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Received November 9, 1976 and in revised form September 19, 1977 and January 4, 1978

Math. Systems Theory **12**, 59–72 (1978)

Mathematical Systems Theory

Doubly-Indexed Dynamical Systems: State-Space Models and Structural Properties

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Abstract. Doubly-indexed dynamical systems provide a state space realization of two-dimensional filters which includes previous state models. Algebraic criteria for testing structural properties (reachability, observability, internal stability) are introduced.

1. Introduction

State space representations of two-dimensional filters are a recent field of investigation; yet there are quite a few contributions [3, 6–12, 16–20, 24].

At first sight these contributions look hard to compare since they are based on state space models having different structures.

If we consider these differences from the realization point of view, it turns out that the state space models we find in the literature realize transfer function classes of different sizes. The recursiveness of the state equations implies the rationality of the transfer function; nevertheless the realization of a generic (strictly causal) rational transfer function cannot be achieved by every model. For instance, the model proposed by Attasi [3] realizes only the subclass of recognizable transfer functions (also called "separable filters").

As proved in [6, 9], the state space models introduced by Roesser [20] and by Fornasini-Marchesini [6, 7, 9] realize the whole class of causal rational functions in two indeterminates. We will show that if we consider any model so far presented in the literature, this can be embedded in the Fornasini-Marchesini model [9] extended to include all causal (not only strictly causal) transfer functions. Moreover it is interesting to notice that the embedding of the Roesser model preserves the dimension of the local state space, whereas the reverse embedding requires in general increasing the dimension of the local state space.

*This work was supported by CNR-GNAS

0025/5661/78/0012-0059$2.80

Recently Kung-Lévy-Morf-Kailath [16, 17] considered the Roesser model as a starting point for extending Rosenbrock's theory of coprimeness [21] to 2-D systems. This approach led to the concepts of modal-controllability and modal-observability and to defining as minimal realizations those which are both modal-controllable and modal-observable. This theory looks very interesting from an algebraic standpoint but unfortunately so far it does not reach a consistent conclusion. Actually the existence of realizations which are both modal-controllable and modal-observable has been only conjectured by Kung-Lévy-Morf-Kailath on the basis of low order examples.

Since the comparison between available state space models indicates that the model introduced by the authors is the most general, we shall focus our attention to analyze its structural properties.

We shall first extend from [7] and [9] the concepts of local reachability and observability and their properties. Then the definition of internal stability will be naturally introduced and we shall develop a stability criterion and connections between internal and external stability.

2. State Space Models

A detailed discussion of the realization theory for two dimensional filters has been presented in [6, 7]. So, in this section we shall introduce directly a state space model without deriving it from the definition of the state via Nerode equivalence classes.

We shall first list some notations:

K arbitrary field
$K[z_1,z_2]$ ring of polynomials in two indeterminates over the field K
$K[[z_1,z_2]]$ ring of formal power series in two indeterminates over the field K
$K[(z_1,z_2)]$ subring of rational power series
$K_0[(z_1,z_2)]$ ideal generated by z_1 and z_2 in $K[(z_1,z_2)]$.

A generic element in $K[[z_1,z_2]]$ will be denoted by

$$s = \Sigma_{h,k}\left(s, z_1^h z_2^k\right) z_1^h z_2^k$$

where $\left(s, z_1^h z_2^k\right) \in K$ is the coefficient of the monomial $z_1^h z_2^k$.

Let us introduce the following definition.

Definition. A *doubly-indexed* linear, stationary, finite-dimensional, *dynamical system* (DIDS) $\Sigma = (A_1, A_2, B_1, B_2, C)$ is defined by the first order partial difference equation

$$x(h+1,k+1) = A_1 x(h+1,k) + A_2 x(h,k+1) + B_1 u(h+1,k) + B_2 u(h,k+1) \tag{1}$$

$$y(h,k) = Cx(h,k)$$

where $u(h,k)$, the input value at (h,k) and $y(h,k)$, the output value at (h,k), are

in K and $h,k \in \mathbb{Z}$, $A_i \in K^{n\times n}$, $B_i \in K^{n\times 1}$, $C \in K^{1\times n}$, $i = 1,2$ and $x \in X = K^n$ (local state space).

