Nash equilibrium with wave dynamics, boundary control

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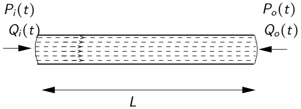
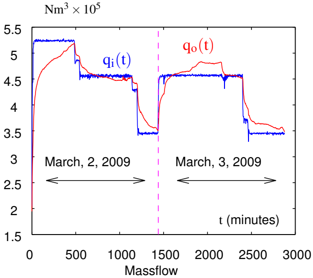
Motivation of the work

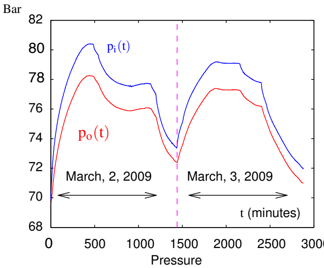
Practical applications exist where distributed boundary control is required.

Example: Gas networks



The gas problem has a repetitive ("periodic") behaviour:





A gas pipeline

Gas dynamics: Hyperbolic PDE

$$\begin{cases} \dfrac{\partial q(t,x)}{\partial t} = -S\dfrac{\partial p(t,x)}{\partial x} - \dfrac{\lambda c^2}{2dS}\dfrac{q^2(t,x)}{p(t,x)}, \\ \dfrac{\partial p(t,x)}{\partial t} = -\dfrac{c^2}{S}\dfrac{\partial q(t,x)}{\partial x}, \end{cases} \tag{1}$$

where

- x is space
- t is time
- p is pressure
- q is mass flow
- S is the cross-sectional area
- d is the pipe diameter
- c is the isothermal speed of sound
- λ is a friction factor.

See (J. Niepłocha, 1988) and (A. Osiadacz, 1987).

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This could be linearised . . .

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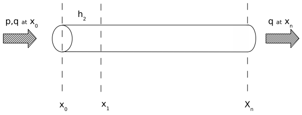
Gas dynamics: Linearisation of the hyperbolic PDE

The linearisation is done around the operational levels: $(\bar{q}, \bar{p}(x))$

- $\bar{q}$ is constant
- $\bar{p}(x)$ is averaged over period of operation T: $\bar{p}(x) = \frac{1}{T}\int_0^T p(x,t)dt\Big|_{x=x_0}$ and

$$\bar{p}(x) = \sqrt{\bar{p}^2(x_0) - \frac{\lambda c^2}{2dS^2}\bar{q}^2\,(x - x_0)}$$



$$\begin{cases} q &= \bar{q} + \Delta q(t,x) \\ p &= \bar{p}(x) + \Delta p(t,x) \end{cases}$$

Δp and Δq are deviations from the reference values

Hence:

$$\frac{q^2}{p} = \frac{(\bar{q} + \Delta q)^2}{\bar{p} + \Delta p} \cong \frac{\bar{q}^2}{\bar{p}(x)} + 2\frac{\bar{q}}{\bar{p}(x)}\Delta q - \frac{\bar{q}^2}{\bar{p}(x)^2}\Delta p. \qquad (2)$$

Linear hyperbolic PDE

Substituting (2) into (1), we obtain:

$$\left\{\begin{array}{lcl} \dfrac{\partial \Delta q(t,x)}{\partial t} & = & -S\dfrac{\partial \Delta p(t,x)}{\partial x} - S\dfrac{\partial \bar{p}(x)}{\partial x} - \dfrac{\lambda c^2}{2dS}\left(\dfrac{\bar{q}^2}{\bar{p}(x)} + 2\dfrac{\bar{q}}{\bar{p}(x)}\Delta q(t,x)\right) \\ & & +\dfrac{\lambda c^2}{2dS}\dfrac{\bar{q}^2}{\bar{p}(x)^2}\Delta p(t,x) \\ \dfrac{\partial \Delta p(t,x)}{\partial t} & = & -\dfrac{c^2}{S}\dfrac{\partial \Delta q(t,x)}{\partial x}. \end{array}\right. \tag{3}$$

Linear hyperbolic PDE

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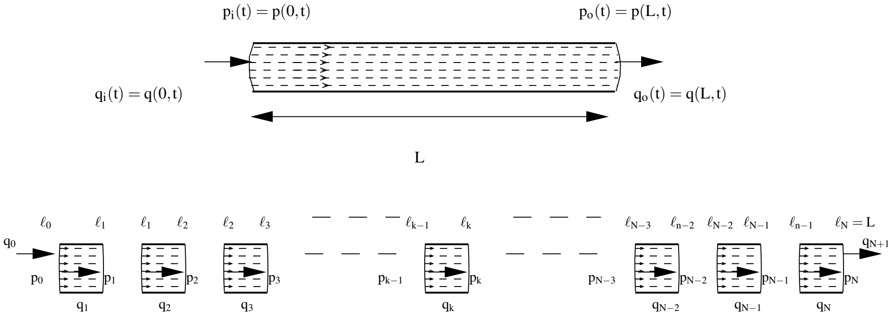
This could be space discretised . . .

