Modelling and structural properties of distributed parameter wind power systems

2016 CIRM ANR MsDos workshop

Hugues Mounier†
in collaboration with Luca Greco†

†L2S, Université Paris Sud, Orsay

Luminy, October, 2016

Outline

- Modelling
- Associated I/O systems
- Differential flatness and associated parametrization

DPS elaboration

- Consider a transmission line and a series of generators
- Generation G_i and power angle change δ_i is continuously distributed over the spatial dimension
- Rotor dynamics of the i^{th} generator:

$$\left(\frac{2H_i}{\Omega_s}\right) G_i\ddot{\delta}_i + \xi\dot{\delta}_i = P_i \tag{1}$$

with

- H_i: inertia constant
- Ω_s: electrical ferquency with 60Hz base
- P_i : real power flowing out the i^{th} machine
- ξ: damping coefficient

DPS elaboration (cont.)

- Real power flow from node i to node $i+1$ over a lossless line

$$P_{i,i+1} = \frac{E_i E_{i+1} \sin(\delta_i - \delta_{i+1})}{x_i}$$

with E_i the voltage magnitude at bus i.

- With small δ_i and $E_i = 1$, one gets

$$P_i = P_{i+1,i} - P_{i,i+1} \frac{(\delta_{i-1} - \delta_i)(\delta_i - \delta_{i+1})}{x_i}$$

- By substitution, one obtains

$$\frac{2}{\Omega_i} \frac{H_i}{\Delta L} \ddot{\delta}_i + \frac{\xi}{\Delta l} \dot{\delta}_i = \frac{\Delta L}{x_i} \frac{\delta_i - \delta_{i-1}}{(\Delta L)^2} - \frac{\Delta L}{x_i} \frac{\delta_i - \delta_{i+1}}{(\Delta L)^2}$$

DPS elaboration (cont.)

- Taking the limit $\Delta L \to 0$, and setting

$$H_T = \frac{1}{L}\int_0^L dH(z) = \frac{H(L)}{L}, \quad \gamma = \frac{x(L)}{L}, \quad \eta = \frac{\xi(L)}{L}$$

yields, with $\nu = \sqrt{377/2H_T G_T \gamma}$

$$\partial_t^2 \delta(z,t) + \eta\, \partial_t \delta(z,t) = \nu^2\, \partial_z^2 \delta(z,t) \tag{2}$$

- The corresponding power flow is

$$P(z,t) = -\frac{1}{\gamma}\, \partial_z \delta(z,t)$$

- This type of model has been used to take into account inter area oscillation phenomena.

Various boundary conditions

- Adding power injection to the previous model leads to

$$\partial_t^2 \delta(z,t) + \eta\, \partial_t \delta(z,t) - \nu^2\, \partial_z^2 \delta(z,t) = W(z,t) \qquad (3)$$

wiht boundary conditions

$$P(0,t) = P(1,t) = 0, \quad \text{or} \quad \partial_z \delta(0,t) = \partial_z \delta(1,t) = 0$$

Various boundary conditions (cont.)

- A first model, used in [1], is a point source injection

$$W(u,t) = \rho P_g(t)\bar{\delta}(z-\alpha)$$

where $\bar{\delta}$ denotes the delta Dirac distribution, and P_g the net power injected.

- Another possible model is a power flow injection

$$W(u,t) = -\gamma P_g(t)\bar{\delta}'(z-\alpha)$$

with $\bar{\delta}'$ is the Dirac's derivative, in the distributional sense.

Various boundary conditions (cont.)

- The previous model (3) with point source injection

$$\partial_t^2\delta(z,t) + \eta\partial_t\delta(z,t) - \nu^2\partial_z^2\delta(z,t) = \rho P_g(t)\bar{\delta}(z-\alpha)$$

is equivalent to the following model

$$
\begin{aligned}
\text{For } z \in [0,\alpha], \quad & \partial_t^2\delta^-(z,t) + \eta\,\partial_t\delta^-(z,t) - \nu^2\,\partial_z^2\delta^-(z,t) = 0 \\
& \partial_z\delta^-(0,t) = 0 && \text{(4a)} \\
& \delta^-(\alpha,t) = \rho P_g(t) && \text{(4b)} \\
\text{For } z \in [\alpha,L], \quad & \partial_t^2\delta^+(z,t) + \eta\,\partial_t\delta^+(z,t) - \nu^2\,\partial_z^2\delta^+(z,t) = 0 \\
& \delta^+(\alpha,t) = \rho P_g(t) && \text{(4c)} \\
& \partial_z\delta^+(L,t) = 0 && \text{(4d)} \\
\text{At } z = \alpha, \quad & \partial_t\delta^-(\alpha,t) = \partial_t\delta^+(\alpha,t) && \text{(4e)}
\end{aligned}
$$

Various boundary conditions (cont.)

