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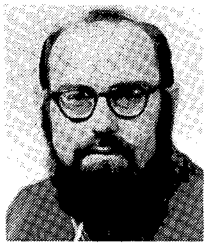

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Some Stability Properties of Two-Dimensional Linear Shift-Invariant Digital Filters

DENNIS GOODMAN

Abstract—**This paper presents a detailed discussion of stability of two-dimensional linear digital filters, and the subtle differences between the one-dimensional and two-dimensional cases. In particular, it is shown that the fact that the impulse response trails off to zero, or more stringently is square summable does not guarantee BIBO (bounded-input–bounded-output) stability.**

Necessary conditions for the impulse response to be bounded and sufficient conditions for it to be square summable and to approach zero geometrically along any fixed column (or row) are stated.

I. Introduction

THE PURPOSE of this paper is to discuss certain stability properties of two-dimensional linear shift invariant digital filters. Most of these properties have no analogs in the one-dimensional case. The effect of the numerator polynomial on stability will be discussed, and an example in which the necessity of Shank's theorem fails will be presented. Two other examples which exhibit behavior different from the one-dimensional case will also be presented; it will be shown that the fact that the impulse response trails off to zero or, more stringently, is square summable, does not guarantee BIBO stability. Lastly, a necessary condition for the impulse response to be bounded, a sufficient condition for the impulse response to be square summable, and a sufficient condition for the impulse response to approach zero geometrically along any fixed column (or row) will be stated.

II. Preliminaries

We will use the symbols ∃, ∋, and ∀ to stand for "there exists," "such that," and "for each," respectively.

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We will be concerned with two-dimensional linear shift invariant filters which are causal (i.e., have a first quadrant impulse response) and have a rational transfer function. Hence our transfer functions will be of the form

$$G(z_1,z_2)=\frac{P(z_1,z_2)}{Q(z_1,z_2)}$$

where $P(z_1,z_2)$ and $Q(z_1,z_2)$ are two variable polynomials in z_1 and z_2. We will assume that $Q(0,0)\neq 0$ so that $Q(z_1,z_2)\neq 0$ in some neighborhood $U_\epsilon^2 \triangleq \{(z_1,z_2):|z_1|<\epsilon, |z_2|<\epsilon\}$ of (0,0), hence in U_ϵ^2 the function $G(z_1,z_2)$ is analytic and has power series expansion

$$G(z_1,z_2)=\sum_{m,n=0}^{\infty} g_{mn} z_1^m z_2^n$$

g_{mn} is defined to be the impulse response of $G(z_1,z_2)$, and it is well known that the digital filter represented by $G(z_1,z_2)$ is bounded-input–bounded-output (BIBO) stable iff $\{g_{mn}\}\in l_1$, i.e., $\sum_{m,n=0}^{\infty}|g_{mn}|<\infty$. We will say the impulse response is square summable if $\{g_{mn}\}\in l_2$, i.e., $\sum_{m,n=0}^{\infty}|g_{mn}|^2<\infty$, and we will say the impulse response is bounded if for some finite M we have $|g_{mn}|<M\ \forall m,n$.

We define $U^2 \triangleq \{(z_1,z_2):|z_1|<1,\ |z_2|<1\}$ to be the open unit bidisc, $\bar{U}^2 \triangleq \{(z_1,z_2):|z_1|\leqslant 1, |z_2|\leqslant 1\}$ to be the closed unit bidisc, and $T^2=\{(z_1,z_2):|z_1|=1, |z_2|=1\}$ to be the distinguished boundary of the unit bidisc. Note that T^2 is a proper subset of the boundary of the unit bidisc.

Certain properties of two variable polynomials and rational functions which will be needed later will now be discussed. It is well known that a two variable polynomial is not in general factorable into first-order polynomials; rather, a two variable polynomial can be factored into irreducible factors which are themselves two variable polynomials but which cannot be further factored [1]. (Of course a given polynomial may itself be irreducible.) These irreducible polynomials are unique up to multiplicative constants [1]. Two polynomials which have no irreducible factors in common are said to be mutually prime. Consider the two variable rational function

$$G(z_1,z_2)=\frac{P(z_1,z_2)}{Q(z_1,z_2)}$$

where $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime. Using the terminology of [2], a point $(z_1,z_2)\ni Q(z_1,z_2)=0$ but $P(z_1,z_2)\neq 0$ will be called a pole or a nonessential singularity of the first kind (such a point is analogous to a pole in the one variable case). A point $(z_1,z_2)\ni Q(z_1,z_2)=P(z_1,z_2)=0$ will be called a nonessential singularity of the second kind (such points have no one variable analog). Clearly, if (z_1,z_2) is a pole, $G(z_1,z_2)=\infty$. If (z_1,z_2) is a nonessential singularity of the second kind, the value of $G(z_1,z_2)$ is undefined and in any open neighborhood Γ of $(z_1,z_2)\exists(z_1^a,z_2^a)\in\Gamma\ni(z_1^a,z_2^a)$ is a pole of $G(z_1,z_2)$. This result may be proved using the argument in [3, pp. 14–15] and follows from the fact that two mutually prime polynomials have at most finitely many common zeros [1].

III. The Effect of the Numerator Polynomial on Stability

Perhaps the most important stability theorem for two-dimensional filters is Shank's theorem which states that $G(z_1,z_2)$ is BIBO stable iff $Q(z_1,z_2)\neq 0\ \forall(z_1,z_2)\in\bar{U}^2$ [4], [5]. Clearly, before applying this theorem, all irreducible factors common to $P(z_1,z_2)$ and $Q(z_1,z_2)$ should first be cancelled to make the numerator and denominator mutually prime; this operation is analogous to cancelling all common poles and zeros in the one-dimensional case. A test for the existence of common factors is given in [6], and an algorithm for extraction of the greatest common factor is given in [7]. Shank's theorem is essentially correct except that cases may arise where $G(z_1,z_2)$ has a nonessential singularity of the second kind on T^2 but $\{g_{mn}\}\in l_1$. Such an example will be given below. For the rest of the paper we will assume that $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime.