Discretisation of the linear hyperbolic PDE



Assumption: constant mass flow in every segment.

Discrete linear hyperbolic PDE

Model (3) becomes:

$$\begin{cases} \Delta q_{k+1}(\ell) & = & \alpha(\ell)\Delta q_k(\ell) + \beta\Delta p_k(\ell-1) + \gamma(\ell)\Delta p(k,\ell) - \beta\Delta p_k(\ell+1) + F(\ell) \\ \Delta p_{k+1}(\ell) & = & \Delta p_k(\ell) + \rho\Delta q_k(\ell+1) - \rho\Delta q_k(\ell-1) \end{cases} \tag{4}$$

where $f(kh_1, \ell h_2) := f_k(\ell)$ and

$$\begin{array}{llllll} \beta & := & \dfrac{Sh_1}{2h_2}, & \xi(\ell) & := & \dfrac{\lambda c^2}{dS}\dfrac{h_1\bar{q}}{\bar{p}(\ell)}, \\ \gamma(\ell) & := & \dfrac{\xi(\ell)}{2\bar{p}(\ell)}\bar{q}, & \alpha(\ell) & := & 1-\xi(\ell), \\ \rho & := & \dfrac{c^2 h_1}{2Sh_2}, & & & \\ F(\ell) & := & -\gamma(\ell)\bar{p}(\ell) - \beta\left(\bar{p}(\ell+1) - \bar{p}(\ell-1)\right). & & & \end{array}$$

Discrete linear hyperbolic PDE

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where $f(kh_1, \ell h_2) := f_k(\ell)$ and

$$\begin{array}{ccc} x_1 & := & \Delta q \\ x_2 & := & \Delta p \end{array} \quad \Longrightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Wave gas model

that is

$$\begin{aligned}
x_{k+1}(\ell) &= A_{-1}x_k(\ell-1) + A_0 x_k(\ell) + A_1 x_k(\ell+1) + \begin{pmatrix} F(\ell) \\ 0 \end{pmatrix} \\
y_k(\ell) &= C x_k(\ell) \\
& k = 0, 1, \ldots, T-1 \\
& \ell = 0, 1, \ldots, L
\end{aligned}$$

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& k = 0, 1, \ldots, T-1 \\
& \ell = -N, \ldots, N \quad \text{and } N := \left[\frac{L}{2}\right]
\end{aligned}
$$

What is missing?

- Boundary conditions: the most convenient regime of operation of the controllable units (or players), i.e., gas pressure and mass flow need to be kept at some desirable levels through time.
- Initial conditions: a starting regime of operation; two possibilities to initialise the flow/pressure vector are:
  (i) using the optimum solution found at the previous period of operation;
  (ii) a starting value could be found in pre-computation.

$$y_k(0) = d_k \text{ and } y_k(L) = g_k, \quad k = 0, 1, \dots, T-1 \tag{5}$$

$$x_0(\ell) = \phi(\ell), \quad \ell = 0, 1, \dots, L \tag{6}$$

d_k is the pumping regime at the inlet
g_k is the contracted delivery level at the offtakes.

Presentation outline

Wave model of length N

$$
\begin{aligned}
x_{k+1}(\ell) &= \sum_{\substack{i=-N \\ |\ell+i|\le N}}^{N} A_i x_k(\ell+i) + \sum_{j=1}^{p-2} B_j u_{j,k}(\ell), \\
y_k(\ell) &= C x_k(\ell) \quad k \in \mathbb{K}, \quad \ell \in \mathbb{L}
\end{aligned}
\tag{7}
$$

See (K.Galkowski, C.Cichy, E. Rogers, 2006), (R. Palucki *et al.*, 2012), (T. Schewerdtfeger, K. Galkowski, A. Kummert, 2013).

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$$y_k(\ell) = C x_k(\ell) \quad k \in \mathbb{K}, \quad \ell \in \mathbb{L}$$

$$\mathbb{L} := [-N, N] \cap \mathbb{Z} \quad \text{with } N = \left[\frac{L}{2}\right] \tag{8}$$

$$\mathbb{K} := \{k | x_k(\ell) = 0, k = T+1, \dots \text{ and } k = \dots, -2, -1\}$$

$$\mathbb{K} \times \mathbb{L} \quad \text{is the compact support of } x_k(\ell), u_{j,k}(\ell), y_k(\ell)$$

See (K.Galkowski, C.Cichy, E. Rogers, 2006), (R. Palucki *et al.*, 2012), (T. Schewerdtfeger, K. Galkowski, A. Kummert, 2013).