Let $\mathcal{P}$ be a partially ordered set. A *cross-cut* $\mathcal{C} \subset \mathcal{P}$ is a set of incomparable points such that if $i \in \mathcal{P}$ exactly one of the following is true [19]:

(a) $i \in \mathcal{C}$
(b) $i > j$ for some $j \in \mathcal{C}$
(c) $i < j$ for some $j \in \mathcal{C}$

The partition induced on $\mathcal{P}$ by a cross-cut $\mathcal{C}$ evidentiates three disjoint sets of points. We shall call *present*, *future*, and *past* with respect to $\mathcal{C}$ the sets of points satisfying (a), (b), (c) respectively.

In $\mathbb{Z}\times\mathbb{Z}$ partially ordered by the product of the orderings, the cross-cut through the point $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ is uniquely determined as the set

$$\{(i+m, j-m),\ m \in \mathbb{Z}\} \triangleq \mathcal{C}_{i+j}$$

Introduce the following notation.

$$\mathcal{X}_r = \{x(h,k) : x(h,k) \in X, (h,k) \in \mathcal{C}_r\}$$

Given a cross-cut $\mathcal{C}_r \subset \mathbb{Z}\times\mathbb{Z}$ (see Fig. 1), the solution of equation (1) in the

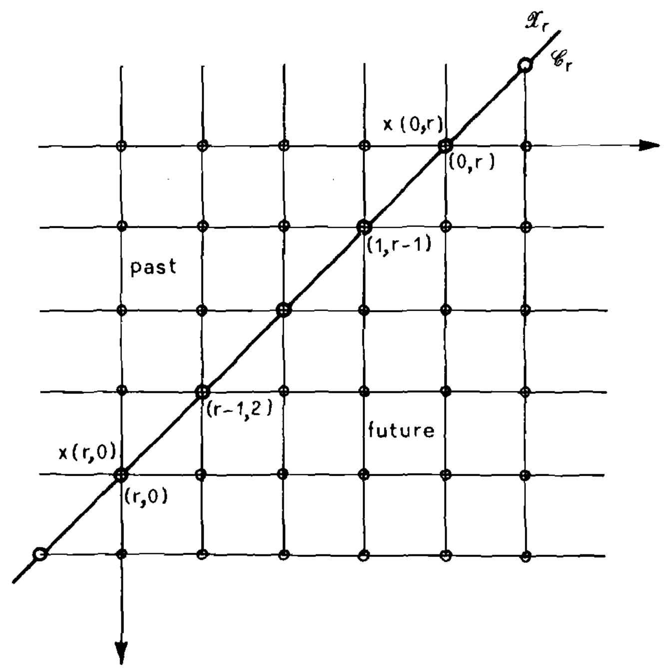


Fig. 1.

future is uniquely determined by $\mathcal{X}_r$ and by the input values on $\mathcal{C}_r$ and on the future set with respect to $\mathcal{C}_r$.

Let $\mathcal{X}_0=0$. The following rational power series:

$$s_\Sigma = C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2) \tag{2}$$

represents the output function of Σ corresponding to the input function $u=\Sigma_{h,k}u(h,k)z_1^hz_2^k=1$.

The series s_Σ is called *transfer function* of Σ.

Let Σ start from $\mathcal{X}_r=0$, then the output function y corresponding to an input u is given by

$$y = s_\Sigma u$$

Definition. A DIDS Σ is a *zero-state realization* of a series $s\in K[[z_1,z_2]]$ if $s=s_\Sigma$. The *dimension of the realization* is the dimension of the local state space X.

Then the following proposition holds:

Proposition 1. *Let $s\in K[[z_1,z_2]]$. Then there exists a* DIDS *which is a zero-state realization of s if and only if $s\in K_0[(z_1,z_2)]$.*

Proof. The necessity is a trivial consequence of (2).

Conversely let $s\in K_0[(z_1,z_2)]$. This means that $s=n(z_1,z_2)p^{-1}(z_1,z_2)$, $n,p\in K[z_1,z_2]$, $n(0,0)=0$ and $p(0,0)=1$. Consider two polynomials ν and π in the ring $K\langle\xi_1,\xi_2\rangle$ of noncommutative polynomials such that their commutative images are n and p respectively.