The sufficiency of Shank's theorem can be proved easily. If $Q(z_1,z_2)\neq 0$ in $\bar{U}^2$, then by the continuity of $Q(z_1,z_2)$, there exists a slightly larger bidisc $U_{1+\epsilon}^2 \triangleq \{(z_1,z_2):|z_1|<1+\epsilon,\ |z_2|<1+\epsilon\}$ such that $Q(z_1,z_2)\neq 0\ \forall(z_1,z_2)\in U_{1+\epsilon}^2\Rightarrow G(z_1,z_2)$ is analytic in $U_{1+\epsilon}^2$ $\Rightarrow\sum_{m,n=0}^{\infty} g_{mn}z_1^m z_2^n$ is absolutely convergent in $U_{1+\epsilon}^2\Rightarrow$ $\sum_{m,n=0}^{\infty}|g_{mn}|<\infty$. The necessity of Shank's theorem may fail to apply under conditions previously mentioned. The following necessary condition does hold, however.

Theorem 1: If $G(z_1,z_2)$ represents a BIBO stable filter, then $G(z_1,z_2)$ has no poles in $\bar{U}^2$, and no nonessential singularities of the second kind on $\bar{U}^2$ except possibly on T^2.

Proof: If $G(z_1,z_2)$ has a pole $(z_1^a,z_2^a)\in\bar{U}^2$, then $G(z_1,z_2)=\sum_{m,n=0}^{\infty} g_{mn}z_1^m z_2^n$ is not absolutely convergent in $\bar{U}^2\Rightarrow\sum_{m,n=0}^{\infty}|g_{mn}|=\infty$, and so the filter is BIBO unstable. To complete the proof we show that if $G(z_1,z_2)$ has a nonessential singularity of the second kind, (z_1^b,z_2^b), in $\bar{U}^2-T^2$ then it has a pole in $\bar{U}^2$. If $(z_1^b,z_2^b)\in U^2$, the result follows from the argument at the end of Section II, and we are left with the case $(z_1^b,z_2^b)\in(\bar{U}^2-T^2)-U^2=\{(z_1,z_2):|z_1|<1,\ |z_2|=1\}\cup\{(z_1,z_2):|z_1|=1,\ |z_2|<1\}$. In the following argument we assume $|z_1^b|=1$, the argument for $|z_2^b|=1$ is similar. Either $Q(z_1^b,z_2)$ is identically zero as a function of z_2 or it is not. If $Q(z_1^b,z_2)\equiv 0$, then any z_2^c with $|z_2^c|\leqslant 1$ and $P(z_1^b,z_2^c)\neq 0$ is such that $(z_1^b,z_2^c)\in\bar{U}^2$ is a pole of $G(z_1,z_2)$. If $Q(z_1^b,z_2)\not\equiv 0$, then setting $\epsilon=\frac{1}{2}(1-|z_2^b|)$, we have by a continuity argument [1] that $\exists\delta>0\ni$ if z_1^c satisfies $|z_1^c-z_1^b|<\delta$ then $\exists z_2^c\ni|z_2^c-z_2^b|<\epsilon$ and $Q(z_1^c,z_2^c)=0$. Choosing z_1^c to satisfy both $|z_1^c|<1$ and $|z_1^c-z_1^b|<\delta$ and then selecting z_2^c as in the previous sentence, we see that (z_1^c,z_2^c) is in U^2 and is either a pole or a nonessential singularity of the second kind of $G(z_1,z_2)$. In either case it follows that $G(z_1,z_2)$ has a pole in $\bar{U}^2$. Q.E.D.

The special case where $Q(z_1,z_2)\neq 0\ \forall (z_1,z_2)\in \overline{U}^2-T^2$ but $G(z_1,z_2)$ has a nonessential singularity of the second kind on T^2 presents problems: it appears that $G(z_1,z_2)$ may or may not be stable. In particular, both

$$G_1(z_1,z_2) \triangleq \frac{(1-z_1)^8(1-z_2)^8}{2-z_1-z_2} \triangleq \frac{P_1(z_1,z_2)}{Q(z_1,z_2)}$$

and

$$G_2(z_1,z_2) \triangleq \frac{(1-z_1)(1-z_2)}{2-z_1-z_2} \triangleq \frac{P_2(z_1,z_2)}{Q(z_1,z_2)}$$

are transfer functions which have mutually prime numerator and denominator polynomials, and $Q(z_1,z_2)\neq 0$ on $\overline{U}^2$ except at $z_1=z_2=1$. Both $G_1(z_1,z_2)$ and $G_2(z_1,z_2)$ have nonessential singularities of the second kind at $z_1=z_2=1$, but we will show that $G_1(z_1,z_2)$ is BIBO stable and $G_2(z_1,z_2)$ is BIBO unstable. $G_2(z_1,z_2)$ will be considered in Section IV; the stability of $G_1(z_1,z_2)$ is proved below.

Example 1: $G_1(z_1,z_2)$ represents a stable filter. In proving the stability of $G_1(z_1,z_2)$ we will use the notation of [3]; page numbers given below will refer to the appropriate page of [3]. We will give two definitions and prove four lemmas in the course of proving stability. The proof of Lemma 1 is long, tedious, and uninteresting, so it has been relegated to the Appendix.

Definition: $G^*(e^{i\theta_1},e^{i\theta_2}) \triangleq \lim_{r\nearrow 1} G(re^{i\theta_1},re^{i\theta_2})$ (p. 24).