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$\mathbb{K} \times \mathbb{L}$ is the compact support of $x_k(\ell), u_{j,k}(\ell), y_k(\ell)$

$x_k(\ell) \in \mathbb{R}^n$	state vector along pass–k	$A_i \in \mathbb{R}^{n\times n}$
$u_{j,k}(\ell) \in \mathbb{R}^{r_j}$	control vectors along pass–k, $j = \overline{1, p-2}$	$B_j \in \mathbb{R}^{n\times r_j}$
$y_k(\ell) \in \mathbb{R}^m$	pass profile vectors along pass–k	$C \in \mathbb{R}^{m\times n}, D_j \in \mathbb{R}^{m\times r_j}$

See (K.Galkowski, C.Cichy, E. Rogers, 2006), (R. Palucki *et al.*, 2012), (T. Schewerdtfeger, K. Galkowski, A. Kummert, 2013).

Autonomous wave model of length N

$$
\begin{aligned}
x_{k+1}(\ell) &= \sum_{\substack{i=-N \\ |\ell+i|\le N}}^{N} A_i x_k(\ell+i) + \sum_{j=1}^{p-2} B_j u_{j,k}(\ell), \\
y_k(\ell) &= C x_k(\ell) \quad k \in \mathbb{K}, \quad \ell \in \mathbb{L}
\end{aligned}
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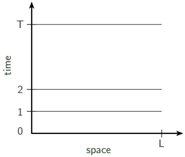

k indexes the pass number

ℓ indexes the steps per pass

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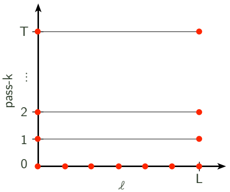
$$y_k(\ell) = C x_k(\ell), \quad k \in \mathbb{K}, \quad \ell \in \mathbb{L} \tag{10}$$

$$x_0(\ell) = \phi(\ell), \quad \ell = -N, \dots, N \tag{11}$$

$$y_k(-N) = d_k \text{ and } y_k(N) = g_k, \quad k = 0, 1, \dots, T-1 \tag{12}$$



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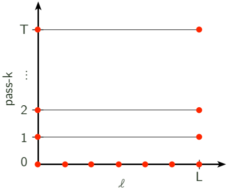

$\mathcal{I} := \mathbb{K} \times \mathbb{L}$
$x_\cdot(.) \in \ell^{2,n}(\mathcal{I}) =: \mathcal{X}$
$u_{j,\cdot}(.) \in \ell^{2,r_j}(\mathcal{I}) =: \mathcal{U}_j$ the controls are admissible
$y_\cdot(.) \in \ell^{2,m}(\mathcal{I})$

$\ell^{2,\nu}(\mathcal{I})$ Hilbert space of ν-dim sequences defined on $\mathcal{I}$ with the standard scalar product.

However, in this presentation we start with the case $T < \infty$.

Operational objectives

There is a quadratic cost functional associated to each player:

$$J_j(u_1,\ldots,u_p,\Phi) = \sum_{\ell=-N+1}^{N-1} x_T^*(\ell)M_jx_T(\ell) + \sum_{k=0}^{T-1}\sum_{\ell=-N+1}^{N-1} x_k^*(\ell)Q_jx_k(\ell) + \sum_{i=1}^{p-2}\sum_{k=0}^{T-1}\sum_{\ell=-N+1}^{N-1} u_{i,k}^*(\ell)R_{ji}u_{i,k}(\ell), \tag{13}$$

where $-^*$ is the hermitian transpose

$M_j, Q_j \in R^{n\times n},\ R_{ji} \in R^{r_j\times r_i};\ j,i = 1,\ldots,p-2,\ k = 0,\ldots,T-1.$

Compacting the notation

$$X_k := \begin{pmatrix} x_k(-N+1) & \cdots & x_k(N-1) \end{pmatrix}^*,$$

$$\Phi := \begin{pmatrix} \phi(-N+1) & \cdots & \phi(N-1) \end{pmatrix}^*, \quad X_k, \Phi \in \mathcal{X}^{(2N-1)}$$

$$U_{j,k} \;:\; = \;\begin{pmatrix} u_{j,k}(-N+1) & \cdots & u_{j,k}(N-1) \end{pmatrix}^* \in \mathcal{U}_j^{(2N-1)}, \; j = 1, \ldots, p$$

$$\mathbb{A} := \begin{pmatrix} A_0 & A_1 & \cdots & A_N & 0 & \ldots & 0 & 0 \\ A_{-1} & A_0 & \cdots & \ldots & A_N & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ A_{-N} & \vdots & \vdots & & \vdots & & \vdots & A_N \\ 0 & A_{-N} & \vdots & \vdots & & & \vdots & A_{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & A_{-N} & \ldots & A_{-1} & A_0 \end{pmatrix}$$

