms employing the methods of elementary decision algebra has been given in [54], [117], and [165]–[167]. As opposed to the exact computational algorithms developed in [54] and [117], many allied problems in system theory can be translated into multidimensional optimization problems where use of the numerous optimization procedures including those based on the gradient of a function or direct random search and search region contraction become applicable. Though computational problems often tend to become unmanageable some work on the solution of the output feedback stabilization and related problems via stochastic optimization has been reported [176].

In addition to the papers [42], [43] cited in Section III, the scope for use of rational functions in complex as well as in real variables occur in the symbolic analysis problems of analog and digital circuits [177], [179], as well as in stability problems of active linear systems [178].

The need for a 2-D rational approximation of a signal spectrum subject to stability constraints of the 2-D recursive filter occur in Markov random field image modeling problems [180, p. 597]. A discussion of difficulties as well as possibilities of extending 1-D linear filtering results to the corresponding problem for multidimensional fields can be found in [181], where the need of the two-parameter martingale calculus [182] with its associated structural richness has been mentioned.

VI. Conclusions and Future Recommendations

This paper presents aspects of a broad class of multidimensional system theory problems which are characterizable by rational functions or matrices in several complex (including the special case of real) variables. To keep this paper within reasonable size, some areas have received more emphasis in the paper than others. In spite of a reasonably large list of references, no attempt has been made here to compile a dictionary of all or most publications in the subject. On the contrary, only those items which have been directly discussed in the paper are referenced, and previous survey papers, books, dissertations, or reports which contain bibliographic materials relevant to the present context, are merely identified. It is evident that the subject of multidimensional system theory provides an arena for application of some of the difficult but fascinating branches of mathematics including function theory of several complex or real variables, decidability theory, algebraic geometry, theory of approximations, and abstract algebra.

Almost throughout the paper the effect of nonessential singularities, especially of the second kind whenever they exist, is brought out. Unlike in the single variable case, where a transfer function has isolated poles which can be extracted via partial fraction expansion, the nonessential singularities of a rational function in n-variables as in (2.1) have as their locus whole ($2n$-2)-dimensional analytic manifold (or manifolds) or whole ($2n$-4)-dimensional analytic manifold (or manifolds), depending upon whether the nonessential singularities are of the first or second kinds. Even though the number of such monogenic analytic configurations which course the neighborhood of a given zero of $Q(p_1, p_2, \ldots, p_n)$ is finite, extraction in general presents a problem. Other properties discussed in Section II-A include the fact that analytic functions of several complex variables are far less capable of adapting themselves to a preassigned region of definition than is the case with the functions of a single variable. In Section II-B the important fact that functions in several variables fail to satisfy, in general, the Haar condition is brought out. This is almost as bad a fundamental curse as that in the factorization problem for $n > 1$, in *Assertion* 2.1. In spite of these fundamental drawbacks, the strides that have been taken (or are yet to be taken) are referred to in the proper context in the preceding sections. One of the implications of the decidability theories, alluded to in Section II-C, that several difficult problems can be solved in a finite number of steps via rational operations has opened up unlimited scope for research into the search for more efficient algorithms.

In particular, the first steps [117] that have been very recently taken towards the simultaneous construction and implementation of a multivariable polynomial global positivity test algorithm, though very encouraging, suffer from the necessity of prohibitive computer storage requirements which limits its applications to polynomials of not too high degree and containing not too many variables. On the other hand, a large number of engineering problems like those associated with generator power-sharing in a complex power grid or in decentralized control of large scale systems, have different requirements—the polynomials in the relevant mathematical formulation or characterization having usually a large number of variables and being, in general, of relatively low degree. It appears that research towards the desired goal via the use of powerful modular methods [9], [173]–[175], is worthy of investigation. It is suggested, as a first step, that the implementation of the multivariable polynomial global positivity test algorithm in [117], modulo irreducible polynomials, and feasibility of exact evaluation of determinants of multivariable or multidimensional polynomial matrices (with elements which are multivariable polynomials having integer coefficients) modulo irreducible polynomials be investigated, to speed up calculation and possibly reduce storage. Each of the elements of the matrix should be represented modulo a set of irreducible polynomials using the multidimensional version of Euclid's algorithm [2], and thus several matrices, each represented modulo an irreducible polynomial, will be obtained. The determinants of each of these matrices can then be evaluated simultaneously using parallel processing and the actual determinant of the original matrix may be recovered using an algorithm analogous to the Chinese remainder algorithm. Actually, it is expected that in this way the resultant as well as the subresultants of the inner matrix required in [117] can be obtained from the representations of the matrix modulo a suitable set of irreducible polynomials, without any extra effort except repeated application of interpolation and Chinese remainder theorem. Computations to calculate the greatest common divisor of two bivariate polynomials modulo irreducible single variable polynomials, for example, have been done [173, p. 394]. The method suggested here is expected to have similar advantages to methods for determinant computation modulo prime numbers in a finite field. As a second step, it may be possible to develop a new global positivity test after representing the given polynomial modulo a suitable set of irreducible polynomials. Preliminary investigation has revealed that formulation of such a direct test using modular methods appears to be feasible, though the computational advantages need to be established. The test appears likely to be conceptually elegant and the computational advantages, it seems, can be exposed by suitable choice of the irreducible polynomials as modulii. Needless to say, such a choice is nonunique as in the case of ordinary prime numbers, and guidelines to obtain an ideal set should be established. Again, the initial stage of representation modulo irreducible polynomials can be implemented via use of the multidimensional version of Euclid's algorithm.

NOMENCLATURE

R	Real number field.
C	Complex number field.
R^n	Cartesian product of n copies of R.
C^n	Cartesian product of n copies of C.
p	$(n \times 1)$ column vector of variables $(p_1, p_2, \cdots, p_n)$.
$\|F(p)\|$	Absolute value of $F(p)$.
$\mathrm{Re}\, p = 0$	$\mathrm{Re}\, p_1 = \mathrm{Re}\, p_2 = \cdots = \mathrm{Re}\, p_n = 0$.
$[L(p)]^t$	Transpose of matrix $L(p)$.
$A_*(p)$	$[A(-p)]^t$.
$\|z\| = 1$	$\|z_1\| = \|z_2\| = \cdots = \|z_n\| = 1$.
p^*	Complex-conjugate of p.
$\mathrm{Re}\, Z(p)$	$Z(p) + [Z^*(p)]^t$.

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