- The previous model (3) with power flow injection

$$\partial_t^2\delta(z,t) + \eta\partial_t\delta(z,t) - \nu^2\partial_z^2\delta(z,t) = -\gamma P_g(t)\bar{\delta}'(z-\alpha)$$

is equivalent to the following model

$$\begin{aligned}
\text{For } z \in [0,\alpha], \quad & \partial_t^2\delta^-(z,t) + \eta\,\partial_t\delta^-(z,t) - \nu^2\,\partial_z^2\delta^-(z,t) = 0 \\
& \partial_z\delta^-(0,t) = 0 && \text{(5a)} \\
& \partial_z\delta^-(\alpha,t) = -\gamma P_g(t) && \text{(5b)} \\
\text{For } z \in [\alpha,L], \quad & \partial_t^2\delta^+(z,t) + \eta\,\partial_t\delta^+(z,t) - \nu^2\,\partial_z^2\delta^+(z,t) = 0 \\
& \partial_z\delta^+(\alpha,t) = -\gamma P_g(t) && \text{(5c)} \\
& \partial_z\delta^+(L,t) = 0 && \text{(5d)} \\
\text{At } z = \alpha, \quad & \delta^-(\alpha,t) = \delta^+(\alpha,t) && \text{(5e)} \\
& \partial_t\delta^-(\alpha,t) = \partial_t\delta^+(\alpha,t) && \text{(5f)}
\end{aligned}$$

Point source model general solution

- Let us consider the point source model for $z \in [0, \alpha]$

$$\partial_t^2 \delta^-(z,t) + \eta\, \partial_t \delta^-(z,t) - \nu^2\, \partial_z^2 \delta^-(z,t) = 0 \qquad (6a)$$
$$\partial_z \delta^-(0,t) = 0 \qquad (6b)$$
$$\delta^-(\alpha,t) = \rho P_g(t) \qquad (6c)$$

- The temporal Laplace transform of (6) yields

$$s^2 \hat{\delta}^-(z,s) + \eta s \hat{\delta}^-(z,s) - \nu^2 \partial_z^2 \hat{\delta}^-(z,s) = 0$$
$$\partial_z \hat{\delta}^-(0,s) = 0$$
$$\hat{\delta}^-(\alpha,s) = \rho \hat{P}_g(s)$$

Point source model general solution (cont.)

- Freezing s leads to an ODE in space:

$$s^2\hat{\delta}^-(z) + \eta s\hat{\delta}^-(z) - \nu^2 \frac{d\hat{\delta}^-}{dz^2}(z) = 0 \tag{7}$$

$$\frac{d\hat{\delta}^-}{dz}(0) = 0, \qquad \hat{\delta}^-(\alpha) = \rho \hat{P}_g(s) \tag{8}$$

where we have kept the symbol δ^- by abuse of notation.

Point source model general solution (cont.)

- The general solution of the previous ODE is investigated through the characteristic equation in ξ:

$$s^2 + \eta s - \nu^2 \xi^2 = 0$$

yielding, with $\sigma = 1/\nu$:

$$\xi = \pm\, \sigma \sqrt{s^2 + \eta s}$$

- Thus, the general solution of (7) is

$$\hat{\delta}^-(z) = e^{\sigma z \sqrt{s^2 + \eta s}}\, \hat{\lambda}_1 + e^{-\sigma z \sqrt{s^2 + \eta s}}\, \hat{\lambda}_2$$

Weak damping case free boundary solution

- For simplicity's sake, consider the weak damping case: $\eta \ll 1$

$$\xi = \pm \sigma s \sqrt{1 + \frac{\eta}{s}} = \pm \sigma s \left(1 + \frac{\eta}{2s}\right) + o(\eta)$$
$$= \pm \sigma s + \frac{\sigma \eta}{2} + o(\eta)$$

- And we shall consider the approximation, where $\zeta = \eta/2$:

$$\xi \approx \pm \sigma s + \sigma \zeta$$

Weak damping case free boundary solution (cont.)

- which corresponds to the following characteristic equation:

$$-\nu^2\xi^2 + (s+\zeta)^2 = -\nu^2\xi^2 + s^2 + 2\zeta s + \zeta^2$$

  or to the following PDE

$$\partial_t^2\delta^- + 2\zeta\,\partial_t\delta^- + \zeta^2\,\delta^- - \nu^2\,\partial_z^2\delta^- = 0 \tag{9}$$

  which can also be considered as a model to exhibit inter area oscillations.

- Then, taking

$$\xi = \pm\sigma s + \sigma\zeta$$

Weak damping case free boundary solution (cont.)

- The general solution of (9) is

$$\begin{aligned}\hat{\delta}^-(z,s) &= e^{\sigma z(s+\zeta)}\hat{\mu}_1(s) + e^{-\sigma z(s+\zeta)}\hat{\mu}_2(s) \\ &= e^{\sigma z\zeta}e^{\sigma zs}\hat{\mu}_1(s) + e^{-\sigma z\zeta}e^{-\sigma zs}\hat{\mu}_2(s) \qquad (10)\end{aligned}$$

- Note that the solution of the undamped wave equation is

$$e^{\sigma zs}\hat{\mu}_1(s) + e^{-\sigma zs}\hat{\mu}_2(s)$$

  which corresponds to the D'Alembert solution (superposition of incoming and outgoing waves).