Compacting the notation

$u_{p-1,k} := d_k$ and $u_{p,k} := g_k$
$\mathbb{B}_{p-1} := \begin{pmatrix} A_{-1} & A_{-2} & \cdots & A_{-N} & 0 & \cdots & 0 \end{pmatrix}^*$
$\mathbb{B}_p := \begin{pmatrix} 0 & \cdots & 0 & A_N & A_{N-1} & \cdots & A_0 \end{pmatrix}^*$

$$
\begin{aligned}
R(m,p) \quad &:= \quad R^{(2N-1)m \,\times\, (2N-1)p} \\
\mathbb{B}_j \quad &:= \quad I_{2N-1} \otimes B_j(t) \in R(n, r_j) \\
\mathbb{R}_{ji} \quad &:= \quad I_{2N-1} \otimes R_{ji} \in R(r_j, r_i), \\
\mathbb{S}_j \quad &:= \quad \mathbb{B}_j R_{jj}^{-1} \mathbb{B}_j^T \in R(n,n) \\
\mathbb{Q}_j \quad &:= \quad I_{2N-1} \otimes Q_j \in R(n,n) \\
\mathbb{M}_j \quad &:= \quad I_{2N-1} \otimes K_j \in R(n,n), j = 1, \ldots, p
\end{aligned}
$$

$\otimes$ is the Kronecker product and I_i is the i-dim unit matrix

Differential game with boundary control

$$\text{Opt} \qquad J_j(u_1, \ldots, u_p, \Phi) = X_T^* \mathbb{M} X_T + \sum_{k=0}^{T-1} X_k^* \mathbb{Q} X_k + \sum_{i=1}^{p} \sum_{k=0}^{T-1} U_{j,k}^* \mathbb{R}_{ji} U_{i,k}^*, \tag{14}$$

$$\text{s.t.} \qquad X_{k+1} = \mathbb{A} X_k + \sum_{j=1}^{p} \mathbb{B}_j u_{j,k} \tag{15}$$

$$X_0 = \Phi \tag{16}$$

The two last players are boundary controls

Differential game with boundary control

$$\text{Opt} \qquad J_j(u_1, u_2, \Phi) = X_T^* \mathbb{M} X_T + \sum_{k=0}^{T-1} X_k^* \mathbb{Q} X_k + \underbrace{\sum_{i=1}^{2} \sum_{k=0}^{T-1} u_{j,k}^* \mathbb{R}_{ji} u_{i,k}^*}_{\mathbb{R}_{ji}=0, j=1,\ldots,p-2}, \tag{17}$$

$$\text{s.t.} \qquad X_{k+1} = \mathbb{A} X_k + \sum_{j=1}^{2} \mathbb{B}_j u_{j,k} \tag{18}$$

$$X_0 = \Phi \tag{19}$$

Differential game with boundary control

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$$X_0 = \Phi \qquad (19)$$

Assumptions: Single pipe: 2 player game
Finite time horizon
OL information structure: the only information is at the initial pass

OL Nash equilibrium: Assumptions

- p-player game

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OL Nash equilibrium: Assumptions

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Open-loop Nash equilibrium

Consider a p-player game, $\Gamma_{p=2}$, on a finite time horizon, $T < \infty$, with OL information structure:

Definition (OL Nash equilibrium)

$(\hat{u}_1, \hat{u}_2)$ is called a (2-player) *OL Nash equilibrium strategy* on the system (9)–(5) if

$$\begin{aligned} J_1(\hat{u}_1, \hat{u}_2, \Phi) &\leq J_1(u_1, \hat{u}_2, \Phi), \\ J_2(\hat{u}_1, \hat{u}_2, \Phi) &\leq J_2(\hat{u}_1, u_2, \Phi) \end{aligned} \tag{20}$$

for all initial states $\Phi \in \mathcal{X}^{(2N-1)}$ and all admissible strategies $u_1, u_2 \in \mathcal{U}_1^{(2N-1)} \times \mathcal{U}_2^{(2N-1)}$.

See (Başar and Olsder, 1995).

Best reply

Definition (Best reply)

An admissible control $\hat{u}_j, j = 1, 2$, is called the best reply of player-j, to any set of admissible controls $u_{\bar{j}} = \{u_i | i \in \{1, 2\} \backslash \{j\}\}$ on system (18)–(19) if

$$J_j \left(\hat{u}_j, u_{\bar{j}}, \Phi \right) \leq J_j \left(u_j, u_{\bar{j}}, \Phi \right)$$

and $J_j, j = 1, 2$, is given in (17).

Corollary (1)

1. *$\hat{u}_1, \hat{u}_2$ is an OL Nash equilibrium in a 2-player game for system (18)–(19) if both players simultaneously achieve their best replies.*
2. *In a one player game, i.e., $p = 1$, the Nash equilibrium coincides with the best reply and is the solution of a standard optimisation problem.*

Value function approach

Value functions for the cost functionals of (17)?

