Computer algebra methods for testing the stability and the stabilizability of multidimensional systems

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Overview

1. Problems under consideration
2. Structural stability of multidimensional systems
3. Solving systems of algebraic equations
4. Stabilizability of multidimensional systems
5. Systems with parameters

Overview

Problems under consideration

- **Input**: MIMO linear n-D systems given under a matrix fraction description

$$P(z) = D^{-1}(z)\, N(z)$$

where $N(z), D(z)$ are n-D polynomial matrices.

- The closed unit polydisc of $\mathbb{C}^n$:

$$\overline{\mathbb{D}}^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1,\ i = 1, \ldots, n,\ \}.$$

- **Stability:** All the entries of the matrix $D^{-1}(z)\, N(z)$ do not have zeros inside $\overline{\mathbb{D}}^n$, i.e.:

$$d(z_1, \ldots, z_n) \neq 0, |z_1| \leq 1, \ldots, |z_n| \leq 1$$

- **Stabilizability:** The reduced minors of $[D(z)\ \ -N(z)]$ do not have common zeros inside $\overline{\mathbb{D}}^n$, i.e.:

$$V(\langle p_1(z_1, \ldots, z_n), \ldots, p_s(z_1, \ldots, z_n)\rangle) \cap \overline{\mathbb{D}}^n = \emptyset$$

1-D linear systems

- Matrices with entries polynomials in $\mathbb{Q}[z]$
- Localization of complex zeros of univariate polynomials
- $\mathbb{Q}[z]$ is an Euclidean domain $\rightsquigarrow$ Remainder sequence, gcd
  - Numerically: Netwon method $\rightsquigarrow$ Non-certified
  - Symbolically: Cauchy index, Sturm sequences $\rightsquigarrow$ certified
- Algebraic stability tests, e.g. Hurwitz, Jury, Bistritz,. . .

$$d(z) := a_n\, z^n + \ldots + a_0 \qquad d^\star(z) := z^n\, d(z^{-1})$$

$$\begin{cases} T_n(z) := d(z) + d^\star(z), \\ T_{n-1}(z) := \dfrac{d(z) + d^\star(z)}{(z-1)}, \\ T_{i-1}(z) := \dfrac{\delta_{i+1}(1+z)T_i(z) - T_{i+1}(z)}{z}, \end{cases}$$

where $\delta_{i+1} := \frac{T_{i+1}(0)}{T_i(0)}$ for $i = n-1, \ldots, 1$.

$\rightsquigarrow$ The system is stable if and only if the sequence is normal and the number of sign variation in $\{T_n(1), \ldots, T_0(1)\}$ is zero.

n-D linear systems

- Matrices involving polynomials in $\mathbb{Q}[z_1, \ldots, z_n]$
- Geometric objects: Algebraic varieties of arbitrary dimension in $\mathbb{C}^n$
- Stability and stabilizability conditions: Semi-algebraic sets in $\mathbb{R}^{2n}$
- Goal: Generalization of the 1-D case
- Existing work:
  - $n = 2$ Several practical algorithms (Bose, Jury, Bistritz, ...)
  - $n \geq 3$ Very few results and no practical criterion
- Our tools: Algebraic-geometric dictionnary (Ideals, Varieties, Variable elimination, Nullstelensatz,...)

Study via semi-algebraic sets

• $z_k := x_k + i\, y_k, \quad x_k,\, y_k \in \mathbb{R}, \quad k = 1, \ldots, n, \quad i^2 = -1.$

Problems are equivalent to the study of semi-algebraic systems:

$$(S) \begin{cases} p_1 := \mathcal{R}_1(x_1, y_1, \ldots, x_n, y_n) + i\, \mathcal{I}_1(x_1, y_1, \ldots, x_n, y_n) \neq 0 \\ \quad \vdots \\ p_s := \mathcal{R}_s(x_1, y_1, \ldots, x_n, y_n) + i\, \mathcal{I}_s(x_1, y_1, \ldots, x_n, y_n) \neq 0 \\ x_1^2 + y_1^2 - 1 \leq 0, \\ \quad \vdots \\ x_n^2 + y_n^2 - 1 \leq 0. \end{cases}$$

• Zero-dimensional systems $\rightsquigarrow$ univariate representation, triangular representation, Gröbner bases.