Definition: $G(z_1,z_2)\in H^\infty$ if $G(z_1,z_2)$ is holomorphic in U^2 and $|G(z_1,z_2)|\leq M<\infty\ \forall (z_1,z_2)\in U^2$ (p. 51).

Lemma 1: $(\partial^2/\partial\theta_2^2)(\partial^2/\partial\theta_1^2)G_1^*(e^{i\theta_1},e^{i\theta_2})$ exists and is continuous (see Appendix).

Lemma 2: $G_1(z_1,z_2)\in H^\infty$.

Proof:

$$|G_1(z_1,z_2)| = \left|\frac{(1-z_1)^8(1-z_2)^8}{2-z_1-z_2}\right| = |1-z_1|^7|1-z_2|^7 \frac{1}{\left|\dfrac{1}{1-z_1}+\dfrac{1}{1-z_2}\right|}.$$

Since

$$\mathrm{Re}\left\{\frac{1}{1-z}\right\} > \frac{1}{2} \qquad \forall z\in U \triangleq \{z:|z|<1\}$$

$$\mathrm{Re}\left\{\frac{1}{1-z_1}+\frac{1}{1-z_2}\right\} > 1 \qquad \forall (z_1,z_2)\in U^2$$

$$\Rightarrow \frac{1}{\left|\dfrac{1}{1-z_1}+\dfrac{1}{1-z_2}\right|} < 1 \qquad \forall (z_1,z_2)\in U^2$$

$$\Rightarrow |G_1(z_1,z_2)| \leq 2^7\cdot 2^7\cdot 1 \qquad \forall (z_1,z_2)\in U^2$$

$$= 4^7$$

therefore $G_1(z_1,z_2)\in H^\infty$.

Lemma 3: The Fourier coefficients of $G_1^*(e^{i\theta_1},e^{i\theta_2})$:

$$\hat{g}_{mn} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} G_1^*(e^{i\theta_1},e^{i\theta_2})e^{-im\theta_1}e^{-in\theta_2}\,d\theta_1\,d\theta_2$$

are equal to the power series coefficients of $G_1(z_1,z_2)$, i.e., $\hat{g}_{mn}=g_{mn}\ \forall m,n$.

Proof: By problem c of p. 53 $G_1(z_1,z_2)\in H^\infty$ (by lemma 2)$\Rightarrow G_1(z_1,z_2)$ is the Poisson integral of $G_1^*(e^{i\theta_1},e^{i\theta_2})$. By Theorem 2.1.4 of p. 21, $G_1(z_1,z_2)$ analytic in $U^2 \Rightarrow \hat{g}_{mn}=0$ except for $(m,n)\in z_+ \triangleq \{m,n: m\geq 0,\ n\geq 0\}$. The series representation of the Poisson integral (p. 17) gives

$$G_1(r_1e^{i\theta_1},r_2e^{i\theta_2}) = \sum_{m,n\in z_+} \hat{g}_{mn} r_1^m e^{im\theta_1} r_2^n e^{in\theta_2}$$

substituting $z_1=r_1e^{i\theta_1}$, $z_2=r_2e^{i\theta_2}$

$$G_1(z_1,z_2) = \sum_{m,n=0}^{\infty} \hat{g}_{mn} z_1^m z_2^n$$

therefore $g_{mn}=\hat{g}_{mn}\ \forall m,n$ by uniqueness of power series.

Lemma 4: $\{\hat{g}_{mn}\}\in l_1$.

Proof:

$$\hat{g}_{mn} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left[G_1^*(e^{i\theta_1},e^{i\theta_2})e^{-im\theta_1}\,d\theta_1\right]e^{-in\theta_2}\,d\theta_2.$$

Integrate the inner integral twice by parts (as in [8, p. 157]) to give

$$\hat{g}_{mn} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left(-\frac{1}{m^2}\right) \cdot \frac{\partial^2}{\partial\theta_1^2} G_1^*(e^{i\theta_1},e^{i\theta_2})e^{-im\theta_1}e^{-in\theta_2}\,d\theta_1\,d\theta_2.$$

Change the order of integration and again integrate the inner integral twice by parts to give

$$\hat{g}_{mn} = \frac{1}{m^2n^2}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left[\frac{\partial^2}{\partial\theta_2^2}\frac{\partial^2}{\partial\theta_1^2}G_1^*(e^{i\theta_1},e^{i\theta_2})\right] \cdot e^{-im\theta_1}e^{-in\theta_2}\,d\theta_1\,d\theta_2 = \frac{1}{m^2n^2}\hat{f}_{mn}$$

where $\hat{f}_{mn}$ are the Fourier coefficients of $(\partial^2/\partial\theta_2^2)(\partial^2/\partial\theta_1^2)G_1^*(e^{i\theta_1},e^{i\theta_2})$. By Lemma 1 this function exists and is continuous; it follows from the Riemann–Lebesgue theorem (in [9, p. 301]) that $\hat{f}_{mn} \underset{m,n\to\infty}{\longrightarrow} 0$. Hence $\exists M<\infty \ni$

$$|\hat{f}_{mn}|\leq M \qquad \forall m,n \Rightarrow |\hat{g}_{mn}| \leq \frac{M}{m^2n^2} \qquad \forall m,n.$$

$\Rightarrow \{\hat{g}_{mn}\}\in l_1$. This completes the proof of Lemma 4 and also the proof that $G_1(z_1,z_2)$ is BIBO stable as $\{\hat{g}_{mn}\}\in l_1 \Rightarrow \{g_{mn}\}\in l_1$ by Lemma 3.

