Modeling and Control of a Sorption Process using 2D Systems Theory

Michael Dymkov, Krzysztof Galkowski, Eric Rogers, Vitali Dymkou and Siarhei Dymkou

***Abstract*— This paper introduces a model for the dynamics of a sorption process from the industrial water supply and sewage treatment industries that is a continuous version of the Roesser state-space model for $2D$ discrete systems. Conditions for unique solvability and the representation formula are then developed together with the solution of an optimization problem using boundary control. The solution of this optimization problem by state feedback is also developed.**

I. INTRODUCTION

Many physical processes must be modeled using representations with $n > 1$ indeterminates. Applications of both practical and/or theoretical interest, such as [1], [2], and [3] – [6], arise, as a representative sample, across the general areas of circuits, image processing, signal processing and control. Also, considering the $2D$ case as a representable example, the propagation of the dynamics in the two independent directions can be a function of i) two discrete variables, ii) a continuous variable in one direction and discrete in the other, or iii) two continuous variables.

Multidimensional, written nD for short, systems cannot, in general, be analyzed by direct extension of techniques from the theory of systems in one indeterminate, also known as $1D$ systems. For example, if a transfer-function representation can be used then coprimeness is a very important analysis tool but in the nD case there is more than one form of primeness. Also there are nD systems theoretic properties that have no $1D$ systems counterparts.

In case of examples under i) above, there has been a very large volume of work has been reported based, in the main, on the Roesser [7] and Fornasini Marchesini [8] state-space models. Repetitive processes [2] are a class of $2D$ systems where information propagation in one of the two directions only occurs over a finite duration, where this is an intrinsic property of the dynamics and not an assumption introduced to simplify analysis. The equation updating the dynamics in one of the two directions is governed by a discrete variable but in the other by either a continuous or discrete variable. Hence these processes fit under i) and ii) above as appropriate. Systems theory for these processes is well developed [2], [10], [11] and they do have applications areas such as iterative learning control where recently control laws have been experimentally validated on a gantry robot [12].

The subject area of this paper is $2D$ systems from iii) above, where previous work, such as [13], [14], has focused on special cases with the work in [14] having a gas pipeline application. A model for a sorption process, which arises in waste water and sewage treatment, in the form of a $2D$ continuous Roesser model, also known in some of the mathematical literature as Goursat-type equations is given. Conditions for its unique solvability and a representation formula for the solution are then developed. The control of this model can only be by boundary action and the paper formulates and develops a solution to a quadratic optimization problem in the form of state feedback.

II. BACKGROUND AND PROBLEM FORMULATION

The term sorption refers to the action of absorption or adsorption, where the former is of interest in this paper and is the incorporation of a substance in one state into another of a different state. Networks and tandems of connected sorption devices are widely used for waste-water treatment in industrial water supply and sewerage. In this paper, the starting point is the mathematical model of a single sorption process under the assumption of non-equilibrium sorption dynamics and linear isotherm [15]. This model can be written as a $2D$ continuous systems model of the form

$$\begin{aligned} \frac{\partial s(x,t)}{\partial t} &= p(x,t) - s(x,t), \quad 0 \le t \le t_1, \\ \frac{\partial p(x,t)}{\partial x} &= s(x,t) - p(x,t), \quad 0 \le x \le l_1, \end{aligned} \tag{1}$$

where l_1 and t_1 are the length and operation period of the considered device, respectively, $s(x,t)$ denotes the density of absorbed substance, that is, the concentration of the polluting substance in the sorbent material, and $p(x,t)$ is the concentration of the polluting substance in the flow at point x and time instant t, see Fig. 1.

The boundary conditions are

$$\begin{aligned} p(0,t) &= \psi(t), \quad 0 \le t \le t_1, \\ s(x,0) &= u(x), \quad 0 \le x \le l_1, \end{aligned} \tag{2}$$

where $\psi(t)$, $0 \le t \le t_1$ is a given continuous function, and $u(x)$ is the control function, which is the sorbent concentration in the device at the initial time $t = 0$. Hence boundary control is the only possibility for this model.