• Systems with positive dimensions $\rightsquigarrow$ cylindrical algebraic decomposition, critical points methods.

Drawback: The number of variables is doubled!

Overview

Structural stability

- Given a polynomial $d(z_1, \dots, z_n) \in \mathbb{R}(z_1, \dots, z_n)$
- **Definition:** d is structurally stable if it is devoid of zero in $\overline{\mathbb{D}}^n$, i.e.:

$$\forall\, z = (z_1, \dots, z_n) \in \overline{\mathbb{D}}^n : \; d(z_1, \dots, z_n) \neq 0. \qquad (1)$$

- The affine algebraic set associated to $d \in \mathbb{R}[z_1, \dots, z_n]$:

$$V_{\mathbb{C}}(d) := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid d(z_1, \dots, z_n) = 0\}.$$

- Condition (1) is equivalent to:

$$V_{\mathbb{C}}(d) \cap \overline{\mathbb{D}}^n = \emptyset.$$

Structural stability : simplified conditions

- Condition (1) is equivalent to the set of conditions [DeCarlo et al.].

$$\begin{cases} d(z_1, 1, \ldots, 1) \neq 0, & |z_1| \leq 1, \\ d(1, z_2, 1, \ldots, 1) \neq 0, & |z_2| \leq 1, \\ \vdots & \vdots \\ d(1, \ldots, 1, z_n) \neq 0, & |z_n| \leq 1, \\ d(z_1, \ldots, z_n) \neq 0, & |z_1| = \ldots = |z_n| = 1. \end{cases}$$

- All the conditions except the last one can be tested using classical univariate stability tests.

- **Focus on the condition:** $d(z_1, \ldots, z_n) \neq 0$, $|z_1| = \ldots = |z_n| = 1$.

$\rightsquigarrow$ Searching for zeros in an n-D subspace of the $2n$-D complex space.

Möbius transformation

- **Definition:** A Möbius transformation is a rational function

$$\begin{array}{rcl} \phi : \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} & \longrightarrow & \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \\ z & \longmapsto & \frac{a\,z+b}{u\,z+v}, \end{array}$$

for $a, b, u, v \in \mathbb{C}$ satisfying $av - bu \neq 0$ $\left(\phi\left(-\frac{v}{u}\right) = \infty,\ \phi(\infty) = \frac{a}{u}\right)$.

- The Möbius transformation $\phi(z) := \frac{z-i}{z+i}$ maps the real line $\overline{\mathbb{R}} := \mathbb{R} \cup \infty$ to the unit complex circle $\mathbb{T}$.

  - $z_k := \dfrac{(x_k - i)}{(x_k + i)}, \quad k = 1, \ldots, n.$

- Let $\mathcal{R}(x_1, \ldots, x_n) + i\,\mathcal{I}(x_1, \ldots, x_n)$ be the numerator of the fraction:

$$d\left(\frac{x_1 - i}{x_1 + i}, \ldots, \frac{x_n - i}{x_n + i}\right).$$

- **Theorem:** $\mathcal{V}_{\mathbb{C}}(d) \cap [\mathbb{T} \setminus \{1\}]^n = \emptyset \iff \mathcal{V}_{\mathbb{R}}(\mathcal{R}, \mathcal{I}) = \emptyset.$

- **Remark:** The total degree of $\mathcal{R}$ and $\mathcal{I}$ is bounded by $\sum_{i=1}^{n} deg_{z_i}(d)$

Real algebraic stability condition

The test of stability reduces to deciding the existence of real zeros of algebraic systems of the form

$$\{\mathcal{R}(x_1, \ldots, x_n) = \mathcal{I}(x_1, \ldots, x_n) = 0\}$$

- The corresponding algebraic varieties are of two types:
- The case of 2-D systems $\rightsquigarrow$ zero-dimensional variety
- The case of n-D systems, $n \geq 3$ $\rightsquigarrow$ Variety of codimension at most 2