The commutative image of the noncommutative series $\sigma=\nu\pi^{-1}$ is the series s. Since σ is recognizable [5], there exist an integer N and matrices $A_1,A_2\in K^{N\times N}$, $B\in K^{N\times 1}$ and $C\in K^{1\times N}$ such that

$$\sigma = C(I-A_1\xi_1-A_2\xi_2)^{-1}B = C\sum_{0}^{\infty}{}_k(A_1\xi_1+A_2\xi_2)^kB = C\sum_{1}^{\infty}{}_k(A_1\xi_1+A_2\xi_2)^kB$$
$$= C(I-A_1\xi_1-A_2\xi_2)^{-1}(B_1\xi_1+B_2\xi_2)$$

where we put $B_1=A_1B$, $B_2=A_2B$.

Since the projection map from the algebra of noncommutative power series $K\langle\langle\xi_1,\xi_2\rangle\rangle$ onto $K[[z_1,z_2]]$ is an algebra homomorphism, the series s can be expressed as

$$s = C(I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)$$

Then the DIDS $\Sigma=(A_1,A_2,B_1,B_2,C)$ is a zero-state realization of s. □

Remark. See [9] for an explicit construction of A_1,A_2,B_1,B_2,C.

We shall now prove that the models investigated by Roesser [11, 12, 20], Kung-Lévy-Morf-Kailath [16], Fornasini-Marchesini [6, 7] and, a fortiori, Attasi [3] can be embedded in (1).

In fact, consider first the model introduced in [6, 7]:

$$\begin{aligned}\bar{x}(h+1,k+1) &= \bar{A}_1\bar{x}(h+1,k)+\bar{A}_2\bar{x}(h,k+1)+\bar{A}_0\bar{x}(h,k)+\bar{B}u(h,k)\\ y(h,k) &= \bar{C}\bar{x}(h,k)\end{aligned} \tag{3}$$

The model of Attasi is a special case of (3) when $\bar{A}_0=-\bar{A}_1\bar{A}_2=-\bar{A}_2\bar{A}_1$.

An embedding of (3) in (1) is accomplished assuming in (1) as local state the following vector:

$$x(h,k) = \begin{bmatrix}\bar{x}(h,k)\\ \bar{x}(h,k-1)\\ u(h,k-1)\end{bmatrix}$$

so that model (3) can be rewritten in form (1) with

$$A_1 = \begin{bmatrix}\bar{A}_1 & 0 & 0\\ I & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad A_2 = \begin{bmatrix}\bar{A}_2 & \bar{A}_0 & \bar{B}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$B_1 = \begin{bmatrix}0\\0\\I\end{bmatrix},\quad B_2 = \begin{bmatrix}0\\0\\0\end{bmatrix},\quad C = [\,\bar{C}\quad 0\quad 0\,].$$

Roesser's model can be described as follows:

$$\begin{bmatrix}x^h(h+1,k)\\ x^v(h,k+1)\end{bmatrix} = \underbrace{\begin{bmatrix}\hat{A}_1 & \hat{A}_2\\ \hat{A}_3 & \hat{A}_4\end{bmatrix}}_{\hat{A}}\begin{bmatrix}x^h(h,k)\\ x^v(h,k)\end{bmatrix} + \underbrace{\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}}_{\hat{B}}u(h,k)$$

$$y(h,k) = \underbrace{[\,\hat{C}_1\quad \hat{C}_2\,]}_{\hat{C}}\begin{bmatrix}x^h(h,k)\\ x^v(h,k)\end{bmatrix} \tag{4}$$

where x^h is called the *horizontal state* and x^v the *vertical state*.

It is clear that assuming in (1) the vector

$$x(h,k) = \begin{bmatrix}x^h(h,k)\\ x^v(h,k)\end{bmatrix}$$

as local state space, model (4) can be recast in form (1) with

$$A_1 = \begin{bmatrix} 0 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ \hat{B}_2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \quad C = [\,\hat{C}_1 \quad \hat{C}_2\,]$$

It is interesting to notice that in Roesser's model the local state is the direct sum of the horizontal and vertical states, so that the embedding above does not require any increasing of dimension. Conversely, embedding (1) in (4) cannot be accomplished in general without increasing the dimension of the state space.

In fact for this embedding we need a preliminary increase of dimension to be able to put matrices A_i, B_i and C of (1) in the partitioned form (5).