M. Dymkov is with the Department of Mathematics, Belarus State Economic University, Partizanskiy Ave 26, 220070 Minsk, Belarus dymkov-m@bseu.by

K. Galkowski is with the Institute of Control and Computation Engineering, University of Zielona Gora, Poland and is currently a Gerhard Mercator Guest Professor (DFG) in The University of Wuppertal k.galkowski@issi.uz.zgora.pl

E. Rogers is with the School of Electronics and Computer Science, University of Southampton, Southampton, SO17 1BJ, UK etar@ecs.soton.ac.uk

V. Dymkou is with the Department of Business Process Analysis and Design of JSC Belagroprombank, Zhukov Ave. 3, 220036 Minsk, Belarus dymkov@mail.ru

S. Dymkou is with the Temasek Laboratories, 5A Engineering Drive 1, National University of Singapore, tslsmd@nus.edu.sg

This paper considers the problem of finding the control function $u(x)$ that minimizes the cost function

$$\begin{aligned} J(u) &= \int_0^{l_1} |u(x)|^2\,dx + \int_0^{l_1}\int_0^{t_1} |p(x,t)|^2 dxdt \\ &+ \int_0^{l_1}\int_0^{t_1} \left[\left|\frac{\partial p(x,t)}{\partial x}\right|^2 + \left|\frac{\partial p(x,t)}{\partial t}\right|^2 \right] dx\,dt, \quad (3) \end{aligned}$$

where $G_i(x,t) > 0, i = 0,1,2,$ and $R_k(x) > 0,\ k = 0,1,$ are given square integrable functions. The aim is to minimize the concentration of the polluting substance in the output flow, sorbent consumption, and the rates of change of these variables.

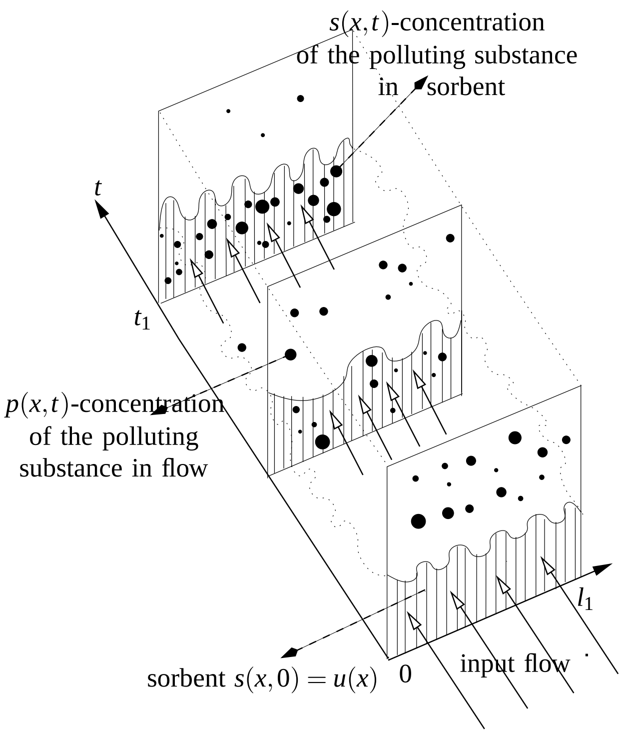

Fig. 1. Schematic diagram of the sorption process.

Remark 1: In (1) unit coefficients for variables is assumed for ease of notation, as have unit weighting terms in the cost function (3).

Suppose that the function $p(x,t)$ is continuously twice differentiable and the first derivative $\frac{\partial s(x,t)}{\partial x}$ is continuous. Then the model described by (1) and (2) can be rewritten as the following partial differential equation

$$\frac{\partial^2 p(x,t)}{\partial x \partial t} + \frac{\partial p(x,t)}{\partial x} + \frac{\partial p(x,t)}{\partial t} = 0, \quad (4)$$

with mixed boundary conditions

$$\begin{aligned} p(0,t) &= \psi(t), \quad 0 \le t \le t_1, \\ p(x,0) + \frac{\partial p(x,0)}{\partial x} &= u(x), \quad 0 \le x \le l_1 \quad (5) \end{aligned}$$

and

$$s(x,t) = p(x,t) + \frac{\partial p(x,t)}{\partial x}.$$

III. Solvability and Representation formula

For the optimization problem defined by (1)–(3), let $C(K)$ denote the space of continuous functions defined on some set K, $AC(K)$ the space of absolutely continuous functions on K, $L^2(K)$ the space of measurable and square integratible functions on K, and $C^1(\Pi)$, $\Pi := L \times T$, the space of differentiable functions $f(x,t)$ defined on some open domain $\Omega \supset \Pi$ with continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial t}$.