Overview

Zero-dimensional systems

- Goal: Numerical isolating boxes around the real solutions $\rightsquigarrow$ answer for the existence of real solutions

- Several methods:

  Numerical: Local analysis, non certified results except for particular systems (e.g. squarefree)

  Symbolic: Global solutions, certified

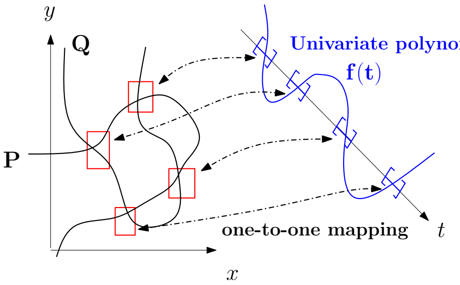
- The principle of symbolic methods is to reduce the problem to a univariate one

- Our tool: Rational Univariate Representation [Rouillier 99]

Zero-dimensional systems : The 2D case

• Consider a zero-dimensional ideal $I := \langle P(x_1, x_2), Q(x_1, x_2) \rangle$

• A Rational Univariate Representation of I is a one-to-one mapping between the points of $V_{\mathbb{C}}(I)$ and the roots of a univariate polynomial



$$\begin{array}{ccc} \mathbf{V}(\{\mathbf{P},\mathbf{Q}\}) & \rightarrow & \mathbf{V}(\mathbf{f}) \\ (\mathbf{x},\mathbf{y}) & \mapsto & \mathbf{x}+\mathbf{a}\mathbf{y} \\ \left(\frac{\mathbf{f_x}(\mathbf{t})}{\mathbf{f_1}(\mathbf{t})}, \frac{\mathbf{f_y}(\mathbf{t})}{\mathbf{f_1}(\mathbf{t})}\right) & \leftarrow & \mathbf{t} \end{array}$$

• Computation:

$\rightsquigarrow$ Linear algebra in the finite-dimensional Q-vector space $\frac{\mathbb{Q}[x_1,x_2]}{I}$

$\rightsquigarrow$ Resultant and subresultant polynomials

Rational Univariate Representation

- $I \subset \mathbb{R}[x_1, \ldots, x_n]$ a zero-dimensional ideal and $V(I) \subset \mathbb{C}^n$ its variety.

A Rational Univariate Representation of I is given by:

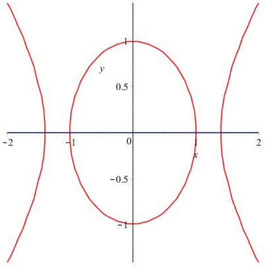
- A linear form $a_1 x_1 + \ldots + a_n x_n$ that separates the points of V.
- A one-to-one mapping between the roots of a univariate polynomial f and the solutions of V:

$$\begin{array}{cccc} \phi_t : & V_{\mathbb{C}}(I) & \approx & V_{\mathbb{C}}(f) \\ & \alpha & \longmapsto & t(\alpha), \\ & \left(\frac{f_{x_1}(\beta)}{f_1(\beta)}, \ldots, \frac{f_{x_n}(\beta)}{f_1(\beta)}\right) & \longleftarrow & \beta. \end{array}$$

- $V(I) \cap \mathbb{R}^n = \emptyset$ if and only if $V(f) \cap \mathbb{R} = \emptyset \rightsquigarrow$ Sturm sequence.