IV. Two Counterexamples

The two filters

$$G_2(z_1,z_2)=\frac{(1-z_1)(1-z_2)}{2-z_1-z_2}$$

$$G_3(z_1,z_2)=\frac{2}{2-z_1-z_2}$$

exhibit behavior not found in the one-dimensional case. In particular, it is well known that a one-dimensional filter with rational transfer function $H(z)=\Sigma_{m=0}^{\infty}h_m z^m$ is BIBO stable iff $\{h_m\}\in l_2$ or $\lim_{m\to\infty}h_m=0$. $G_2(z_1,z_2)$ is of interest for three reasons: first, because although it is similar to G_1 in that its only singularity on $\bar{U}^2$ is a nonessential singularity of the second kind at $z_1=z_2=1$, it is BIBO unstable; second, because it serves as a warning to anyone who might mistakenly interpret a nonessential singularity of the second kind on T^2 as a "pole-zero cancellation;" third, because although $\{g_{mn}\}\notin l_1$, $\{g_{mn}\}\in l_2$ (hence also $\lim_{m,n\to\infty}g_{mn}=0$). G_3 is interesting because it shows that impulse response behavior different from the one-dimensional case is not confined to filters having nonessential singularities of the second kind; we will show that the impulse response of $G_3(z_1,z_2)$ trails off to zero.

Example 2:

$$G_2(z_1,z_2)\triangleq\frac{(1-z_1)(1-z_2)}{2-z_1-z_2}$$

is BIBO unstable but has square summable impulse response.

Proof: Writing

$$G_2(z_1,z_2)=\frac{1}{\dfrac{1}{(1-z_1)}+\dfrac{1}{(1-z_2)}}$$

and arguing as in Lemma 2; we see that $|G_2(z_1,z_2)|\leqslant 1$ $\forall(z_1,z_2)\in U^2\Rightarrow G_2\in H^\infty$. Noting that $G_2(z_1,z_2)$ is analytic in U^2 and arguing as in Lemma 3 we see that the power series coefficients g_{mn} of $G_2(z_1,z_2)$ equal the Fourier series coefficients $\hat{g}_{mn}$ of $G_1^*(e^{i\theta_1},e^{i\theta_2})$ for all m,n. Since

$$G_2^*(1,1)=\lim_{r\to1}\frac{(1-r)(1-r)}{2-2r}=\lim_{r\to1}\frac{(1-r)}{2}=0$$

and $G_2(z_1,z_2)$ is continuous everywhere else on T^2

$$G_2^*(e^{i\theta_1},e^{i\theta_2})=\begin{cases}G_2(e^{i\theta_1},e^{i\theta_2}), & (\theta_1,\theta_2)\neq(0,0)\\ 0, & (\theta_1,\theta_2)=0.\end{cases}$$

But taking the limit as $(\theta_1,\theta_2)\to0$ from two different directions

$$\lim_{\theta\to0}G_2^*(e^{i\theta},e^{-i\theta})=\lim_{\theta\to0}\frac{(1-e^{i\theta})(1-e^{-i\theta})}{2-e^{i\theta}-e^{-i\theta}}$$

$$=\lim_{\theta\to0}\frac{2-2\cos\theta}{2-2\cos\theta}=1$$

$$\lim_{\theta\to0}G_2^*(e^{i\theta},e^{i\theta})=\lim_{\theta\to0}\frac{(1-e^{i\theta})(1-e^{i\theta})}{2-e^{i\theta}-e^{i\theta}}$$

$$=\lim_{\theta\to0}\frac{1-e^{i\theta}}{2}=0$$

we conclude $\{\hat{g}_{mn}\}\notin l_1$ and so $G_2(z_1,z_2)$ is not BIBO stable.

We now show $\{g_{mn}\}\in l_2$. For $r<1$

$$G_2(re^{i\theta_1},re^{i\theta_2})=\sum_{m,n=0}^{\infty}g_{mn}(re^{i\theta_1})^m(re^{i\theta_2})^n$$

$$=\sum_{m,n=0}^{\infty}[g_{mn}r^m r^n]e^{im\theta_1}e^{in\theta_2}.$$

Using Parseval's relation (in [9, p. 301]) and the fact that $|G_2(z_1,z_2)|\leqslant1$ $\forall(z_1,z_2)\in U^2$ gives

$$\sum_{m,n=0}^{\infty}|g_{mn}r^m r^n|^2=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|G_2(re^{i\theta_1},re^{i\theta_2})|^2\,d\theta_1\,d\theta_2$$

$$\leqslant1,\qquad r\in[0,1)$$

hence

$$\sum_{m,n=0}^{\infty}|g_{mn}|^2r^{2(m+n)}\leqslant1,\qquad r\in[0,1).$$

Since

$$\sum_{m,n=0}^{M}|g_{mn}|^2r^{2(m+n)}$$

is continuous in r for any M,

$$\sum_{m,n=0}^{M}|g_{mn}|^2r^{2(m+n)}\leqslant1$$

for $r\in[0,1],\forall M$

$$\Rightarrow\sum_{m,n=0}^{M}|g_{mn}|^2\leqslant1\ \forall M\Rightarrow\sum_{m,n=0}^{\infty}|g_{mn}|^2$$

$$\leqslant1\Rightarrow\{g_{mn}\}\in l_2.$$

Example 3:

$$G_3(z_1,z_2)\triangleq\frac{2}{2-z_1-z_2}=\frac{1}{1-\frac{1}{2}z_1-\frac{1}{2}z_2}$$

is BIBO unstable but has an impulse response $\{g_{mn}\}$ such that $\lim_{m,n\to\infty}g_{mn}=0$.