Lemma 1: The system model described by (1) and (2) has an unique absolutely continuous solution $p(x,t)$, $s(x,t)$ for any initial function $\psi(\cdot) \in C[0,t_1]$ and control input $u(\cdot) \in L^2[0,l_1]$.

Proof: It is routine to verify by differentiation that if the absolutely continuous function $p(x,t)$ is a solution of the following integral equation

$$\begin{aligned} p(x,t) = e^{-(x+t)} \int_0^x \int_0^t e^{\xi+\eta} p(\xi,\eta) d\xi d\eta + \\ e^{-(x+t)} \int_0^x e^z u(z) dz + e^{-x}\psi(t), \end{aligned} \quad (6)$$

then $p(x,t)$ and

$$s(x,t) = \int_0^t e^{\eta - t} p(x,\eta) d\eta + e^{-t} u(x) \quad (7)$$

satisfy (1) and (2). Also it can be shown that the operator T given by

$$\begin{aligned} (Tf)(x,t) = \int_0^x \int_0^t e^{\xi+\eta} f(\xi,\eta) d\xi d\eta + \\ e^{-(x+t)} \int_0^x e^z u(z) dz + e^{-x}\psi(t), \end{aligned} \quad (8)$$

is contractive [16] in the corresponding Sobolev space $W_2^1(\Pi)$ for any given functions $\psi(\cdot) \in C[0,t_1]$ and $u(\cdot) \in L^2[0,l_1]$.

The fixed-point theorem provides the solution of the integral equation (6), the corresponding $s(x,t)$ can be found from integral expression (7), and the proof is complete. ■

To obtain the representation formula for $p(x,t)$ introduce the following parametric integral equation

$$\begin{aligned} p(x,t) = \mu e^{-(x+t)} \int_0^x \int_0^t e^{\eta+\xi} p(\eta,\xi) d\eta d\xi \\ + e^{-(x+t)} \int_0^x e^z u(z) dz + e^{-x}\psi(t),\ t \in [0,t_1],\ x \in [0,l_1], \end{aligned} \quad (9)$$

with respect to the unknown function $p(x,t)$, where μ is some scalar parameter. This equation is of the Volterra type whose right-hand side is a contractive operator for any given functions $\psi(\cdot) \in C[0,t_1]$ and $u(\cdot) \in L^2[0,l_1]$. Hence, the existence of an unique solution of (9) again follows from the fixed-point theorem [16].

The solution of the integral equation (9) can also be written as a power series in the parameter μ as

$$p(x,t) = \sum_{n=0}^{\infty} p_n(x,t)\mu^n, \tag{10}$$

where

$$p_0(x,t) = e^{-x}\psi(t) + e^{-(x+t)} \int_0^x e^z u(z)dz,$$

$$p_n(x,t) = e^{-(x+t)} \int_0^x \int_0^t K_n(x,t,\xi,\eta) p_0(\xi,\eta) d\xi d\eta, \quad n = 1,2,\ldots \tag{11}$$

and the kernels $K_n(x,t,\xi,\eta)$ are defined by the following recursion formula

$$K_{n+1}(x,t,\xi,\eta) = \int_\xi^x \int_\eta^t K_n(z,\tau,\xi,\eta) dz d\tau, \tag{12}$$

$$n = 0,1,2,\ldots, \; K_1(x,t,\xi,\eta) = e^{\xi+\eta}.$$

Moreover, the solution of the integral equations in (12) can be written in the form

$$\begin{aligned} K_{n+1}(x,t,\xi,\eta) &= e^{-(x+t)} e^{\xi+\eta} \frac{(x-\xi)^n}{n!} \frac{(t-\eta)^n}{n!} \\ 0 &\le x,\xi \le l_1, \quad 0 \le t,\eta \le t_1. \end{aligned} \tag{13}$$