Example. Consider the rational function $(z_1+z_2)(1-z_1-z_2)^{-1}$. A realization in form (1) is $\Sigma=(1,1,1,1,1)$. Clearly the dimension of a realization in Roesser's form is at least two.

The idea of splitting the local state space X in horizontal and vertical components, which leads to Roesser's model, implies that the structure of the updating equations is not invariant under similarity transformations in X. Clearly equations (1) keep their structure under such transformations.

3. Structural Properties of State Space Representations

Reachability and observability notions for DIDS have been introduced in [6, 9, 20]. We shall now adjust them to model (1) for obtaining reachability and observability criteria.

We say that a local state $\bar{x} \in X$ is reachable from zero initial states if there exists an input $u \in K[[z_1,z_2]]$ and integers $i>0, j>0$ such that $x(i,j)=\bar{x}$, when Σ starts from $\mathcal{X}_0$ identically zero.

Since the DIDS we consider are stationary, we introduce the following definitions:

Definition . A state $x \in X$ is *reachable* if $x=((I-A_1z_1-A_2z_2)^{-1}(B_1z_1-B_2z_2)u,1)$ for some $u \in K[z_1^{-1},z_2^{-1}]$.

The *reachable local state space* is

$$X^R = \{x : x=((I-A_1z_1-A_2z_2)^{-1}(B_1z_1+B_2z_2)u,1), u \in K[z_1^{-1},z_2^{-1}]\}$$

The realization Σ is *L-reachable* if $X=X^R$.

We introduce the following matrices $M_{ij} \in K^{n\times n}$:

$$M_{ij} = ((I-A_1z_1-A_2z_2)^{-1}, z_1^iz_2^j).$$

(i.e., $M_{00}=I$, $M_{10}=A_1$, $M_{01}=A_2$, $M_{20}=A_1^2$, $M_{11}=A_1A_2+A_2A_1,\ldots$).

Then the columns of the infinite matrix

$$\mathcal{R}_\infty = [\,B_1\,B_2\,M_{10}B_1\,M_{10}B_2+M_{01}B_1\ldots M_{i-1,j}B_1+M_{i,j-1}B_2\ldots\,]$$

span X^R. Consequently system Σ is L-reachable if $\mathcal{R}_\infty$ is full rank.

Also the notion of indistinguishable states is extended to this system.

Definition. A state $x \in X$ is *indistinguishable* from the state $0 \in X$ if

$$C(I-A_1z_1-A_2z_2)^{-1}x = 0$$

The *indistinguishable local state space* X^I is defined as:

$$X^I = \{x : x \in X, C(I-A_1z_1-A_2z_2)^{-1}x=0\}$$

The subspace X^I coincides with the null space of the matrix:

$$\mathcal{O}_\infty = \begin{bmatrix} C \\ CM_{10} \\ CM_{01} \\ CM_{20} \\ \vdots \end{bmatrix}$$

The realization Σ is *L-observable* if $X^I=\{0\}\subset X$, i.e. if $\mathcal{O}_\infty$ is full rank.

The rank evaluation of $\mathcal{O}_\infty$ and $\mathcal{R}_\infty$ can be reduced to compute the rank of two finite dimensional submatrices, by using the following extension of the Cayley-Hamilton theorem.

Proposition 2. *Let $(I-A_1z_1-A_2z_2)^{-1}=\Sigma_{ij}M_{ij}z_1^iz_2^j$. Then the M_{ij} with $i+j\geqslant n$ are linear combinations of the M_{ij} with $i+j<n$, i.e.*

$$\mathrm{span}(M_{ij}, i,j \in \mathbb{Z}) = \mathrm{span}(M_{ij}, i,j\geqslant 0, i+j\leqslant n-1)$$

The proof is a straightforward consequence of the identity:

$$\Sigma_{ij}M_{ij}z_1^iz_2^j\det(z_1^{-1}z_2^{-1}I-A_1z_2^{-1}-A_2z_1^{-1})$$
$$= z_1^{-1}z_2^{-1}\,\mathrm{Adj}(z_1^{-1}z_2^{-1}I-A_1z_2^{-1}-A_2z_1^{-1})$$