- The temporal form of (10) is ($e^{\sigma z\zeta}$, $e^{-\sigma z\zeta}$ being close to 1):

$$\delta^-(z,t) = e^{\sigma z\zeta}\mu_1(t+\sigma z) + e^{-\sigma z\zeta}\mu_2(t-\sigma z) \qquad (11)$$

Weak damping case free boundary solution (cont.)

- The functions μ_1 and μ_2 are determined through the boundary conditions.
- Note that the analysis which follows could also have been conducted without the assumption $\zeta \ll 1$.
- The only difference is that the associated operators are more involved, being distributed instead of point delays.
- The assumption has been kept for pedagogical reasons.

Boundary value problem solution

- The general solution (9) is rewritten as

$$\hat{\delta}^{-}(z,s) = C_z^{-}(s)\hat{\mu}_1(s) + S_z^{-}(s)\hat{\mu}_2(s), \qquad \text{where}$$

$$C_z^{-}(s) = \cosh\left(\sigma z\left(s+\zeta\right)\right), \qquad S_z^{-}(s) = \frac{\sinh(\sigma z\left(s+\zeta\right))}{\sigma\left(s+\zeta\right)}$$

- The sole advantage of using these operators is that:

$$C_0^{-}(s) = 1, \qquad S_0^{-}(s) = 0$$

which simplifies the boundary conditions expressions.

Boundary value problem solution (cont.)

- The spatial derivatives of C_z^- and S_z^- are:

$$\partial_z C_z^-(s) = (\sigma s + \zeta)^2 S_z^-(s), \quad \partial_z S_z^-(z) = C_z^-$$

- And the spatial derivative of $\hat{\delta}^-$ is

$$\partial_z \hat{\delta}^-(z,s) = (\sigma s + \zeta)^2 S_z^- \hat{\mu}_1 + C_z^- \mu_2$$

- The boundary conditions of the point source model (6)

$$\begin{aligned} \partial_t^2 \delta^-(z,t) + \eta\, \partial_t \delta^-(z,t) - \nu^2\, \partial_z^2 \delta^-(z,t) &= 0 \\ \partial_z \delta^-(0,t) &= 0 \\ \delta^-(\alpha,t) &= \rho P_g(t) \end{aligned}$$

Boundary value problem solution (cont.)

are then expressed as

$$\partial_z \hat{\delta}^-(0,s) = \hat{\mu}_2 = 0$$
$$\hat{\delta}^-(\alpha,s) = C_\alpha^- \hat{\mu}_1 + S_\alpha^- \mu_2 = C_\alpha^- \hat{\mu}_1 = \rho \hat{P}_g(s)$$

- And the general solution is

$$\hat{\delta}^-(z,s) = C_z^- \hat{\mu}_1$$

Power flow model general solution

- Let us consider the power flow model for $z \in [0, \alpha]$

$$\partial_t^2 \delta^-(z,t) + \eta\, \partial_t \delta^-(z,t) - \nu^2\, \partial_z^2 \delta^-(z,t) = 0 \tag{12a}$$

$$\partial_z \delta^-(0,t) = 0 \tag{12b}$$

$$\partial_z \delta^-(\alpha,t) = -\gamma P_g(t) \tag{12c}$$

- The general solution remains the same as for the point source injection model

$$\hat{\delta}^-(z,s) = C_z^-(s)\hat{\mu}_1(s) + S_z^-(s)\hat{\mu}_2(s), \qquad \text{recalling that}$$

$$C_z^-(s) = \cosh\left(\sigma z\,(s+\zeta)\right), \qquad S_z^-(s) = \frac{\sinh(\sigma z\,(s+\zeta))}{\sigma\,(s+\zeta)}$$

Boundary value problem solution

- The boundary conditions of this power flow model (12)

$$\begin{aligned}
&\partial_t^2\delta^-(z,t) + \eta\,\partial_t\delta^-(z,t) - \nu^2\,\partial_z^2\delta^-(z,t) = 0 \\
&\partial_z\delta^-(0,t) = 0 \\
&\partial_z\delta^-(\alpha,t) = -\gamma P_g(t)
\end{aligned}$$

are then expressed as

$$\begin{aligned}
\partial_z\hat{\delta}^-(0,s) &= \hat{\mu}_2 = 0 \\
\partial_z\hat{\delta}^-(\alpha,s) &= (\sigma s+\zeta)^2 S_\alpha^-\,\hat{\mu}_1 + C_\alpha^-\,\mu_2 \\
&= (\sigma s+\zeta)^2 S_\alpha^-\,\hat{\mu}_1 = -\gamma\hat{P}_g(s)
\end{aligned}$$

- And the general solution is

$$\hat{\delta}^-(z,s) = C_z^-\,\hat{\mu}_1$$

Point source I/O system

- Recalling the controlled boundary condition and general solution for the point source model:

$$\hat{\delta}^-(\alpha, s) = C_\alpha^- \hat{\mu}_1 = \rho \hat{P}_g(s)$$
$$\hat{\delta}^-(z, s) = C_z^- \hat{\mu}_1$$

- Multiplying the first equation by C_z^- and the second by C_α^- yields the I/O system

$$C_\alpha^-(s)\,\hat{\delta}^-(z, s) = \rho C_z^-(s)\,\hat{P}_g(s)$$

- Or, what is the same

$$\cosh(\sigma\alpha(s+\zeta))\,\hat{\delta}^-(z, s) = \cosh(\sigma z(s+\zeta))\,\hat{P}_g(s)$$

Point source I/O system (cont.)