Since the functions $p_n(x,t)$ are bounded on the domain Π the power series (10) is absolutely and uniformly convergent for each finite parameter value μ. Also, under the given assumptions, for any finite parameter μ and for $t \ge \tau$ the following power series

$$R(x,t,\xi,\eta,\mu) = \sum_{n=0}^{\infty} K_n(x,t,\xi,\eta)\mu^n \tag{14}$$

is absolutely and uniformly convergent to the resolvent function $R(x,\xi,t,\tau,\mu)$. It is easy to check that this function satisfies the both integral equation

$$R(x,\xi,t,\eta,\mu) = e^{\xi+\eta} + \mu \int_\xi^x \int_\eta^t R(z,\tau,\xi,\eta) dz d\tau, \tag{15}$$

and the integro-differential equation

$$\frac{\partial R(x,t,\xi,\eta,\mu)}{\partial x} = \mu \int_\eta^t R(x,\tau,\xi,\eta,\mu) d\tau, \tag{16}$$

with initial conditions

$$R(x,t,x,\eta,\mu) = e^{x+\eta}, \quad R(x,t,\xi,t,\mu) = e^{\xi+t}. \tag{17}$$

Setting $\mu = 1$ in these last two formulas yields the solution $p(x,t)$ of (1) and (2), and the corresponding $s(x,t)$ is given by (7). The following theorem is a formal statement of these facts.

Theorem 1: For the given admissible control function $u(x)$ the system described by (1) and (2) has a unique solution $(p(x,t), s(x,t))$ given by

$$\begin{aligned} p(x,t) &= e^{-x}\psi(t) + e^{-(x+t)} \int_0^x e^z u(z)dz \\ &+ e^{-(x+t)} \Bigg[\int_0^x \int_0^t e^{-\xi} R(x,t,\xi,\eta)\psi(\eta) d\xi d\eta \\ &+ \int_0^x e^z u(z) \Big[\int_z^x \int_0^t e^{-(\xi+\eta)} R(x,t,\xi,\eta) d\xi d\eta \Big] dz \Bigg], \\ s(x,t) &= \int_0^t e^{\eta-t} p(x,\eta) d\eta + e^{-t} u(x), \end{aligned} \tag{18}$$

where the function $R(x,t,\xi,\eta)$ satisfies (15) and (17) with $\mu = 1$.

IV. Optimality conditions

In this section the representation formula of (18) and the operator setting are used to develop a solution of the optimization problem (1)–(3) in the form of the next theorem.

Theorem 2: For any initial function $\psi(\cdot) \in C[0,t_1]$ the optimization problem (1)–(3) has an unique optimal solution $u^0(\cdot) \in L^2[0,l_1]$.

Proof: Introduce the inner product

$$(u,v)_2 \triangleq \int_0^{l_1} u(x)v(x)dx, \quad \forall\, u,v \in L^2[0,l_1], \tag{19}$$

on the space $L^2[0,l_1])$, and define the inner product on the space $AC(\Pi)$ as

$$\begin{aligned} (\phi,\psi)_1 &\triangleq \int_0^{t_1}\int_0^{l_1} \varphi(x,t)\psi(x,t) dx dt + \\ &\int_0^{t_1}\int_0^{l_1} \left[\frac{\partial\varphi(x,t)}{\partial x}\frac{\partial\psi(x,t)}{\partial x} + \frac{\partial\varphi(x,t)}{\partial t}\frac{\partial\psi(x,t)}{\partial t} \right] dx dt. \end{aligned} \tag{20}$$

Also introduce the operator $\mathscr{L}: L^2[0,l_1] \to AC(\Pi)$

$$\begin{aligned} (\mathscr{L}u)(x,t) &= e^{-(x+t)} \int_0^x e^z u(z)dz \\ &+ \int_0^x e^z u(z) \Big[\int_z^x \int_0^t e^{-(\xi+\eta)} R(x,t,\xi,\eta) d\xi d\eta \Big] dz, \end{aligned} \tag{21}$$

and the operator $\mathscr{F}: C[0,t_1] \to AC(\Pi)$

$$\begin{aligned}\big(\mathscr{F}\big)(x,t) &= e^{-x}\psi(t) \\ &+ e^{-(x+t)}\int_0^x\int_0^t e^{-\xi}R(x,t,\xi,\eta)\psi(\eta)d\xi d\eta.\end{aligned} \tag{22}$$