An immediate application of Proposition 2 is the result that the rank of $\mathcal{R}_\infty$ coincides with the rank of the $n\times\frac{1}{2}(n+2)(n-1)$ submatrix of $\mathcal{R}_\infty$:

$$\mathcal{R} = [\,B_1\,B_2\,M_{10}B_1\,M_{10}B_2+M_{01}B_1\ldots M_{0,n-1}B_2\,]$$

Analogously the rank of $\mathcal{O}_\infty$ is the rank of its $\frac{1}{2}(n+1)n\times n$ submatrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CM_{10} \\ CM_{01} \\ \vdots \\ CM_{0,n-1} \end{bmatrix}$$

Let $\Sigma=(A_1,A_2,B_1,B_2,C)$ be a realization of a rational series s and assume that Σ is not L-reachable. An L-reachable realization having as state space the reachable state space X^R of Σ, can be obtained following a procedure analogous to that outlined in [9,10]. In a similar way it is possible to derive an L-observable realization whose dimension is the rank of $\mathcal{O}$.

In [16, 17], Kung-Lévy-Morf-Kailath considered the controllability problems of Roesser's model through the extension of the coprimeness property to matrices with entries in $K[z_1,z_2]$.

The transfer function of Roesser's state space description has the following structure:

$$s_R = [\hat{C}_1\,\hat{C}_2]\begin{bmatrix} z_1^{-1}I-\hat{A}_1 & -\hat{A}_2 \\ -\hat{A}_3 & z_2^{-1}I-\hat{A}_4 \end{bmatrix}^{-1}\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \tag{6}$$

which is a particular form of (2), as we can see comparing (5) and (6).

The system matrix appearing in (6) shows the peculiar property of being partitioned in block-matrices each containing either z_1 or z_2 separately.

Kung-Lévy-Morf-Kailath were motivated by this fact to define the system (4) to be *modal-controllable* and *modal-observable* if the matrix pairs

$$\left(\begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix}-\hat{A},\hat{B}\right) \quad \text{and} \quad \left(\hat{C},\begin{bmatrix} z_1^{-1}I & 0 \\ 0 & z_2^{-1}I \end{bmatrix}-\hat{A}\right)$$

are left-coprime and right-coprime respectively.

The analysis of modal-controllability and modal-observability can be based on the following interesting coprimeness criterion [Kung-Lévy-Morf-Kailath]:

Let $M(z_1,z_2)$ and $N(z_1,z_2)$ be polynomial matrices of size $n\times n$ and $m\times n$ with entries in $K[z_1,z_2]$. Then $M(z_1,z_2)$ and $N(z_1,z_2)$ are right-coprime if and only if

$$\operatorname{rank}\begin{bmatrix} M(\zeta_1,\zeta_2) \\ N(\zeta_1,\zeta_2) \end{bmatrix} = n$$

for any generic point (ζ_1,ζ_2) of any algebraic curve generated by the irreducible factors of $\det M(z_1,z_2)$.

In this framework, the interesting problem to be solved relies in establishing whether realizations both modal-controllable and modal-observable do exist.

This kind of realization would be rather interesting since the dimension of the local state space would be minimal with respect to Roesser's model. Of course, since Roesser's models are a subclass of models (1), a modal-controllable and modal-observable Roesser's realization of a transfer function is not in general minimal in the class of realization (1). However, Kung-Lévy-Morf-Kailath failed to prove the existence of such realizations.

4. Stability

The stability problem for two-dimensional filters in input-output form has been investigated by several authors [1, 2, 4, 14, 15, 23]. Attasi [3] was considering the stability of realizations of separable two-dimensional filters, i.e. DIDS having transfer functions with structure $\bar{C}(I-\bar{A}_1z_1^{-1})^{-1}(I-\bar{A}_2z_2^{-1})^{-1}\bar{B}$ and $\bar{A}_1\bar{A}_2=\bar{A}_2\bar{A}_1$. Obviously, the factorized form of the system matrix in the product $(I-\bar{A}_1z_1)^{-1}(I-\bar{A}_2z_2)^{-1}$ reduces the stability problem to the stability analysis of $\bar{A}_1$ and $\bar{A}_2$ separately.

In this section we shall deal with the stability of DIDS represented by model (1).

From now on we assume that $K=\mathbb{R}$ and the euclidean norm in $X=\mathbb{R}^n$. Moreover we introduce

$$\|\mathcal{X}_r\| = \sup_{n\in\mathbf{Z}}\|x(r-n,n)\|$$

We therefore have the following definition.