- With

$$\begin{aligned}\cosh(\sigma z(s+\zeta)) &= \frac{1}{2}\left(e^{\sigma\zeta z}e^{\sigma zs} - e^{-\sigma\zeta z}e^{-\sigma zs}\right)\\ &= \frac{1}{2}\left(\beta_{-z}\hat{\Delta}_{-z} - \beta_z\hat{\Delta}_z\right)\end{aligned}$$

- with

The damping term	$\beta_z = e^{-\sigma\zeta z}$
The delay	$\hat{\Delta}_z = e^{-\sigma zs}$

Point source I/O system (cont.)

- The I/O system

$$C_\alpha^-(s)\,\hat{\delta}^-(z,s) = \rho C_z^-(s)\,\hat{P}_g(s)$$

is then rewritten as

$$\left(\beta_{-\alpha}\Delta_{-\alpha} + \beta_\alpha\Delta_\alpha\right)\,\delta^-(z,t) = \rho\left(\beta_{-z}\Delta_{-z} + \beta_z\Delta_z\right)\,P_g(t) \tag{13}$$

- This is an I/O system between any point z of the line and the control.

- Thus, the Distributed system is viewed as the collection of the previous systems for all $z \in [0,\alpha]$.

Point source I/O system (cont.)

- By multiplication of β_α and action of Δ_α:

$$(1 - \beta_{2\alpha}\Delta_{2\alpha})\ \delta^-(z,t) = \rho\left(\beta_{\alpha-z}\Delta_{\alpha-z} + \beta_{\alpha+z}\Delta_{\alpha+z}\right)\ P_g(t)$$

- which, in developed form, is given by

$$\delta^-(z,t) = \beta_{2\alpha}\delta^-(z,t-2\sigma\alpha) + \rho\Big[\beta_{\alpha-z}P_g(t-\sigma(\alpha-z)) + \beta_{\alpha+z}P_g(t-\sigma(\alpha+z))\Big] \quad (14)$$

- This system is purely a difference equation, i.e. it has no dynamics as a delay system.

Power flow I/O system

- Recalling the controlled boundary condition and general solution of the power flow system:

$$\partial_z \hat{\delta}^-(\alpha, s) = (\sigma s + \zeta)^2 S_\alpha^- \hat{\mu}_1 = -\gamma \hat{P}_g(s)$$
$$\hat{\delta}^-(z, s) = C_z^- \hat{\mu}_1$$

- Taking cross products of the operators $(\sigma s + \zeta)^2 S_\alpha^-$ and C_z^- yields the I/O system

$$(\sigma s + \zeta)^2 S_\alpha^- \, \hat{\delta}^-(z, s) = -\gamma C_z^-(s) \, \hat{P}_g(s)$$

- which can be rewritten as

$$(\beta_{-\alpha}\Delta_{-\alpha} + \beta_\alpha \Delta_\alpha) \left(\sigma \dot{\delta}^- - \zeta \delta^-\right) = -\gamma \left(\beta_{-z}\Delta_{-z} + \beta_z \Delta_z\right) P_g \quad (15)$$

Power flow I/O system (cont.)

- By multiplication of β_α and action of Δ_α:

$$(1-\beta_{2\alpha}\Delta_{2\alpha})\left(\sigma\dot{\delta}^- - \zeta\delta^-\right) = -\gamma\left(\beta_{\alpha-z}\Delta_{\alpha-z} + \beta_{\alpha+z}\Delta_{\alpha+z}\right) P_g$$

- Or, in developed form:

$$\sigma\dot{\delta}^-(z,t) = \sigma\beta_{2\alpha}\dot{\delta}^-(z,t-2\sigma\alpha) + \zeta\delta^-(z,t) - \tag{16}$$

$$\zeta\beta_{2\alpha}\delta^-(z,t-2\sigma\alpha) - \gamma\Big[\beta_{\alpha-z}P_g(t-\sigma(\alpha-z)) + \beta_{\alpha+z}P_g(t-\sigma(\alpha+z))\Big] \tag{17}$$

- This system is a differential difference equation, more precisely it is a neutral delay system.