Then the solution $p \in AC(\Pi)$ of the first equation of (1) with the initial conditions (2) can be rewritten in operator form as

$$p = \mathscr{L}u + \mathscr{F}\varphi, \tag{23}$$

and hence the cost function (3) can be written as

$$\begin{aligned} J(u) &= \Big(p,p\Big)_1 + \Big(u,u\Big)_2 \\ &= \Big((\mathscr{L}u+\mathscr{F}\varphi),(\mathscr{L}u+\mathscr{F}\varphi)\Big)_1 + \Big(u,u\Big)_2 \\ &= \Big(u,(\mathscr{L}^*\mathscr{L}+\mathscr{E})u\Big)_2 + 2\Big(u,(\mathscr{L}^*\mathscr{F})\varphi\Big)_2 \\ &+ \Big(\varphi,\mathscr{F}^*\mathscr{F}\varphi\Big)_2,\end{aligned} \tag{24}$$

where $\mathscr{E}$ denotes the identity operator in $L^2[0,l_1]$ and $\mathscr{L}^*: AC(\Pi) \to L^2[0,l_1]$ denotes the adjoint operator of the operator $\mathscr{L}$ with respect to the scalar products defined above.

Since the operator $\mathscr{L}^*\mathscr{L}+\mathscr{E}$ is invertible, the following control function in operator form can be introduced

$$u^0 = -(\mathscr{L}^*\mathscr{L}+\mathscr{E})^{-1}\mathscr{L}^*\mathscr{F}\varphi, \tag{25}$$

and in order to prove that u^0 is an optimal solution of the problem it is sufficient to check the inequality $J(u)-J(u^0) \geq 0$ for all admissible $u \in L^2[0,l_1]$. Let $\Gamma = (\mathscr{L}^*\mathscr{L}+\mathscr{E})$. Then

$$J(u)-J(u^0) = (\Gamma(u-u^0),(u-u^0))_2,$$

and, since $\Gamma = \mathscr{L}^*\mathscr{L}+\mathscr{E} > 0$,

$$\begin{aligned}&J(u)-J(u^0) = \\ &((\mathscr{E}+\mathscr{L}^*\mathscr{L})(u-u^0),(u-u^0))_2 > 0,\end{aligned}$$

for any admissible u, $u \neq u^0$. This last fact means that the function u^0 given by (25) is the unique optimal control and $p^0 = \mathscr{L}u^0 + \mathscr{F}\varphi$ is the corresponding optimal state for the problem (1)–(3), and the proof is complete. ■

The solution of the linear quadratic optimal control problem for $1D$ linear systems can be written as state feedback. The next theorem shows that the solution of the optimization problem considered in this paper can also be written as state feedback.

Theorem 3: The optimal control $u^0(z)$, $0 \leq z \leq l_1$, for the problem (1)–(3) can be written as

$$\begin{aligned} u^0(z) = e^z\Bigg[&\int_0^{t_1} e^{-t}\frac{\partial p^0(z,t)}{\partial z}dt + \int_z^{l_1}\int_0^{t_1}\Bigg(e^{-(x+t)}p^0(x,t) \\ &+ (t-e^{-(x+t)})\frac{\partial p^0(x,t)}{\partial x} + (x-z-e^{-(x+t)})\frac{\partial p^0(x,t)}{\partial t} + \\ &\int_z^x\int_0^t e^{-(\xi+\eta)}\Bigg(p^0(x,t)R(x,t,\xi,\eta) + \frac{\partial p^0(x,t)}{\partial x}\frac{\partial R(x,t,\xi,\eta)}{\partial x} \\ &+ \frac{\partial p^0(x,t)}{\partial t}\frac{\partial R(x,t,\xi,\eta)}{\partial t}\Bigg)d\xi d\eta\Bigg)dxdt\Bigg],\end{aligned}$$

where $R(x,t,\xi,\eta)$ satisfies (15) and (17), with $\mu = 1$ and $p^0(x,t)$ is solution of (1) and (2) at optimality.