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$$V_j(k) := \frac{1}{2}X_k^* E_j(k)X_k + e_j^*(k)X_k + d_j(k), \quad k = 0, \ldots, T, \quad j = 1, 2 \tag{21}$$

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- $E_j(k) \in R(n, n)$

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where

- $E_j(k) \in R(n, n)$
- $X_k, e_j(k) \in R(n, 1)$

Value function approach

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$$V_j(k) := \frac{1}{2} X_k^* E_j(k) X_k + e_j^*(k) X_k + d_j(k), \quad k = 0, \ldots, T, \quad j = 1, 2 \tag{21}$$

where

- $E_j(k) \in R(n, n)$
- $X_k, e_j(k) \in R(n, 1)$
- $d_j(k) \in R$

To make functions V_j value functions for J_j

Theorem

Let solutions $E_j(k)$ of the symmetric standard discrete time matrix Riccati equations (SSRDE)

$$\begin{aligned} 0 &= \mathbb{A}^* E_j(k+1)\mathbb{A} - E_j(k) + \mathbb{Q}_j - \\ &\quad -\mathbb{A}^* E_j(k+1)\mathbb{B}_j \times \left(R_{jj} + \mathbb{B}_j^* E_j(k+1)\mathbb{B}_j\right)^{-1} \mathbb{B}_j^*(t) E_j(k+1)\mathbb{A} \qquad (22) \\ E_j(T) &= \mathbb{M}_j, \quad j = 1, 2. \end{aligned}$$

exist for $k = 0, \ldots, T$

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exist for $k = 0, \ldots, T$ (hence necessarily $S_j(k) := R_{jj} + \mathbb{B}_j^* E_j(k+1)\mathbb{B}_j$ is invertible).

To make functions V_j value functions for J_j

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exist for $k = 0, \ldots, T$ (hence necessarily $S_j(k) := R_{jj} + \mathbb{B}_j^* E_j(k+1)\mathbb{B}_j$ is invertible).

Then, for admissible controls u_1, u_2 the difference equations:

$$\begin{aligned} 0 &= \mathbb{B}_j^* e_j(k+1) - S_j(k) b_j(k) + \mathbb{B}_j^* E_j(k+1)\gamma_j(k) \\ 0 &= -\mathbb{A}^* E_j(k+1)\mathbb{B}_j b_j(k) + \mathbb{A}^*(t) E_j(k+1)\gamma_j(k) + \mathbb{A}^* e_j(k+1) - e_j(k) \\ 0 &= e_j(T) = b_j(T), \quad j = 1, 2, \end{aligned} \tag{23}$$

are solvable backwards, where $\gamma_j(k) = \sum_{s \neq j} \mathbb{B}_s u_{s,k}$.

To make functions V_j value functions for J_j

Furthermore, with $d_j(k)$ a solution of the simple difference equation:

$$\begin{array}{rcl} d_j(k+1) - d_j(k) - \frac{1}{2} b_j^*(k) S_j(k) b_j(k) + \frac{1}{2} \sum_{s \neq j} u_{s,k}^* R_{js} u_{s,k} + & & \\ + \frac{1}{2} \gamma_j(k)^* E_j(k) \gamma_j(k) + e_j^*(k+1) \gamma_j(k) & = & 0 \\ d_j(T) & = & 0, \quad j = 1, 2, \end{array} \tag{24}$$

we obtain for $j = 1, 2$

$$J_j = \frac{1}{2} \left(X_0^* E_j(0) X_0 + e_j^*(0) X_0 + d_j(0) + \sum_{k=0}^{T-1} \| u_{j,k} + c_j(k) \|_{S_j}^2 \right), \tag{25}$$

where we used $c_j(k) = S_j^{-1}(k) \mathbb{B}_j^* E_j(k+1) \mathbb{A} X_k + b_j(k)$ and X_k is the solution of (17)–(19).

Convexity conditions

Remark

In case of convexity assumptions, i.e. if $\mathbb{Q}_j \geq 0$, $\mathbb{M} \geq 0$ *and* $R_{jj} > 0$, $j = 1, 2$, *the SSRDE (22) is always solvable (see [Kandil, Freiling, Ionescu, Jank, 2003]), hence we always can obtain the representation (25) of the cost functionals.*

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However, such convexity assumptions appear to be too restrictive in real application problems, since they are violated, for example, in zero-sum games or in general rather conflicting game situations.

Unique best reply representation

Player j obtains a *unique best reply* to any action of the other players if $S_j(k) > 0$ and

$$\hat{u}_{j,k} = -c_j(k) = -S_j^{-1}(k)\mathbb{B}_j^* E_j(t+1)\mathbb{A}X_k - b_j(k).$$

The existence of a minimum of J_j in (25) necessarily implies $S_j(k) \geq 0$.

Sufficient conditions for existence of Nash eq.

Theorem

Let $E_j(k)$ *be a solution of SSRDE such that* $S_j(k) = R_{jj} + \mathbb{B}_j^* E_j(k+1)\mathbb{B}_j > 0$, *for* $k = 0, \dots, T-1, j = 1, 2$. *Then controls*

$$u_{j,k} = -S_j^{-1}(k)\mathbb{B}_j^* E_j(k+1)\mathbb{A}X_k - b_j(k), \quad j = 1, 2, \tag{26}$$