Definition. Let Σ be described by equations (1). The system Σ is *asymptotically stable* if assuming $u=0$ and $\|\mathcal{X}_0\|$ finite $\|\mathcal{X}_i\|\to 0$ as $i\to+\infty$.

As is well known, the asymptotic stability analysis of discrete time linear systems reduces to investigating the position of the zeros of the characteristic polynomial of the one-step state transition matrix A.

The asymptotic stability of a DIDS $\Sigma=(A_1,A_2,B_1,B_2,C)$ is related to the algebraic curve defined in $\mathbb{C}\times\mathbb{C}$ by the equation

$$\det(I-z_1A_1-z_2A_2)=0$$

as stated in the following Proposition.

Proposition 3. *Let Σ be as in* (1). *Then Σ is asymptotically stable if and only if the polynomial* $\det(I-A_1z_1-A_2z_2)$ *is not zero in the closed polydisc*:

$$\mathcal{P}_1 = \{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}: |z_1|\leqslant 1, |z_2|\leqslant 1\}$$

Proof. *Sufficiency*. Let $\det(I-z_1A_1-z_2A_2)\neq 0$ in $\mathcal{P}_1$ and call V the algebraic curve defined by $\det(I-A_1z_1-A_2z_2)=0$. Since V and $\mathcal{P}_1$ are closed, $V\cap\mathcal{P}_1=\varnothing$

implies that there exists $\varepsilon>0$ such that the polydisc

$$\mathcal{P}_{i+\varepsilon} = \{(z_1,z_2)\in\mathbb{C}\times\mathbb{C}: |z_1|\leqslant 1+\varepsilon, |z_2|\leqslant 1+\varepsilon\}$$

does not intersect V.

Then the rational matrix $(I-A_1z_1-A_2z_2)$ can be inverted in $\mathcal{P}_{1+\varepsilon}$ and its McLaurin series expansion, given by

$$(I-A_1z_1-A_2z_2)^{-1} = \Sigma_{ij}M_{ij}z_1^iz_2^j \tag{7}$$

converges normally in the interior of $\mathcal{P}_{1+\varepsilon}$ [13].

It follows that the series $\Sigma_{ij}\|M_{ij}\|$ converges. Consequently $\Sigma_{i+j=r}\|M_{ij}\|\to 0$ as $r\to\infty$, [22]. This implies the asymptotic stability of Σ. For, assume $\|\mathcal{X}_0\|$ finite and pick in $\mathcal{X}_r$, $r>0$, any local state $x(m,r-m)$, then

$$\|x(m,r-m)\| = \left\| \sum_{i+j=r} M_{ij}x(m-i,r-m-j)\right\|$$

$$\leqslant \sum_{i+j=r}\|M_{ij}\|\,\|x(m-i,r-m-j)\| \leqslant \|\mathcal{X}_0\|\sum_{i+j=r}\|M_{ij}\|$$

Necessity. Assume Σ be asymptotically stable. Then for any $x\in X$, $M_{ij}x\to 0$ as $i+j>\infty$. This fact and

$$\|M_{ij}\| \leqslant \sum_{1}^{n}{}_k\|M_{ij}e_k\|,$$

(with $\{e_k\}_1^n$ the standard basis in $X=\mathbb{R}^n$) imply

$$\lim_{i+j\to\infty}\|M_{ij}\| \leqslant \lim_{i+j\to\infty}\sum_{1}^{n}{}_k\|M_{ij}e_k\| = 0$$

By Abel's Lemma, the series $\Sigma_{ij}M_{ij}z_1^iz_2^j$ converges in the interior of $\mathcal{P}_1$. Then $(I-A_1z_1-A_2z_2)$ is invertible in the interior of $\mathcal{P}_1$.