Proof: From (25)

$$\mathscr{L}^*\Big(\mathscr{L}u^0 + \mathscr{F}\varphi\Big) + \mathscr{E}u^0 = 0, \tag{26}$$

and using (23) it follows that $p^0 \in AC(\Pi)$ satisfies

$$p^0 = \mathscr{L}u^0 + \mathscr{F}\varphi.$$

Also (26) can be rewritten as

$$\mathscr{L}^*p^0 + \mathscr{E}u^0 = 0,$$

and hence

$$u_0 = -\mathscr{L}^*p^0. \tag{27}$$

The adjoint operator $\mathscr{L}^*$ of $\mathscr{L}$ of (21) satisfies, with $u \in L_2[0,l_1]$, $\varphi AC(\Pi)$ and the scalar product of (19),

$$(\varphi,\mathscr{L}u)_1 = (\mathscr{L}^*\varphi,u)_2. \tag{28}$$

In particular

$$\begin{aligned}(\varphi,\mathscr{L}u)_1 = &\int_0^{l_1}\int_0^{t_1}\Big(\varphi(x,t),(\mathscr{L}u)(x,t)\Big)dxdt + \\ &\int_0^{l_1}\int_0^{t_1}\frac{\partial\varphi(x,t)}{\partial x}\frac{\partial(\mathscr{L}u)(x,t)}{\partial x} + \frac{\partial\varphi(x,t)}{\partial t}\frac{\partial(\mathscr{L}u)(x,t)}{\partial t}dxdt,\end{aligned}$$

where

$$\begin{aligned}\big(\mathscr{L}u\big)(x,t) &= e^{-(x+t)}\int_0^x e^z u(z)dz \\ &+ \int_0^x e^z u(z)\Big[\int_z^x\int_0^t e^{-(\xi+\eta)}R(x,t,\xi,\eta)d\xi d\eta\Big]dz,\end{aligned}$$

and it is routine to verify that

$$\begin{aligned}\frac{\partial(\mathscr{L}u)(x,t)}{\partial t} &= e^{-t}u(x) + (t-e^{-(x+t)})\int_0^x e^z u(z)dz \\ &+ \int_0^x e^z u(z)\Bigg(\int_x^x\int_0^t e^{-(\xi+\eta)}\frac{\partial R(x,t,\xi,\eta)}{\partial t}d\xi d\eta\Bigg)dz.\end{aligned}$$

Calculating the required derivative $\dfrac{\partial(\mathscr{L}u)(x,t)}{\partial x}$ in the same manner as above, substituting in the last formula, and interchanging the order of integration gives $\mathscr{L}^*: AC(\Pi) \to L^2[0,l_1]$ as

$$(\mathscr{L}^*\varphi)(z) = e^z \Bigg[\int_0^{t_1} e^{-t} \frac{\partial \varphi(z,t)}{\partial z} dt + \int_z^{l_1} \int_0^{t_1} \Bigg(e^{-(x+t)} \varphi(x,t)$$
$$+ (t - e^{-(x+t)}) \frac{\partial \varphi(x,t)}{\partial x} + (x - z - e^{-(x+t)}) \frac{\partial \varphi(x,t)}{\partial t}$$
$$+ \int_z^x \int_0^t e^{-(\xi+\eta)} \Bigg(\varphi(x,t) R(x,t,\xi,\eta) + \frac{\partial \varphi(x,t)}{\partial x} \frac{\partial R(x,t,\xi,\eta)}{\partial x}$$
$$+ \frac{\partial \varphi(x,t)}{\partial t} \frac{\partial R(x,t,\xi,\eta)}{\partial t} \Bigg) d\xi d\eta \Bigg) dx dt. \Bigg]$$

Substitution in (27) gives the required formula and the proof is complete. ∎

V. CONCLUSIONS AND FURTHER WORK

In this paper the sorption process has been modeled as a $2D$ system described by the continuous variable form of the Roesser model. Moreover, an optimal control problem has been formulated and solved. Further research should aim to extend to more complicated sorption networks, which are much more relevant for applications.

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