Module

Definition

A ring $(R, +, .)$ is a group $(R, +)$ with distributivity of multiplication wrt addition

$$\begin{array}{rrl} \forall r_1 \in R, \exists r_2 \in R, & r_1 + r_2 = 0, & \text{inverse for } + \\ \exists e \in R, \forall r \in R, & r + e = e + r = r, & \text{élt. neutre for } + \\ \exists \epsilon \in R, \forall r \in R, & r.\epsilon = \epsilon.r = r, & \text{élt. neutre for } . \\ \forall r_1, r_2, r_3 \in R, & r_1.(r_2 + r_3) = r_1.r_2 + r_1.r_3, & \text{ditributivity} \end{array}$$

- Examples : $(\mathbb{R}[x], +, .)$, $(C^\infty, +, *)$
- A field is a ring with inverse for the multilication ($\mathbb{R}$, $\mathbb{R}(x)$)

Module (cont.)

- We consider a commutative ring R with unity elt for . and without zero divisors

Definition

An *R–module* M is a commutative group together with an action on R, i.e. a map $R \times M \to M$, written $(r, m) \mapsto rm$, such that, for all $r, s \in R$ and $m, n \in M$, we have:

$$
\begin{aligned}
r(sm) &= (rs)m && \text{(associativity)} \\
r(m+n) &= rm + rn && \\
(r+s)m &= rm + sm && \text{(distributivity)} \\
1m &= m && \text{(identity)}
\end{aligned}
$$

Module (cont.)

- A module has the same axioms as a vector space, but its scalars are taken in a field instead of in a ring.

Definition

An R-system is a finitely generated R-module.

Module : examples

- Hence one gets less simple properties, since the scalars cannot be necessarily inverted
- Example

$$\dot{y} = Ty + u$$

corresponds to a module over $\mathbb{R}[\frac{d}{dt}]$. Integration is not authorized.

- On the contrary, within a transfer function

$$s\hat{y} = T\hat{y} + u \quad \text{writes} \quad \hat{y} = \frac{1}{s - T}\hat{u}$$

and we have a vector space over $\mathbb{R}(s)$. Any differential equation integration is allowed.

Controllability notions

- An R-system Λ is called R-torsion free (resp. R-projective, R-free) controllable if Λ is torsion free (resp. projective, free).
- An R-module is torsion free if it contains no torsion element, i.e. no element w satisfying $pw = 0$, with $p \in R, p \neq 0$.
- A torsion element satisfies a differential equation not influenced by the input.

Controllability notions (cont.)

- This is impossible in a vector space, since $pw = 0$ implies $w = 0$, p being invertible.
- For example in

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 + u \end{aligned} \qquad \begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

the first system is $\mathbb{R}[\frac{d}{dt}]$-torsion free controllable and the second is not, since x_1 is torsion.

Controllability notions (cont.)

- An R-module is projective if any presentation matrix admits a right inverse.
- For example in

$$\begin{pmatrix} \frac{d}{dt} & -1 & 0 \\ 0 & \frac{d}{dt} - 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} = 0$$

We have

$$\begin{pmatrix} \frac{d}{dt} & -1 & 0 \\ 0 & \frac{d}{dt} - 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Directly related to the existence of Bézout equations.

Controllability notions (cont.)

- An R-module is free if it admits a basis, i.e. a R linearily independant and generator set.
- For example in

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 + u \end{aligned} \qquad \begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

  The first system is $\mathbb{R}[\frac{d}{dt}]$-free controllable, since it admits x_1 as a basis; indeed, $x_2 = \frac{d}{dt}x_1$, $u = -\frac{d}{dt}x_1 + \frac{d^2}{dt^2}x_1$.
  The second is not, since x_1 is torsion; indeed
- R-free (resp. projective) controllability implies R-projective (resp. torsion free) controllability.

Controllability notions (cont.)

- For example, in

$$\frac{d}{dt}y = -y + u$$

we have

$$\hat{y} = \frac{1}{1+s}\,\hat{u}$$

- The corresponding $\mathbb{R}[\frac{d}{dt}]$-module is free, with basis y:

$$u = \left(\frac{d}{dt} + 1\right) y$$

- Enables a very easy trajectory tracking; being given $y_d(t)$, the open loop control $u_d(t)$ is directly given by

$$u_d(t) = \dot{y}_d(t) + y_d(t)$$

Class of systems

- For simplicity's sake, we shall restrict ourselves to
- $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l$ and $\boldsymbol{u} = (u_1, \ldots, u_m)$ (concentrated) s.t. :

$$\partial_x \boldsymbol{w}_i = A_i \boldsymbol{w}_i + B_i \boldsymbol{u}, \quad \boldsymbol{w}_i : \Omega_i \to ({\mathcal{E}'}^*)^2, \quad \boldsymbol{u} \in ({\mathcal{E}'}^*)^m$$
$$A_i \in (\mathbb{R}[s])^{2\times 2},\ B_i \in (\mathbb{R}[s])^{2\times m},\ i \in \{1, \ldots, l\} \tag{18a}$$

where ${\mathcal{E}'}^*$ is a compact support ultradistribution space.

- The matrices $A_1, \ldots, A_l$ have the charcteristic polynomial:

$$\det(\lambda 1 - A_i) = \lambda^2 - \sigma, \quad \sigma = as^2 + bs + c \neq 0, \quad a, b, c \in \mathbb{R}, \quad a \geqslant 0.$$

Class of systems (cont.)