Proof: $G_3(z_1,z_2)$ is BIBO unstable by Theorem 1. The impulse response of such a filter was calculated in [10] using Cauchy's theorem and was shown to be $g_{mn}=(\frac{1}{2})^{m+n}((m+n)!/m!n!)$. Writing $m+n=k$, $g_{mn}=(\frac{1}{2})^k\binom{k}{n}$. Since the binomial coefficients have the properties:

for k even: $\binom{k}{l}\leqslant\binom{k}{k/2}$, $\quad 0\leqslant l\leqslant k$

and

$$\text{for k odd:}\quad \binom{k}{l} \leqslant \binom{k}{(k+1)/2} < \binom{k+1}{(k+1)/2}, \qquad 0 \leqslant l \leqslant k.$$

We may conclude

$$0 < g_{mn} \leqslant \left(\frac{1}{2}\right)^k \frac{k!}{\left(\frac{k}{2}!\right)^2}, \qquad m+n=k,\ k \text{ even}$$

$$0 < g_{mn} \leqslant \left(\frac{1}{2}\right)^k \frac{(k+1)!}{\left[\left(\frac{k+1}{2}\right)!\right]^2}, \qquad m+n=k,\ k \text{ odd}.$$

Therefore, to show $\lim_{m,n\to\infty} g_{mn}=0$, it suffices to show $\lim_{k\to\infty}(\frac{1}{2})^{2k}[(2k)!/(k!)^2]=0$. Applying [11, eq. (6), p. 211]

$$n! = \sqrt{2\pi n}\, n^n e^{-n} e^{[\Omega(n)/12(n+1)]}$$

where $0<\Omega(n)<1\ \forall n$, we see that

$$\lim_{k\to\infty}\left(\frac{1}{2}\right)^{2k}\frac{(2k)!}{(k!)^2}$$

$$= \frac{1}{\sqrt{\pi}} \lim_{k\to\infty} \frac{1}{\sqrt{k}} \exp\left\{\frac{\Omega(2k)}{24k-12} - \frac{\Omega(k)}{6k-6}\right\}.$$

Since $\lim_{k\to\infty} 1/\sqrt{k}=0$, and $\lim_{k\to\infty} \exp\{(\Omega(2k)/24k-12) - (\Omega(k)/6k-6)\} = 1$; $\lim_{k\to\infty}(\frac{1}{2})^{2k}(2k)!/(k!)^2 = 0$, therefore, we conclude $\lim_{m,n\to\infty} g_{mn}=0$. It is also interesting to note that on the diagonal $g_{nn}=(\frac{1}{2})^{2n}n!/(n!)^2$, hence $g_{nn}\to 0$ as $1/\sqrt{n}$. This is behavior contrasting with that in the one-dimensional case where if the impulse response approaches zero, it must do so geometrically. It is also interesting to note $\{g_{mn}\}\notin l_2$.

V. Some Stability Theorems

Theorem 2: If $G(z_1,z_2)$ has a bounded impulse response, then $G(z_1,z_2)$ is analytic in U^2 (i.e., $Q(z_i,z_2)\neq 0$ in U^2).

Proof: $|g_{mn}|<M\ \forall m,n \Rightarrow \Sigma g_{mn} z_1^m z_2^n$ is absolutely convergent $\forall (z_1,z_2)\in U^2 \Rightarrow G(z_1,z_2)$ has no poles in $U^2 \Rightarrow G(z_1,z_2)$ is analytic in U^2 (recall that in Theorem 1 we argued that if $\exists$ a nonessential singularity of the second kind $\in U^2$, $\exists$ a pole $\in U^2$).

Remark: From Example 3 one might be tempted to conclude the converse of Theorem 2; however, the converse is false, e.g.,

$$G_4(z_1,z_2) = \frac{1}{(1-z_1z_2)^2}$$

is analytic in U^2 but has impulse response

$$g_{mn} = \begin{cases} 0, & m\neq n \\ m, & m=n. \end{cases}$$

Theorem 3: If $G(z_1,z_2)$ is bounded in U^2, then $\{g_{mn}\}$ is square summable.

Proof: $G(z_1,z_2)\leqslant M<\infty$ in $U^2 \Rightarrow G(z_1,z_2)$ has no poles in $U^2 \Rightarrow G(z_1,z_2)$ has no nonessential singularities of the second kind in $U^2 \Rightarrow G(z_1,z_2)$ is analytic in $U^2 \Rightarrow G(z_1,z_2)$ has power series expansion $\Sigma_{m,n=0}^{\infty} g_{mn} z_1^m z_2^n$ which is absolutely convergent in U^2. Using the same argument as in Example 2:

$$\sum_{m,n=0}^{\infty} |g_{mn}|^2 r^{2(m+n)} = \frac{1}{(2\pi)^2}$$

$$\cdot \int_0^{2\pi}\int_0^{2\pi} |G(re^{i\theta_1}, re^{i\theta_2})|^2\, d\theta_1\, d\theta_2 \leqslant M^2, \qquad \forall r\in[0,1).$$

It follows that $\Sigma|g_{mn}|^2<\infty$ or $\{g_{mn}\}\in l_2$.

Theorem 4: Let $G(z_1,z_2)=P(z_1,z_2)/Q(z_1,z_2)$. If $Q(z_1,0)\neq 0$ for $z_1\in \bar{U} \triangleq \{z_1 \ni |z_1|\leqslant 1\}$, then for any fixed $n, g_{mn}\to 0$ geometrically in m and $\Sigma_{m=0}^{\infty}|g_{mn}|<\infty$.