determine a Nash equilibrium for any solution of the following BVP

$$\begin{aligned}
0 &= \mathbb{B}_j^* e_j(k+1) - S_j(k)b_j(k) + \\
&\quad -\mathbb{B}_j^* E_j(k+1) \sum_{s \neq j} \mathbb{B}_s \left(S_s^{-1}(k)\mathbb{B}_s^* E_s(k+1)\mathbb{A}X_k + b_s \right) \\
0 &= +\mathbb{A}^* E_j(k+1)\mathbb{B}_j b_j(k) + \\
&\quad +\mathbb{A}^* E_j(k+1)\gamma_j(k) + \mathbb{A}^* e_j(k+1) - e_j(k) \qquad (27) \\
0 &= e_j(T) = b_j(T), \quad j = 1, 2, \\
X_{k+1} &= \mathbb{A}X_k - \textstyle\sum_{j=1}^{p} \mathbb{B}_j \left(S_j^{-1}(k)\mathbb{B}_j^* E_j(k+1)\mathbb{A}X_k + b_j \right) \\
X_0 &= \Psi.
\end{aligned}$$

Sufficient conditions for existence

Theorem

Consider that solutions $E_j(k)$ *of SSRDE (22) exist for* $k = 0, \ldots, T; j = 1, 2$.
If the BVP

$$\begin{aligned} \psi_j(k) &= \mathbb{Q}_j X_k + \mathbb{A}^* \psi_j(k+1) \\ \psi_j(T) &= \mathbb{M}_j X_T, \quad (\psi_j(T+1) = 0) \\ X_{k+1} &= \mathbb{A} X_k - \textstyle\sum_{s=1}^{p} \mathbb{B}_s R_{ss}^{-1} \mathbb{B}_s^* \psi_s(k+1) \\ X_0 &= \Psi \end{aligned} \tag{28}$$

admits a solution then $e_j(k), b_j(k), X_k$ *are a solution of the BVP (27) if we set*

$$\begin{aligned} e_j(k) &= \psi_j(k) - E_j(k) X_k \\ b_j(k) &= S_j^{-1}(k) \mathbb{B}_j^*(k) \left[E_j(k+1) \textstyle\sum_{s \neq j} \mathbb{B}_s \hat{u}_{s,k} + e_j(k+1) \right], \end{aligned} \tag{29}$$

where

$$\hat{u}_{j,k} = -R_{jj}^{-1} \mathbb{B}_j^* \psi_j(k+1), \quad t = 0, \ldots, T-1. \tag{30}$$

On the other hand, if $e_j(k), b_j(k), X_k$ *are a solution of the BVP (27) then, with the settings (29),(30), we obtain a solution of the BVP (28).*

Sufficient conditions for existence/uniqueness

Corollary

Let SSRDE (22) admit solutions $E_j(k)$ *such that*

$$S_j(k) = R_{jj} + \mathbb{B}_j^* E_j(k+1)\mathbb{B}_j > 0$$

for all $k = 0, \ldots, T-1$ *and* $j = 1, 2$.

1. *The functions* $u_{j,k}$ *in (30) are a Nash equilibrium if and only if the BVP (28) is solvable.* *This is an explicit condition for playability as it was obtained in the operator based approach [Same authors, Controlo'08].*
2. *Nash equilibrium is unique iff BVP (28) is uniquely solvable.*
3. *Nash costs for each player can be calculated from (25):*

$$\frac{1}{2}\left[X_0^* E_j(0) X_0 + e_j^*(0) X_0 + d_j(0)\right],$$

where $e_j(0)$ *was defined in (29) and* $d_j(0)$ *is obtained by solving (24).*

Sufficient condition for existence/uniqueness

Theorem

Let SSRDE (22) admit solutions $E_j(k)$ such that $S_j(k) > 0$ for $k = 0, \dots, T-1, j = 1, 2$.

Furthermore, if the discrete time OL Nash Riccati difference equation (OLNRDE)

$$\begin{aligned} K_j(k) &= \mathbb{Q}_j + \mathbb{A}^* K_j(k+1)\Omega^{-1}\mathbb{A}, \\ K_j(T) &= \mathbb{K}_j, \quad j = 1, 2, \quad k = 0, \dots, T-1, \end{aligned} \tag{31}$$

admits a solution, where $\Omega := \left(I + \sum_{s=1}^{p} \mathbb{B}_s R_{ss}^{-1} \mathbb{B}_s^* K_s(k+1) \right)$*, then there exists a unique OL Nash equilibrium defined in quasi-feedback form by*

$$\hat{u}_{j,k} = -R_{jj}^{-1}\mathbb{B}_j^* \left(K_j(k+1)X_{k+1} + D_j(k+1) \right), k = 0, \dots, T-1,$$

whence $D_j(k), G_j(k)$ are defined as:

$$D_j(k) = \mathbb{A}^* D_j(k+1) + G_j(k), \quad D_j(T) = 0, \tag{32}$$

$$G_j(k) = -\mathbb{A}^* K_j(k+1)\Omega^{-1} \sum_{s=1}^{p} \mathbb{B}_s R_{ss}^{-1} \mathbb{B}_s^* D_j(k+1) \tag{33}$$

See (T-P Azevedo Perdicoúlis & G. Jank, 2008).