- The intervals Ω_i ($i = 1, \ldots, l$) are open sets of

$$\tilde{\Omega}_i = [x_{i,0}, x_{i,1}], \quad \ell_i = x_{i,1} - x_{i,0} = q_i \ell, \; q_i \in \mathbb{Q}, \; \ell \in \mathbb{R}. \tag{18b}$$

- The boundary conditions are

$$\sum_{i=1}^{l} L_i \boldsymbol{w}_i(0) + R_i \boldsymbol{w}_i(\ell_i) + D\boldsymbol{u} = 0 \tag{18c}$$

where $D \in (\mathbb{R}[s])^{q\times m}$ and $L_i, R_i \in (\mathbb{R}[s])^{q\times 2}$.

- Remark: The study can be extended to any PDE system where the matrices A_i are $\xi \times \xi$, $\xi > 0$, such that the associated characteristic polynomials $\lambda^{\xi} - \sigma(s)$, with $\sigma(s)$ a polynomial of order ξ in s yielding solutions σ_i such that $e^{x\sigma_i}$ corresponding to C^{∞} functions of Ω in a compact support ultradistributions ring $\mathcal{E}'^{*}$.

Cauchy problem solution

- Cauchy problem with initial conds in $x = \xi$

$$\partial_x \boldsymbol{w} = A\boldsymbol{w} + B\boldsymbol{u}, \quad \boldsymbol{w}(\xi) = \boldsymbol{w}_\xi \tag{19}$$

- Joint initial value problem:

$$(\partial_x^2 - \sigma)v(x) = 0, \quad v(0) = v_0, \quad (\partial_x v)(0) = v_1, \tag{20}$$

associated with the characteristic equation
$\det(\lambda 1 - A) = \lambda^2 - \sigma$ avec $\sigma = as^2 + bs + c \neq 0$

- Let S be a non trivial solution of (20) and $C = \partial_x S$
- Let's suppose that S and C correspond to C^∞ functions of Ω in the compact support ultradistributions ring $\mathcal{E}'^*$.

Cauchy problem solution (cont.)

- The unique solution $x \mapsto \Phi(x, \xi)$ of the intial value problem

$$\partial_x \Phi(x, \xi) = A\Phi(x, \xi), \quad \Phi(\xi, \xi) = 1,$$

with 1 designating the identity of $\mathbb{R}^{2\times 2}$, is

$$\Phi(x, \xi) = AS(x - \xi) + 1C(x - \xi). \tag{21}$$

with the characteristic polynomial compagnon matrix A, i.e.,

$$A = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}, \quad \Phi(x, \xi) = \begin{pmatrix} C(x - \xi) & S(x - \xi) \\ \sigma S(x - \xi) & C(x - \xi) \end{pmatrix}, \tag{22}$$

Cauchy problem solution (cont.)

- The solution of the problem associated with the inhomogeneous equation

$$\partial_x \Psi(x,\xi) = A\Psi(x,\xi) + B \tag{23}$$

is obtained through constants variation

- This yields

$$\Psi(x,\xi) = \int_\xi^x \Phi(x,\zeta)d\zeta B. \tag{24}$$

- The general solution of the problem (19) is then

$$\boldsymbol{w}(x) = \Phi(x,\xi)\boldsymbol{w}_\xi + \Psi(x,\xi)\boldsymbol{u}.$$

Cauchy problem solution (cont.)

- or, equivalently

$$\boldsymbol{w}(x) = W(x,\xi)\boldsymbol{c}, \quad W(x,\xi) = \begin{pmatrix} \Phi(x,\xi) & \Psi(x,\psi) \end{pmatrix}, \quad \boldsymbol{c}_\xi = \begin{pmatrix} \boldsymbol{w}_\xi \\ \boldsymbol{u} \end{pmatrix}$$

- The components of the matrix Φ belong to $\mathbb{C}[s, C, S]$
- On the contrary, and after (24), the components of Ψ may contain integrals of S and C.

$$\int_0^x C(\zeta)dx = S(x), \quad \int_0^x S(\zeta)dx = (C(x) - 1)/\sigma.$$

Module associated to the system

- Injecting the solutions of the initial value problem into the boundary conditions, we get

$$\boldsymbol{w}(x) = W_{\boldsymbol{\xi}}(x)\boldsymbol{c}_{\boldsymbol{\xi}}, \quad P_{\boldsymbol{\xi}}\boldsymbol{c}_{\boldsymbol{\xi}} = 0. \tag{25}$$

- Here, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ is *arbitrary but fixed*, $\boldsymbol{c}_{\boldsymbol{\xi}}^T = (\boldsymbol{w}_1^T(\xi_1) \cdots \boldsymbol{w}_l^T(\xi_l), \boldsymbol{u}^T)$,

$$W_{\boldsymbol{\xi}} = \begin{pmatrix} \Phi_1(x,\xi_1) & 0 & 0 & \Psi_1(x,\xi_1) \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & \Phi_l(x,\xi_l) & \Psi_l(x,\xi_l) \end{pmatrix}, \quad P_{\boldsymbol{\xi}} = \left(P_{\boldsymbol{\xi},1} \cdots P_{\boldsymbol{\xi},l+1}\right)$$