Proof: Since $Q(0,0)\neq 0$ is assumed, $\exists \epsilon>0 \ni$

$$\sum_{m,n=0}^{\infty} |g_{mn} z_1^m z_2^n| < \infty, \qquad \forall (z_1,z_2)\in U_\epsilon^2$$

$$U_\epsilon^2 \triangleq \{(z_1,z_2): |z_1|<\epsilon,\ |z_2|<\epsilon\}$$

so for $(z_1,z_2)\in U^2$ we may iterate the sum to give

$$G(z_1,z_2) = \sum_{n=0}^{\infty} z_2^n \sum_{m=0}^{\infty} g_{mn} z_1^m$$

$$G(z_1,z_2) = \sum_{n=0}^{\infty} z_2^n g_n(z_1)$$

where $g_n(z_1)=\Sigma_{m=0}^{\infty} g_{mn} z_1^m$. For $(z_1,z_2)\in U_\epsilon^2$ we may differentiate term-wise k times to give

$$\frac{\partial^k}{\partial z_2^k}[G(z_1,z_2)] = \sum_{n=0}^{\infty} \frac{\partial^k}{\partial z_2^k} z_2^n g_n(z_1)$$

setting $z_2=0$ gives

$$\left.\frac{\partial^k}{\partial z_2^k}[G(z_1,z_2)]\right|_{z_2=0} = k!\, g_k(z_1).$$

The left-hand side is a rational function of the form

$$\frac{P_k(z_1)}{[Q(z_1,0)]^k}$$

so we conclude

$$\sum_{m=0}^{\infty} g_{mn} z_1^m = g_n(z_1) = \frac{1}{n!}\frac{P_n(z_1)}{[Q(z_1,0)]^n}$$

and since $g_n(z_1)$ is a rational function of one variable with denominator nonzero on $\bar{U}$, we conclude $g_{mn} \xrightarrow[m\to\infty]{} 0$ geometrically for any fixed n, and $\Sigma_{m=0}^{\infty}|g_{mn}|<\infty$.

Remark: A filter satisfying $Q(z_1,0)\neq 0\ \forall z_1\in \overline{U}$ and $Q(0,z_2)\neq 0\ \forall z_2\in \overline{U}$ is not necessarily stable. For example,

$$G(z)=\frac{1}{1-z_1z_2}$$

has impulse response

$$g_{mn}=\begin{cases}1, & m=n\geqslant 0\\ 0, & \text{otherwise}\end{cases}$$

and hence is unstable, but

$$\sum_{m=0}^{\infty} g_{mn}=1,\qquad \forall n \text{ and } \sum_{n=0}^{\infty} g_{mn}=1,\qquad \forall m.$$

VI. Conclusion

In this section we summarize the results discussed in this paper. Let

$$G(z_1,z_2)=\frac{P(z_1,z_2)}{Q(z_1,z_2)}=\sum_{m,n=0}^{\infty} g_{mn}z_1^mz_2^n$$

where $P(z_1,z_2)$ and $Q(z_1,z_2)$ are mutually prime and $Q(0,0)\neq 0$. Then the following relationships hold:

a) BIBO stability	$\Leftrightarrow$	$\{g_{mn}\}\in l_1$
b) $Q(z_1,z_2)\neq 0$ in $\overline{U}^2$	$\not\Leftarrow$ $\Rightarrow$	BIBO stability
c) $Q(z_1,z_2)\neq 0$ in $\overline{U}^2-T^2$	$\Leftarrow$ $\not\Rightarrow$	BIBO stability
d) $\{g_{mn}\}\in l_2$	$\Leftarrow$ $\not\Rightarrow$	BIBO stability
e) $\lim g_{mn}=0$	$\Leftarrow$ $\not\Rightarrow$	$\{g_{mn}\}\in l_1$ or $\{g_{mn}\}\in l_2$
f) $Q(z_1,z_2)\neq 0$ in U^2	$\Leftarrow$ $\not\Rightarrow$	$\lvert g_{mn}\rvert\leqslant M<\infty,\quad \forall m,n$
g) $\lvert G(z_1,z_2)\rvert\leqslant N<\infty$ in U^2	$\Rightarrow$	$\{g_{mn}\}\in l_2$
h) $Q(z_1,0)\neq 0$ in $\overline{U}$	$\Rightarrow$	$\sum_{m=0}^{\infty}\lvert g_{mn}\rvert<\infty,\quad \forall n.$

Appendix

Proof of Lemma 1: We first determine the values of $G_1^*(e^{i\theta_1},e^{i\theta_2})$. Since $G_1(z_1,z_2)$ is continuous on T_1^2 except at $z_1=z_2=1$, $G_1^*(e^{i\theta_1},e^{i\theta_2})=G_1(e^{i\theta_1},e^{i\theta_2})$ except at $(\theta_1,\theta_2)=(0,0)$. At $(0,0)$, $G_1^*(e^{i0},e^{i0})=\lim_{r\nearrow 1}G_1(r,r)=(1-r)^8(1-r)^8/2(1-r)=0$. Hence

$$G_1^*(e^{i\theta_1},e^{i\theta_2})=\begin{cases}\dfrac{(1-e^{i\theta_1})^8(1-e^{i\theta_2})^8}{2-e^{i\theta_1}-e^{i\theta_2}}, & (\theta_1,\theta_2)\neq(0,0)\\ 0, & (\theta_1,\theta_2)=(0,0).\end{cases}$$

Since $P_1(e^{i\theta_1},e^{i\theta_2})\triangleq(1-e^{i\theta_1})^8(1-e^{i\theta_2})^8$ and $Q(e^{i\theta_1},e^{i\theta_2})\triangleq 2-e^{i\theta_1}-e^{i\theta_2}$ are both differentiable in θ_1 and θ_2 and $Q(e^{i\theta_1},e^{i\theta_2})\neq 0$ except at $(\theta_1,\theta_2)=(0,0)$, applying the formula $(x/y^n)'=(x'y-ny'x)/y^{n+1}$ ($y\neq 0$, n any integer $\geqslant 1$) four times we conclude that $(\partial^2/\partial\theta_2^2)\cdot(\partial^2/\partial\theta_1^2)G_1^*(e^{i\theta_1},e^{i\theta_2})$ exists and is continuous except possibly at $(0,0)$.

Because the notation gets messy in proving existence and continuity at $(0,0)$, we let $\theta\triangleq\theta_1$, $\phi\triangleq\theta_2$ and $G(\theta,\phi)\triangleq G_1^*(e^{i\theta},e^{i\phi})$. The proof proceeds as follows: show, in order

1) $G_\theta(0,0)=0$
2) $G_{\theta\theta}(0,0)=0$
3) $G_{\theta\theta\phi}(0,0)=0$
4) $G_{\theta\theta\phi\phi}(0,0)=0$
5) $G_{\theta\theta\phi\phi}(\theta,\phi)$ is continuous at $(0,0)$.