Team controllability

Definition (Team controllability)

Let Γ_2 be a 2-player game. We say that the game is *team controllable* if for any initial and terminal states $X_0, X_1 \in X$ and initial time $k_0 \in \mathbb{K}$ there exist a terminal time $k_1 > k_0$ and a set of control functions $u_{j,k} \in \mathcal{U}_j, j = 1, 2$, such that for the solution of the difference equation

$$X_{k+1} = f(k, X_k, u_{j,k}, \hat{u}_{\bar{j},k}) = \mathbb{A}X_k + \mathbb{B}_j u_{j,k} + \mathbb{B}_{\bar{j}} \hat{u}_{\bar{j},k}, \qquad X_0 = \Phi$$

$X(k_1) = X_1$ holds.

See (Kun,2000) and (T. Perdicoulis, nDS2013)

Definition (Individual controllability)

Let Γ_2 be a 2-player game. Suppose that strategies are chosen such that $(\hat{u}_1, \hat{u}_2)$ is an equilibrium for Γ_2. Then, we say that the game is controllable at this equilibrium point, from the point of view of the jth player, if the control system

$$X_{k+1} = f(k, X_k, u_{j,k}, \hat{u}_{\bar{j},k}) = \mathbb{A}X_k + \mathbb{B}_j u_{j,k} + \mathbb{B}_{\bar{j}} \hat{u}_{\bar{j},k}$$

is controllable in the admissible set of $u_{j,k}, j = 1, 2$.

See (Kun,2000).

Characterisation of individual controllability

Lemma

Let Γ_2 be a linear OL quadratic differential game. Suppose $(\hat{u}_1, \hat{u}_2)$ (and $\hat{X}$ its respective trajectory) to be a Nash eq. for Γ_2, based on the solutions $K_j(k), j = 1, 2$, of the correspondent OLNRDE, then Γ_2 is individually controllable for the jth player iff any triple $(k_0, \Phi, \Phi) \in \mathbb{K} \times \mathcal{X}^{(2N-1)} \times \mathcal{X}^{(2N-1)}$ of the following linear control system

$$\begin{pmatrix} X_{k+1} \\ \hat{X}_{k+1} \end{pmatrix} = \begin{pmatrix} \Omega_{\bar{j}}^{-1} A & 0 \\ 0 & \Omega^{-1} A \end{pmatrix} \begin{pmatrix} X_k \\ \hat{X}_k \end{pmatrix} + \begin{pmatrix} \Omega_{\bar{j}}^{-1} \mathbb{B}_j \\ 0 \end{pmatrix} u_{j,k}, \tag{34}$$

with

$$\Omega_{\bar{j}} := I + \sum_{\substack{s=1 \\ s \neq j}}^{2} \mathbb{B}_s R_{ss}^{-1} \mathbb{B}_s^* K_s(k+1) \quad \text{and} \quad \Omega := I + \sum_{s=1}^{2} \mathbb{B}_s R_{ss}^{-1} \mathbb{B}_s^* K_s(k+1)$$

can be controlled to a pass $X_f \times \mathcal{X}^{(2N-1)}$ for all $X_f \in \mathcal{X}^{(2N-1)}$.

Proof: See (Kun,2000).

Definition (Individual pass controllability)

System (9) is (completely) pass boundary controllable for player j in $k_0, k_0+1, \ldots, k_1$ with $k_0, k_1 \in \mathbb{K}$ if for any initial conditions $\phi(-N+1), \ldots, \phi(0), \ldots, \phi(N-1)$ in (6) and any vector pass $x_f(\ell), \ell \in \mathbb{L}$, if there exists sequences of boundary data d_k (or g_k), $k = k_0, \ldots, k_1$ such that $x_{k_1}(\ell) = x_f(\ell), \ell \in \mathbb{L}$.

Individual pass controllability

Theorem

The wave model (9) is completely pass controllable on $0, 1, \ldots, T$*, if and only if the grammian matrix*

$$G_T = \sum_{s=0}^{T-1} M(s) M(s)^* \tag{35}$$

is positive definite, and where

$$M(s) := \begin{pmatrix} \Omega_{\bar{j}}^{-1} A & 0 \\ 0 & \Omega^{-1} A \end{pmatrix}^s \begin{pmatrix} \Omega_{\bar{j}}^{-1} \mathbb{B}_j \\ 0 \end{pmatrix}.$$

Individual pass controllability

Proof.

Consider the linear control system (34) written in terms of the initial pass and recall that the boundary conditions are written as controls in (18). Hence:

$$\begin{pmatrix} X_k \\ \hat{X}_k \end{pmatrix} = \begin{pmatrix} \Omega_{\mathrm{J}}^{-1}A & 0 \\ 0 & \Omega^{-1}A \end{pmatrix}^k \begin{pmatrix} \Phi \\ \Phi \end{pmatrix} + \sum_{s=1}^{k} \begin{pmatrix} \Omega_{\mathrm{J}}^{-1}A & 0 \\ 0 & \Omega^{-1}A \end{pmatrix}^{k-1} \begin{pmatrix} \Omega_{\mathrm{J}}^{-1}\mathbb{B}_j \\ 0 \end{pmatrix} u_{j,s-1}.$$

Then the grammian G_T is defined in terms of the transition matrix $\begin{pmatrix} \Omega_{\mathrm{J}}^{-1}A & 0 \\ 0 & \Omega^{-1}A \end{pmatrix}$ and the output matrix $\begin{pmatrix} \Omega_{\mathrm{J}}^{-1}\mathbb{B}_j \\ 0 \end{pmatrix}$.