Module associated to the system (cont.)

with

$$P_{\boldsymbol{\xi},i} = L_i \Phi_i(0,\xi_i) + R_i \Phi_i(\ell_i,\xi_i), \quad i = 1,\dots,I$$
$$P_{\boldsymbol{\xi},I+1} = D + \sum_{i=1}^{I} L_i \Psi_i(0,\xi_i) + R_i \Psi_i(\ell_i,\xi_i).$$

- The system is represented by a module generated by $\boldsymbol{c}_\xi$ with a presentation given by (30)
- The coefficient ring must contain $W_{\boldsymbol{\xi}}(x)$ and $P_{\boldsymbol{\xi}}$, whose entries are values of C, S and their spatial integrals
- A possible ring is $\mathcal{R}^I_{\mathbb{R}}[s, \mathfrak{S}, \mathfrak{S}^I]$, isomorphic to a subring of $\mathcal{E}'^*$ through inverse Laplace transform

Module associated to the system (cont.)

- For all $\mathbb{X} \subseteq \mathbb{R}$, we denote $\mathcal{R}^I_{\mathbb{X}} = \mathbb{C}[\mathfrak{S}_{\mathbb{X}}, \mathfrak{S}^I_{\mathbb{X}}]$, with

$$\begin{aligned} \mathfrak{S} &= \{C, S\}, & \mathfrak{S}_{\mathbb{X}} &= \{C(z\ell), S(z\ell) | z \in \mathbb{X}\}, \\ \mathfrak{S}^I &= \{C^I, S^I\}, & \mathfrak{S}^I_{\mathbb{X}} &= \{C^I(z\ell), S^I(z\ell) | z \in \mathbb{X}\}, \end{aligned}$$

  ℓ defined as in (18b), and

$$S^I(x) = \int_0^x S(\zeta) d\zeta, \quad C^I(x) = \int_0^x C(\zeta) d\zeta.$$

- To simplify the analysis of the module theoretic properties, we shall use, instead of $\mathcal{R}^I_{\mathbb{R}}$, a slightly larger ring, given by $\mathcal{R}_{\mathbb{R}} = \mathbb{C}(s)[\mathfrak{S}_{\mathbb{R}}] \cap \mathcal{O}$ (where $\mathcal{O}$ designates the entire functions ring in s).

Module associated to the system (cont.)

Definition

The *convolution system* $\Sigma = \Sigma_{\mathbb{R}}$ associated to the boundary problem (18) is the module generated by $\boldsymbol{c}_{\boldsymbol{\xi}}$ over $\mathcal{R}_{\mathbb{R}}$ with $P_{\boldsymbol{\xi}}$ as presentation matrix. The module $\Sigma_{\mathbb{Q}}$ will designate the same system, but over $\mathcal{R}_{\mathbb{Q}}$.

Boundary value system of point source model

- The boundary values of the point source model are

$$\hat{C}_\alpha \hat{\delta}_{p0}^{+} + \hat{S}_\alpha \hat{\delta}_{p0}^{+'} = \hat{C}_\alpha \hat{\delta}_{p0}^{-} \tag{26a}$$

$$\sigma^2 \hat{S}_L \hat{\delta}_{p0}^{+} + \hat{C}_L \hat{\delta}_{p0}^{+'} = 0 \tag{26b}$$

with $\hat{C}_z(s) = \cosh\left(\sigma z\left(s+\zeta\right)\right)$, $\hat{S}_z(s) = \frac{\sinh(\sigma z(s+\zeta))}{\sigma(s+\zeta)}$

- The presentation of $\Lambda_{\mathbb{Q}}^{p}$ is then

$$\begin{pmatrix} -\hat{C}_\alpha & \hat{C}_\alpha & \hat{S}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{p0}^{-} \\ \hat{\delta}_{p0}^{+} \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = 0 \tag{27}$$

viewed as an $\mathcal{R}_{\mathbb{Q}}$-module $\Lambda_{\mathbb{Q}}^{p}$ generated by $= [\hat{\delta}_{p0}^{-}, \hat{\delta}_{p0}^{+}, \hat{\delta}_{p0}^{+'}]_{\mathcal{R}_{\mathbb{Q}}}$.

- where $\mathcal{R}_{\mathbb{Q}} = \mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{R}}] \cap \mathcal{E}'^{*}$, $\mathfrak{S}_{\mathbb{X}} = \{C(z\ell), S(z\ell) | z \in \mathbb{X}\}$ and $\mathcal{E}'^{*}$ a space of Gevrey ultradistributions.