Systems with positive dimension

- Goal: Deciding the existence of real zeros
- Principle: Search for one real zero in each connected component
- Example: $f(x, y) = (x^2 - y^2 - 2) * (x^2 + y^2 - 1) = 0$, a curve in $\mathbb{C}^2$



$$\begin{array}{cccc} \pi : & \mathbb{C}^2 & \to & \mathbb{C} \\ & (x, y) & \mapsto & x \end{array}$$

Critical points of π: $\left\{ \begin{array}{l} f(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{array} \right.$

$$\rightsquigarrow \left\{ \begin{array}{l} f(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{array} \right.$$

$$\rightsquigarrow \left\{ \begin{array}{l} (x^2 - y^2 - 2) * (x^2 + y^2 - 1) = 0 \\ -4 * y^3 - 2 * y = 0 \end{array} \right.$$

- The critical points of π are $(-\sqrt{2}, 0), (-1, 0), (1, 0), (\sqrt{2}, 0)$

Critical point methods

• **Principle:** Computation of the critical points of a polynomial application Φ restricted to the algebraic set $\mathcal{V} := V(\langle \mathcal{R}, \mathcal{C} \rangle)$.

• **Theorem:** Under mild conditions, the set of critical points of Φ is finite and meets the algebraic set $\mathcal{V}$ on each of its real connected components.

• Compute zero-dimensional systems that encode these critical points and check if they admit real solutions.

$\rightsquigarrow$ Rational Univariate Representation (RUR).

The overall algorithm

Procedure: IsStable

```
begin
    Data : D(z_1, ..., z_n) ∈ R[z_1, ..., z_n]
    Result : return True if V(D(z_1, ..., z_n)) ∩ 𝔻^n = ∅
    for k = 0 to n − 2 do
        Compute S_k, the set of polynomials obtained from D(z_1, ..., z_n) after
        substituting k variables by 1
        foreach D_k in S_k do
            {R, C} = Möbius_transform(D_k)
            if V_ℝ({R, C}) ≠ ∅ then
                return False
            end
        end
    end
    if all the univariate polynomials in S_{n−1} are stable then
        return True
        else
            return false
        end
    end
end
```

Implementation

- A Maple procedure is provided based on:
  - The univariate stability test of Bistritz.
  - The library RS for the real zero of 2D systems
  - The Maple routine HasRealRoots for the study of real zeros of polynomial algebraic systems

nb var \ degree		3	5	8	10
2	sparse	0.074	0.087	0.21	0.38
	dense	0.078	0.13	0.61	1.82
3	sparse	0.31	0.51	2.31	4.71
	dense	0.36	1.05	9.77	36.70
4	sparse	2.03	4.87	19.68	32.64
	dense	3.32	75.71	350	t/o

Table: CPU times in seconds of `IsStable` run on random polynomials in 2,3 and 4 variables with rational coefficients.

Overview

Stabilizability

- $\{p_1, \ldots, p_s\}$ are the reduced minors of the matrix $[D(z) - N(z)]^T$
- $I = \langle p_1, \ldots, d_s \rangle \subset \mathbb{R}[z_1, \ldots, z_n]$ is the ideal generated by these polynomials
- The associated algebraic variety $V_{\mathbb{C}}(I)$ is given as

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid p_1(z_1, \ldots, z_n) = \cdots = p_s(z_1, \ldots, z_n) = 0\}.$$

- **Definition:** P is Stabilizable if

$$V_{\mathbb{C}}(I) \cap \overline{\mathbb{D}}^n = \emptyset.$$

- No simplified conditions in the general case
- We restrict the study to zero-dimensional ideal $I := \langle p_1, \ldots, p_s \rangle$:

$$\sharp V_{\mathbb{C}}(I) < \infty$$

Stabilizability through RUR compuation

- Compute a Univariate Representation of $\langle p_1, \ldots, p_s \rangle$

$$\begin{cases} f(t) & = & 0 \\ z_1 & = & \frac{f_{z_1}}{f_1}(t) \\ \vdots & \vdots & \vdots \\ z_n & = & \frac{f_{z_n}}{f_1}(t) \end{cases}$$

- Isolate solutions into pair of intervals $z_k = [a_{k,1}, a_{k,2}] + i\,[b_{k,1}, b_{k,2}]$
- Compute the sign of $[a_{k,1}, a_{k,2}]^2 + [b_{k,1}, b_{k,2}]^2 - 1$

  $\rightsquigarrow$ May requires some refinements
- What if some solutions are close to the poly-circle ?