The proofs of 1), 2), 3), are similar to the proof of 4) and will be omitted. Assuming 3) has been proved

$$G_{\theta\theta\phi}(\theta,\phi)=\begin{cases}\dfrac{\partial}{\partial\phi}\dfrac{\partial^2}{\partial\theta^2}\dfrac{(1-e^{i\theta})^8(1-e^{i\phi})^8}{2-e^{i\theta}-e^{i\phi}}, & (\theta,\phi)\neq(0,0)\\ 0, & (\theta,\phi)=(0,0).\end{cases}$$

Now

$$\frac{\partial}{\partial\phi}\frac{\partial^2}{\partial\theta^2}\frac{(1-e^{i\theta})^8(1-e^{i\phi})^8}{2-e^{i\theta}-e^{i\phi}}=\frac{(1-e^{i\theta})^6(1-e^{i\phi})^7A(e^{i\theta}e^{i\phi})}{(2-e^{i\theta}-e^{i\phi})^3}$$

where $A(e^{i\theta},e^{i\phi})$ is some polynomial in $e^{i\theta}$ and $e^{i\phi}$, and

$$\begin{aligned}G_{\theta\theta\phi\phi}(0,0)&=\lim_{\epsilon\to 0}\frac{G_{\theta\theta\phi}(0,\epsilon)-G_{\theta\theta\phi}(0,0)}{\epsilon}\\ &=\lim_{\epsilon\to 0}\frac{\dfrac{(1-1)^6(1-e^{i\epsilon})^7A(1,e^{i\epsilon})}{(1-e^{i\epsilon})^3}-0}{\epsilon}\\ &=\lim_{\epsilon\to 0}\frac{0-0}{\epsilon}\\ &=0.\end{aligned}$$

Hence

$$G_{\theta\theta\phi\phi}(\theta,\phi)=\begin{cases}\dfrac{\partial^2}{\partial\theta^2}\dfrac{\partial^2}{\partial\theta^2}\dfrac{(1-e^{i\theta})^8(1-e^{i\theta})^8}{2-e^{i\theta}-e^{i\theta}}, & (\theta,\phi)\neq(0,0)\\ 0, & (\theta,\phi)=0\end{cases}$$

and it remains to show the continuity of $G_{\theta\theta\phi\phi}$ at $(0,0)$.

Since for $(\theta,\phi)\neq(0,0)$

$$G_{\theta\theta\phi\phi}(\theta,\phi)=\frac{(1-e^{i\theta})^6(1-e^{i\phi})^6B(e^{i\theta},e^{i\phi})}{(2-e^{i\theta}-e^{i\phi})^4}$$

where $B(e^{i\theta},e^{i\phi})$ is some polynomial in $e^{i\theta}$ and $e^{i\phi}$, it is sufficient to show that

$$\lim_{\|(\theta,\phi)\|\to 0}\frac{|1-e^{i\theta}|^6|1-e^{i\phi}|^6}{|2-e^{i\theta}-e^{i\phi}|^4}=0$$

where $\|(\theta,\phi)\|\triangleq\max\{|\theta|,|\phi|\}$. Simplifying,

$$\frac{|1-ei^{\theta}|^6|1-e^{i\phi}|^6}{|2-e^{i\theta}-e^{i\phi}|^4}=4\frac{[1-\cos\theta]^3[1-\cos\phi]^3}{[3-2\cos\theta-2\cos\phi+\cos(\theta-\phi)]^2}.$$

Claim: Let $M(\theta,\phi)\triangleq 3-2\cos\theta+2\cos\phi+\cos(\theta-\phi)$. Then $\exists\epsilon>0\ni 0<\|(\theta,\phi)\|<\epsilon\Rightarrow M(\theta,\phi)>\frac{1}{4}\|(\theta,\phi)\|^4$.

Proof of Claim: In the proof $0(\cdot)$ will denote a function which satisfies $\lim_{x\to 0}0(x)/x=0$. We will show $\exists\epsilon>0\ni$ for (θ,ϕ) with $0<\|\theta,\phi\|<\epsilon$ and $|\theta|>0$, $M(\theta,\phi)>\frac{1}{4}\theta^4$. Since $M(\theta,\phi)=M(\phi,\theta)$ this suffices to prove the claim as reversing the roles of θ and ϕ we get for the same ϵ.

$$0<\|(\theta,\phi)\|<\epsilon,\ |\phi|>0\Rightarrow M(\theta,\phi)>\tfrac{1}{4}\phi^4$$

and so,

$$0<\|(\theta,\phi)\|<\epsilon\Rightarrow M(\theta,\phi)>\max\{\tfrac{1}{4}\theta^4,\tfrac{1}{4}\phi^4\}=\tfrac{1}{4}\|(\theta,\phi)\|^4.$$

Taking partials with respect to ϕ

$$M_\phi(\theta,\phi)=2\sin\phi-\sin(\phi-\theta)$$
$$M_{\phi\phi}(\theta,\phi)=2\cos\phi-\cos(\phi-\theta).$$