Then the proof is the same as in classical systems and therefore omitted here. □

See (T-P Azevedo Perdicoúlis & G. Jank,2008) and (Knobloch & Kwakernaak, 1985).

Individual initial pass controllability

Definition

System (9) is completely pass controllable by initial pass control if for any boundary conditions $d_0, d_1, \ldots, d_T$ and $g_0, g_1, \ldots, g_T$ in (5) and any vector pass $x_f(\ell), \ell \in \mathbb{L}$, there exists a sequence of initial data $\phi(-N+1), \ldots, \phi(0), \ldots, \phi(N-1)$, subsumed in Φ, such that $X_T = X_f$.

Theorem (Simple criterion)

System (9) is completely pass controllable by initial pass control if $\begin{pmatrix} \Omega_j^{-1} A & 0 \\ 0 & \Omega^{-1} A \end{pmatrix} \in R(2n, 2n)$ *has full rank.*

Proof: See (T-P Azevedo Perdicoúlis & G. Jank, 2010).

Observability

Definition (Boundary observability)

System (9)–(10) is pass-boundary observable in $\{0, 1, \ldots, T\}$, if for all $t_1 \in \mathbb{N}, 0 < t_1 \leq T$ and boundary data Φ for any two trajectories $X_k, \tilde{X}_k, 0 < k \leq t_1$, corresponding to the same input $u_{j,k}, j = 1, 2, 0 < k \leq t_1$, from

$$\mathbb{C}X_k = \mathbb{C}\tilde{X}_k, 0 < k \leq t_1,$$

it follows necessarily that $X_k = \tilde{X}_k, 0 < k \leq t_1$.

Theorem (pass-boundary observable)

System (9) and (10) is pass-boundary observable in $\{0, 1, \ldots, T\}$, *if* $\mathrm{rank}\left(\mathbb{C}\mathbb{A}^{k-1}\mathbb{B}_s\right) = n$.

Observability

Proof.

Using the compact notation, we define $\mathbb{C} = \mathrm{diag}\{C, \ldots, C\}$.
If we set $\hat{X}_k = X_k - \tilde{X}_k, k = 1, \ldots, T$, i.e., $\hat{X}_k$ is the solution of the homogeneous equation, then pass boundary observable is equivalent to the condition:

$$\mathbb{C}\hat{X}_k = 0 \implies \hat{X}_k = 0, 0 < k \leq t_1,$$

considering $\Phi = 0$. Hence:

$$\begin{aligned} \hat{Y}_k &= Y_k - \tilde{Y}_k = \mathbb{C}X_k - \mathbb{C}\tilde{X}_k \\ &= \mathbb{C}\sum_{s=1}^{2}\sum_{i=0}^{k-1}\mathbb{A}^{k-1}\mathbb{B}_s\left(u_{s,i} - \tilde{u}_{s,i}\right) \end{aligned}$$

Considering $\hat{Y}_k = 0, k = 1, 2, \ldots, t_1$, we obtain:

$$\begin{aligned} \hat{Y}_1 &= \mathbb{C}\sum_{s=1}^{2}\mathbb{B}_s\left(u_{s,0} - \tilde{u}_{s,0}\right) = 0 \implies u_{s,0} = \tilde{u}_{s,0} \\ \hat{Y}_2 &= \mathbb{C}\sum_{s=1}^{2}\mathbb{B}_s\left(u_{s,0} - \tilde{u}_{s,0}\right) + \mathbb{A}\mathbb{B}_s\left(u_{s,1} - \tilde{u}_{s,1}\right) = 0 \\ &\qquad \implies u_{s,1} = \tilde{u}_{s,1} \\ &\vdots \end{aligned}$$

Then we have that the boundary controls are uniquely defined by a measured output. □

Conclusions

- Formulation of a wave RP as an OL Nash game where the strategies are the boundary settings.
- We state sufficient conditions for the existence/uniqueness of the equilibrium strategies.
- These sufficient conditions are suitable for numerical calculations.
- We study structural properties of the equilibrium strategies.

Future Work

- Consider the same problem for the infinite time horizon/moving horizon.
- Then, questions such as individual stabilisation of the solution by the different players become relevant as well as uniqueness of the equilibrium strategies.
- Consider other type of information structures and equilibria for the same problem.
- Consider a system whose parameters are not constant but depend on k, ℓ, instead.
- Extend the wave model/differential game to a complex network

Thank you!