The proof will be completed by showing that $\det(I-A_1z_1-A_2z_2)\neq 0$ on the boundary $\rho\mathcal{P}_1$ of $\mathcal{P}_1$. For, let (a_1,a_2) belong to $\rho\mathcal{P}_1$ and assume that

$$\det(I-A_1a_1-A_2a_2) = 0$$

Hence there exists a nonzero vector $v\in\mathbb{C}^n$ which satisfies $v=A_1a_1v+A_2a_2v$. It is not restrictive to assume that $|a_1|=1$, so that it makes sense to consider $\mathcal{X}_0=\{x_{n,-n}\}$ with

$$x_{n,-n}\begin{cases} =0 & \text{if } n<0\\ =\alpha a_1^{-n}a_2^nv+\bar\alpha\bar a_1^{-n}\bar a_2^n\bar v & \text{if } n\geqslant 0,\ \alpha\in\mathbb{C}\end{cases}$$

Assume now that $\alpha=e^{j\psi}$ and write $a_1=e^{-j\phi}$ and $v=r+jw$. Then the state values on $(k,0)$, $k=0,1,2,\ldots$ are given by the sequence

$$x(k,0) = 2r\cos(k\phi+\psi)-2w\sin(k\phi+\psi), \qquad k=0,1,2,\ldots.$$

and it is always possible to select a phase ψ which makes the sequence not infinitesimal as k goes to infinity. □

As far as stability criteria are concerned, the result presented in Proposition 3 makes those tests elaborated for input-output stability [1, 2, 4, 14, 15, 23] suitable for asymptotic stability analysis. In fact for a two-dimensional filter, with transfer function $p(z_1,z_2)/q(z_1,z_2)$, $q(0,0)=1$, to be input-output stable it is necessary and sufficient that $q(z_1,z_2)$ not be zero in $\mathcal{P}_1$.

Coprimeness properties are relevant in analyzing the relations between input-output stability and asymptotic stability of DIDS. For this, it is important to note [16] that if $\Sigma=(A_1,A_2,B_1,B_2,C)$ is a realization of a transfer function $p(z_1,z_2)/q(z_1,z_2)$ with p and q relatively prime and

(i) $(C,I-A_1z_1-A_2z_2)$ are left-coprime

and

(ii) $(I-A_1z_1-A_2z_2,B_1z_1+B_2z_2)$ are right-coprime,

then $\det(I-A_1z_1-A_2z_2)=q(z_1,z_2)$.

Realizations satisfying (i) and (ii) will be called *coprime*.

For DIDS, input-output stability and internal stability are related as shown in the following Corollary:

Corollary. *Let $\Sigma=(A_1,A_2,B_1,B_2,C)$. Then we have the following implications*:

Σ asymptotically stable $\to$ Σ input-output stable

Σ asymptotically stable $\leftarrow$ Σ input-output stable + Σ coprime

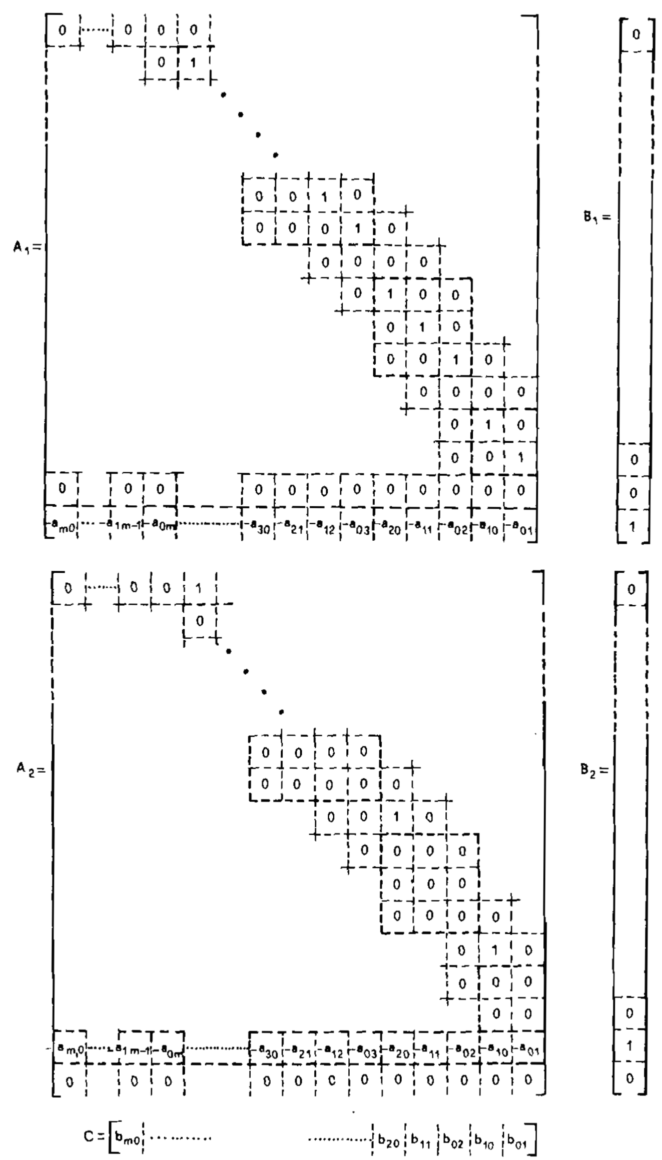
In the Appendix we shall show that any transfer function $p(z_1,z_2)/q(z_1,z_2)$ admits coprime realizations, so it is always possible to construct asymptotically stable realizations starting from stable transfer functions.