  $\rightsquigarrow$ Cannot conclude
- Construct an algebraic system that characterize these solutions
- Apply $z_k = \frac{x-i}{x+i}, i = 1, \ldots, n \rightsquigarrow$ Real zeros of $\{p_1(x), \ldots, p_s(x)\}$

Stabilizability and stabilization

- To summarize, testing the stabilizability resumes to test that

1. $V_{\mathbb{C}}(\langle p_1, \ldots, p_s \rangle) \cap \mathbb{U}^n = \emptyset$
2. $V_{\mathbb{C}}(\langle p_1, \ldots, p_s \rangle) \cap \mathbb{D}^n = \emptyset$

Theorem (Polydisk Nullstelensatz)

Let $p_1, \ldots, p_s \in \mathbb{Q}[z_1, \ldots, z_s]$ be such that $V_{\mathbb{C}}(\langle p_1, \ldots, p_s \rangle) \cap \overline{\mathbb{D}}^n = \emptyset$, then there exists a polynomial S as well as $u_1, \ldots, u_s$ in $\mathbb{Q}[z_1, \ldots, z_s]$ and an integer $e > 0$ such that

$$S^e(z_1, \ldots, z_n) = \sum_{i=1}^{s} u_i(z_1, \ldots, z_n)\, p_i(z_1, \ldots, z_n)$$

and $V_{\mathbb{C}}(S(z_1, \ldots, z_n)) \cap \overline{\mathbb{D}}^n = \emptyset$

- $S(z_1, \ldots, z_n)$ is used to construct a stabilizing compensator
- First constructive proof for the zero-dim case (**Guillaume talk**)

Overview

Stability of 2D systems with parameters

• $\frac{N(z_1,z_2,U)}{D(z_1,z_2,U)}$ is a transfert function where $N, D \in \mathbb{R}[U][z_1, z_2]$ and $U = [U_1, \ldots, U_k]$ is a set of real parameters.

• **Goal:** Compute regions in the parameter's space $\mathbb{R}^k$ in which the underlying system (after substitution of the parameters) is stable.

• $\bigcup_i \mathcal{U}_i$ such that $\mathcal{U}_i$ are semi-algebraic sets in $\mathbb{R}^k$ and $\forall (u_1, \ldots, u_k) \in \mathcal{U}_i$

$$D(z_1, z_2, u_1, \ldots, u_k) \neq 0 \text{ for } |z_1| \leq 1, |z_2| \leq 0$$

Or, according to Decarlo et al.

$$\begin{cases} D(z_1, 1, u1, \ldots, u_k) \neq 0, |z_1| \leq 1, \\ D(1, z_2, u1, \ldots, u_k) \neq 0, |z_1| \leq 1, \\ D(z_1, z_2, u1, \ldots, u_k) \neq 0, |z_1| = |z_2| = 1. \end{cases} \quad (3)$$

- Compute regions in the parameter's space $\mathbb{R}^k$ such that
  - $D(z, U) \neq 0 \mid |z| \leq 1$
  - $\mathcal{S} := \{\mathcal{R}(x, y, U) = \mathcal{I}(x, y, U) = 0\}$ does not have real zeros.

- **Approach:** Use elimination to compute a set of polynomials in $\mathbb{Q}[U]$ that decomposes $\mathbb{R}^k$ into the desired regions.

- We focus in the sequel on the second condition

- Decompose $\mathbb{R}^k$ depending on the number of real solutions of $\mathcal{S}$ and select the region for which this number is zero.

Discriminant variety

- Generalization of the classical discriminant.

- $\Pi_U: \quad \mathcal{V} := V(\mathcal{S}) \rightarrow \mathbb{C}^k$, $\quad (x, y, U) \mapsto U$ — The canonical projection

Definition [D. Lazard and F. Rouillier, 04]

- $D(\mathcal{V}) \subset \mathbb{C}^k$ s.t. for all connected open set $\mathcal{U} \subset \mathbb{C}^k / D(\mathcal{V})$:

$(\Pi_U^{-1}(\mathcal{U}) \cap \mathcal{V}, \Pi_U)$ is an analytic covering of $\mathcal{U}$.