Boundary value system of power flow model

- The boundary values of the power flow model are

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{+} + \hat{C}_\alpha \delta_{f0}^{+'} = \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \tag{28a}$$

$$\sigma^2 \hat{S}_L \hat{\delta}_{f0}^{+} + \hat{C}_L \delta_{f0}^{+'} = 0 \tag{28b}$$

- The presentation of $\Lambda_{\mathbb{Q}}^f$ is then

$$\begin{pmatrix} -\sigma^2 \hat{S}_\alpha & \sigma^2 \hat{S}_\alpha & \hat{C}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{f0}^{-} \\ \hat{\delta}_{f0}^{+} \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = 0 \tag{29}$$

viewed as an $\mathcal{R}_{\mathbb{Q}}$-module $\Lambda_{\mathbb{Q}}^p$ generated by $= [\hat{\delta}_{p0}^{-}, \hat{\delta}_{p0}^{+}, \hat{\delta}_{p0}^{+'}]_{\mathcal{R}_{\mathbb{Q}}}$.

Controllability of PDE systems

Proposition

The ring $\mathcal{R}_{\mathbb{Q}} = \mathbb{C}(s)[\mathfrak{S}_{\mathbb{Q}}] \cap \mathcal{O}$ *is a Bézout domain, i.e., such that any finitely generated ideal is principal.*

- This type of ring can be built as $\widetilde{\mathcal{R}}_{\mathbb{X}} := \mathbb{C}(s)[\widetilde{C}_a, \widetilde{S}_a; a \in \mathbb{X}]/\mathfrak{a}$
- with the ideal $\mathfrak{a}$ generated by $(\sigma \in \mathbb{C}(s), \quad a, b \in \mathbb{X})$

$$\widetilde{C}_a \widetilde{C}_b \pm \sigma \widetilde{S}_a \widetilde{S}_b - \widetilde{C}_{a \pm b},\ \widetilde{S}_a \widetilde{C}_b \pm \widetilde{C}_a \widetilde{S}_b - \widetilde{S}_{a \pm b},\ \widetilde{C}_0 - 1,\ \widetilde{S}_0$$

Controllability of PDE systems (cont.)

Proposition

The convolution system Σ defined by the $\mathcal{R}_{\mathbb{R}}$-module ($\mathcal{R}_{\mathbb{R}} = \mathbb{C}(s)[\mathfrak{S}_{\mathbb{R}}] \cap \mathcal{O}$) generated by $\boldsymbol{c}_\xi$ and admitting

$$P_\xi \boldsymbol{c}_\xi = 0 \quad (\text{avec } \boldsymbol{w}(x) = W_\xi(x)\boldsymbol{c}_\xi) \tag{30}$$

as presentation is free, if, and only if it is torsion free. More generally $\Sigma = \mathrm{t}\Sigma \oplus \Sigma/\mathrm{t}\Sigma$ where $\mathrm{t}\Sigma$ is torsion and $\Sigma/\mathrm{t}\Sigma$ is free.

Controllability results

Theorem (Point source system)

The $\mathcal{R}_{\mathbb{Q}}$-system $\Lambda^p_{\mathbb{Q}}$ is $\mathcal{R}_{\mathbb{Q}}$-free controllable if, and only if, $\hat{C}_{L-\alpha}$ and $\hat{C}_{\alpha}$ have no common zeros in $\mathbb{C}$, i.e. iff

$$\frac{L-\alpha}{\alpha} \neq \frac{1+2k_1}{1+2k_2}, \quad \text{for any} \quad k_1, k_2 \in \mathbb{N}$$

Theorem (Power flow system)

The system $\mathcal{R}^{\sigma}_{\mathbb{Q}} \otimes_{\mathcal{R}_{\mathbb{Q}}} \Lambda^f_{\mathbb{Q}}$ is $\mathcal{R}^{\sigma}_{\mathbb{Q}}$-free controllable if, and only if, $\hat{S}_{\alpha}$ and $\hat{S}_{L-\alpha}$ have no common zeros in $\mathbb{C}$, i.e. iff

$$\frac{L-\alpha}{\alpha} \neq \frac{k_1}{k_2}, \quad \text{for any} \quad k_1, k_2 \in \mathbb{N}$$

Conclusion

- We have examined two possible modelisations for inter area oscillations.
- One with a point source power injection leads to a delay system with no dynamics.
- Another one, with power flow injection, leads to a neutral delay system.
- The first model bears some resemblance with a pure transport equation.
- Whereas the second one exhibits some dynamics probably expected in a wave equation model.
- Both associated modules are free with some conditions.

End

- Thank you for your attention.
- I'll be glad to answer questions, if any.

K.S.T. Magar, M.J. Balas, D.F. Gayme, Adaptive Suppression of Inter-Area Oscillation using Multiple Wind Power Systems in a Distributed Parameter Control Methodology, proc. of *19th IFAC World Congress*, Cape Town, South Africa, 2014.

S. Chandra, D.F. Gayme, A. Chakrabortty, Coordinating Wind Farms and Battery Management Systems for Inter-Area Oscillation Damping: A Frequency-Domain Approach, *IEEE Transactions on Power Systems*, 2014.