Appendix

A coprime realization $\Sigma=(A_1,A_2,B_1,B_2,C)$ of the transfer function

$$\frac{p(z_1,z_2)}{q(z_1,z_2)} = \frac{b_{10}z_1+b_{01}z_2+\cdots+b_{0m}z_2^m}{1+a_{10}z_1+a_{01}z_2+\cdots+a_{0m}z_2^m}$$

is given by:



By direct inspection we can see that the realization above satisfies

$$\det(I - A_1 z_1 - A_2 z_2) = q(z_1, z_2)$$

This identity tells us that the realization Σ is coprime. In fact if, for instance, C and $(I - A_1 z_1 - A_2 z_2)$ were not left-coprime, a square matrix S with entries in $K[z_1, z_2]$ and $\deg_{z_1}(\det S) > 0$ or $\deg_{z_2}(\det S) > 0$ would exist such that

$$C = VS$$
$$(I - A_1 z_1 - A_2 z_2) = TS$$

Thus

$$q(z_1, z_2) = \det(I - A_1 z_1 - A_2 z_2) = \det T \det S$$

and

$$\frac{p(z_1, z_2)}{q(z_1, z_2)} = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2)$$

$$= VSS^{-1}T^{-1}(B_1 z_1 + B_2 z_2) = \frac{1}{\det T} V \operatorname{adj} T(B_1 z_1 + B_2 z_2)$$

This would imply that $\det S$ is a common factor of p and q, contradicting the assumption of the relative primeness of p and q.

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Received September 1, 1977 and in revised form February 2, 1978 and March 20, 1978

Math. Systems Theory **12**, 73–101 (1978)

Mathematical Systems Theory

A Model for Pattern Perception with Musical Applications* Part III: The Graph Embedding of Pitch Structures

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Abstract. This is the third paper of a series which begins by treating the perception of pitch relations in musical contexts and the perception of timbre and speech. The preceding papers dealt with those properties of musical scales which allow them to function as reference frames which provide both for the measurement of intervals and for the identification of their elements as scale degrees. The effect of these properties upon the perceptibility of various musical relations and properties has been discussed. Here we extend the treatment to systems of different scales (as exist in many musical cultures) where a listener's recognition of any one scale in the system interacts with his ability to recognize the others. Reading of the two previous papers is required.

16. The Directed Graph, G

Thus far we have assumed that a listener has learned only one scale (measuring set) and will classify (measure intervals in) any stimulus (set of pitches) by embedding it in a key of this scale. Now we will consider the classification of stimuli by a listener who has learned *many* scales (e.g. a sophisticated Western listener or an Indian familiar with the enormous number of distinct "ragas" in use). That is, we have dealt with the problem of determining x in $P_v(x)$ (Part 2, Sec. 14), given a set of points which is a subset of some $P_v(x)$. Now we consider the problem of determining both v and x in $P_v(x)$ when the given points may be a subset of *any* of several given $P_v(x)$, $v = v_1, v_2 \dots$.

Note that $P_u(x)$ may be a subset of $P_v(y)$, $u \neq v$, where x may or may not equal y. Indeed, if we denote by $P_1(0)$ that $P_v(x)$ corresponding to all the points

*This research was supported in part by grants and contracts AF-AFOSR 881-65, AF 49(638)-1738 and AF-AFOSR 68-1596.

0025/5661/78/0012-0073$5.80