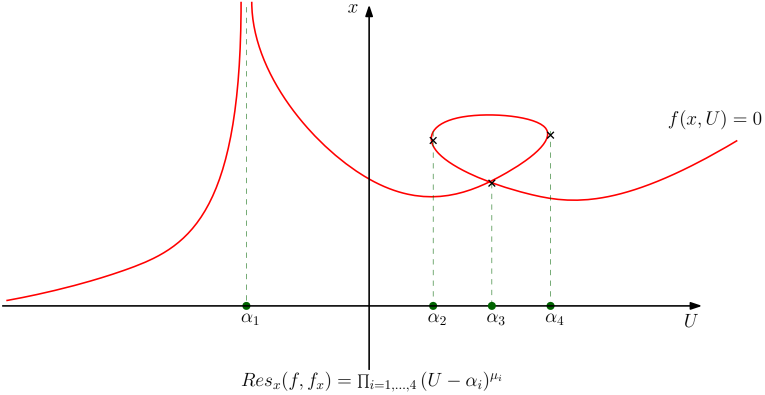
Key property in the real

- For all connected open set $\mathcal{U} \subset \mathbb{C}^k / D(\mathcal{V})$:

Number of real zeros of $\mathcal{S}_u$ is constant for all $u \in \mathcal{U}$

Discriminant variety : a simple case

- Let $f(x, U) \in \mathbb{Q}[U][x]$ be a polynomial in one variable x.



- The discriminant is the resultant of f and its derivative w.r.t x.
- $\forall u_0$ in any open interval (α_i, α_{i+1}), the number of real roots of $f(x, u_0)$ is constant.

Discriminant variety: computation

- In our setting, the discriminant variety of $\{\mathcal{R} = \mathcal{I} = 0\}$ is union of:
  - O_{mult} Projection of the multiple solutions.
  - O_{∞} Projection of the solutions at infinity.

Computation of the discriminant variety

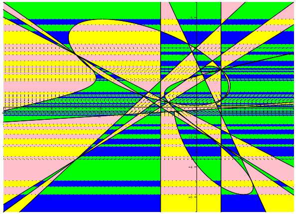
- $O_{mult} = \Pi_U(V(\mathcal{R}, \mathcal{I}, Jac_{x,y}(\mathcal{R}, \mathcal{I}))) \rightsquigarrow$ Elimination via Gröbner bases
- O_{∞}: The leading coefficients of some Gröbner basis.
- $D(\mathcal{V}) = O_{mult} \cup O_{\infty}$

Computation of $\mathcal{U} \subset \mathbb{R}^k / D(\mathcal{V})$

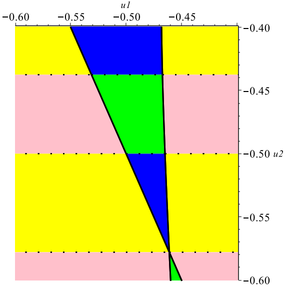
- Cylindrical Algebraic Decomposition adapted to $\mathcal{I}(D(\mathcal{V}))$

Example

- $D(z_1, z_2) = (4u_1 + 2u_2 + 3)z_1 + (-2u_1 + 1)z_2 + (4u_1 - 2u_2 - 2)z_1\, z_2 + (2u_1 - 2u_2 + 4){z_1}^2\, z_2 + (-u_1 - u_2 + 1)z_1\, {z_2}^2$.
- DV consists of an union of 10 lines, 2 quadrics and one curve of degree 6.
- Decomposing the parameter's space w.r.t this DV yields 1161 cells



zoom $\longrightarrow$



- 1043 cells correspond to stable systems, e.g. the cell corresponding to the point $(u_1 = -.5952602220, u_2 = -.5389591122)$