We note that $\exists\epsilon_1>0\ni$ if θ,ϕ satisfy $0\leqslant\|(\theta,\phi)\|<\epsilon_1$, then $M(\theta,\phi)\geqslant 0$, $M_{\phi\phi}(\theta,\phi)>0$ with $M(\theta,\phi)=0$ only at (0,0). Next consider $M_\phi(\theta,\phi)$: since $M_\phi(0,0)=0, M_{\phi\phi}(0,0)=1, M_{\phi\theta}(0,0)=1$, and $M_\phi(\theta,\phi)$ is analytic in θ and ϕ, it follows by the implicit function theorem (pp. 270–273 of [12], the analyticity follows from 10.2.4 on p. 272) that $\exists\epsilon_2>0$ and a unique analytic $\Phi(\theta)\ni M_\phi(\theta,\Phi(\theta))=0$ for $|\theta|<\epsilon_2$. Furthermore, $\Phi(0)=0$ and $d\Phi(\theta)/d\theta|_{\theta=0}=-M_{\phi\theta}(0,0)/M_{\phi\phi}(0,0)=-1$ so that $\Phi(\theta)=-\theta+0(\theta)$. We next select $\epsilon_3>0$, $\epsilon_3\leqslant\min\{\epsilon_1,\epsilon_2\}$ such that $0<|\theta|<\epsilon_3\Rightarrow$

a) $|\Phi(\theta)|<\epsilon_1$

b) $-1.1\leqslant\dfrac{\Phi(\theta)}{\theta}\leqslant-0.9.$

It follows that for $\phi,\theta\ni 0<\|(\theta,\phi)\|<\epsilon_3$ with $|\theta|>0$ we have

$$M(\theta,\phi)\geqslant M(\theta,\Phi(\theta))>0$$

and $(\theta,\Phi(\theta))$ is in the sector bounded by lines of slope -1.1 and -0.9, i.e., $\Phi(\theta)=k\theta$ for some $k\ni-1.1\leqslant k\leqslant-0.9$ (see Fig. 1). We now take the Taylor expansion of $M(\theta,\phi)$ to give

$$M(\theta,\phi)=\left[\theta^2+\phi^2-\tfrac{1}{2}(\theta-\phi)^2\right]+\tfrac{1}{24}\left[(\theta-\phi)^4-2\theta^4-2\phi^4\right]$$
$$+\left\{0_1(\theta^5)+0_2(\phi^5)+0_3(\theta,\phi)^5\right\}$$

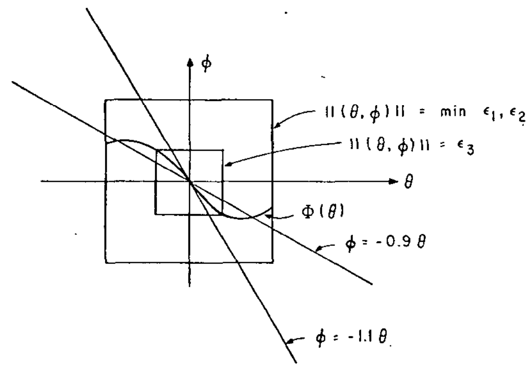

Fig. 1. Behavior of $\Phi(\theta)$ in neighborhood of $(\theta,\phi)=(0,0)$.

set $\phi=k\theta$ to give

$$M(\theta,k\theta)=(1-k)^2\theta^2+\tfrac{1}{24}\left[(1-k)^4-2(1-k^4)\right]\theta^4$$
$$+\left\{0_1(\theta^5)+0_2\left[(k\theta)^5\right]+0_3\left[(1-k)^5\theta^5\right]\right\}$$
$$\geqslant\tfrac{1}{24}\left[(1-k)^4-2(1-k^4)\right]\theta^4$$
$$+\left\{0_1(\theta^5)+0_2\left[(k\theta)^5\right]+0_3\left[(1-k)^5\theta^5\right]\right\}$$

and for $-1.1\leqslant k\leqslant-0.9$

$$M(\theta,k\theta)\geqslant\tfrac{1}{2}\theta^4+\left\{0_1(\theta^5)+0_2(k^5\theta^5)+0_3\left[(1-k)^5\theta^5\right]\right\}$$

therefore $\exists\epsilon_4>0\ni$

$$M(\theta,k\theta)\geqslant\tfrac{1}{4}\theta^4,\qquad\forall\theta:|\theta|<\epsilon_4\quad k:-1.1\leqslant k\leqslant 0.9.$$

Finally, let $\epsilon=\min\{\epsilon_3,\epsilon_4\}$ from which we conclude: for $(\theta,\phi)\ni 0<\|(\theta,\phi)\|<\epsilon_1$, $|\theta|>0: M(\theta,\phi)\geqslant M(\theta,\Phi(\theta))\geqslant\theta^4/4$. As discussed previously, this completes the proof of our claim that for $(\theta,\phi)\ni 0<\|(\theta,\phi)\|<\epsilon$, $M(\theta,\phi)>\frac{1}{4}\|(\theta,\phi)\|^4$. Next, consider the function $1-\cos\theta=+\theta^2/2+0(\theta^3)$.It follows that $\exists\epsilon_5>0\ni[1-\cos\theta]<\theta^2$ for $|\theta|<\epsilon_5$. Hence, for $(\theta,\phi)\ni 0<\|(\theta,\phi)\|<\epsilon_0=\min\{\epsilon,\epsilon_5\}$

$$0\leqslant\frac{4[1-\cos\theta]^3[1-\cos\phi]^3}{[3-2\cos\theta-\cos\phi+\cos(\theta-\phi)]^2}\leqslant\frac{4[\theta^2]^3[\phi^2]^3}{\left[\dfrac{\|(\theta,\phi)\|^4}{4}\right]^2}$$
$$\leqslant 64\frac{\|\theta,\epsilon\|^{12}}{\|\theta,\phi\|^8}$$
$$\leqslant 64\|\theta,\phi\|^4$$

$$\Rightarrow\lim_{\|\theta,\phi\|\to 0}\frac{4[1-\cos\theta]^3[1-\cos\phi]^3}{[3-2\cos\theta-2\cos\phi+\cos(\theta-\phi)]^2}=0$$

$\Rightarrow G_{\theta\theta\phi\phi}(\theta,\phi)$ is continuous at (0,0).

Q.E.D.

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