On embedded FIR filter models for identifying continuous-time and discrete-time transfer functions: the RPM approach [1]

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Abstract

For identifying a continuous-time (CT) transfer function model, data filtering is a solution which provides the necessary unmeasurable input-output derivative approximations. In discrete-time (DT) system identification, the well-known ARX model can be used successfully if the estimate is performed with suitable prefiltered data. This paper describes the reinitialized partial moment (RPM) model which embeds implicitly a finite impulse response filter in both CT and DT domains. With knowledge of the important role of data prefiltering in standard methods, this RPM model embedded filter gives particular properties to this original tool. Although both the CT RPM model and the DT RPM model present an embedded filter, the formulation and the implementation in the CT and the DT domains are different. Therefore, the aim of this paper is to present a tutorial on the RPM models and to give an overview of all the applications.

Mots-clés

Continuous-time model, Discrete-time model, Reinitialized partial moments, RPM filter, RPM model, System identification, Transfer function model

1 Introduction

The moments are useful tools in statistics and probability theory, and a large body of literature has been dedicated on this topic for a century [6]. The well-known moment method is used in a very

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wide application field, especially in control system and model reduction [1, 2, 16]. In system identification, [9] first used the time moments to depict the impulse response and the transient behaviour of a linear system. Based on Kalman's work, minimal realizations using moments have been proposed in [3, 27]. However, the straightforward application in system identification has been limited by two main problems : the restriction to an impulse or step input and the necessity to calculate the moments on an infinite time interval. To solve these two problems, the partial moment has been introduced in [35] which consists of considering the moment only on a finite time interval. The main property of this formulation is its minimum variance for an optimal time interval. Hence, to keep this property throughout the time domain, the reinitialized partial moment (RPM) has been introduced [36].

The contribution of this paper is to present a tutorial on the RPM models in the continuous-time (CT) and the discrete-time (DT) domains. The main property of the RPM models is an embedded finite impulse response (FIR) filter. In system identification, data filtering is a well-known notion and is often a necessary step for an efficient model estimation. For instance, for identifying a CT model, data filtering is a solution which provides the necessary unmeasurable input-output derivative approximations [14], or again, in DT system identification, the well-known ARX model can be successfully used if the estimate is performed with suitable prefiltered data [19]. The RPM model, which can be formulated in both CT and DT domains, embeds implicitly a FIR filter that plays the same role as the explicit filter mentioned in [14, 19].

Although both the CT RPM model and the DT RPM model present an embedded filter, the formulation and the implementation in the CT and the DT domains are different. The goal of this paper is to describe in two separate sections all the mathematical developments which have led to the CT RPM model and the DT RPM model [36, 31].

The RPM models with the embedded FIR filter are a powerful tool. They are alternative solutions for the output error method [2] initialization problems mentioned in [19, 22, 40], and the performance of the DT RPM model in this context has been shown in [33, 25]. By considering the increasing interest for CT model approaches [15, 20], the CT RPM model is a well-tested method in different applications [4, 5, 7, 8]. Moreover, recent developments have used the embedded CT RPM filter to build a CT subspace-based identification method [24]. In addition, new algorithms based on the RPM properties have been introduced in [31]. Lastly, recent papers [10, 11, 12] have introduced a CT algebraic framework similar to the partial moment introduced in [35]. The present paper also contributes an overview of all the RPM applications.

This paper is organized as follows. The RPM approach is described in Section 2 and an illustrative example is given in Section 3 to compare the RPM model with basic models. Sections 4 and 5 present the CT RPM models and the DT RPM models, respectively. In Section 6, the choice of the design parameter is discussed. The conclusion is given in Section 7.

2 The RPM approach

In system identification, the equation error methods, which consist of a linear regression formulation, have to solve a specific problem in both CT and DT cases. For the CT system identification, the unmeasurable input-output derivatives must be approximated to allow the linear regression formulation and to calculate the least-squares estimate. For the DT system identification with the ARX model, the input-output samples are available and there are no problems in calculating the least-squares estimate. But in this case, it is well known that data filtering is necessary to reduce the bias according to the nature of the noise. The RPM approach is a unifying tool that solves both problems. In the CT framework, the RPM is equivalent to an integration which allows a reformulation of the differential equation, hence, the unmeasurable input-output derivatives vanish. In the DT framework, a reformulation of the RPM shows an implicit filtering which plays the same role as the explicit filter

2. Output error methods and equation error methods in the sense of the classification introduced by [17, 21]

for the ARX model.
The starting point of the RPM approach is the partial moment which consists of considering the CT or DT moment on a finite time interval. The mathematical developments, which give the CT RPM model and the DT RPM model, follow the same steps :

(1) A fundamental equation describes the system behaviour on a fixed finite time interval from input-output data and from the partial moments. This general equation allows computations for any system order.
(2) A formulation based on partial moment is deduced from the fundamental equation. This formulation has the property of minimum variance for an optimal time interval.
(3) A RPM formulation allows the retention of the minimum variance property at each instant. The RPM formulation can be seen as an integration on a sliding window whose width is the optimal time interval.
(4) A formulation with an embedded FIR filter can be obtained from the RPM formulation.

Because specific computations are used in both CT and DT cases, the CT RPM model and the DT RPM model are presented separately in Sections 4 and 5.
The RPM models need the choice of a design parameter called the reinitialization parameter. As for the design parameters of other equation error methods, this choice needs a certain expertise which will be described in Section 6.

3 Illustrative example

Before describing the mathematical developments for both CT RPM and DT RPM models, consider a simple example to illustrate the RPM performance and to compare it with two basic approaches. For the CT case, the state-variable filter (SVF) approach [37] is considered and, for the DT case, a comparison with the ARX model [18] is presented.
Consider a second order oscillating CT system defined by

$$G(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \tag{1}$$

with $\zeta = 0.4$ and $\omega_n = 1rad/s$.
The sampling period t_s is assumed to be $0.2s$. The output response of $G(s)$ to a square input signal with a period of $40s$ is simulated. One thousand samples are considered. The performance of models is evaluated by a Monte Carlo simulation with 1000 realizations of an output white noise and a signal-to-noise ratio of $10dB$.
To evaluate the estimation quality, consider a fitting index defined by

$$FIT = 100 \times \left(1 - \frac{\|y - \hat{y}\|}{\|y - mean(y)\|}\right) \tag{2}$$

where y is the measured output and $\hat{y}$ is the estimated model output. Consider also the normalized root mean squared error defined by

$$NRMSE = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}\left(\frac{\theta_j^0 - \hat{\theta}_j(i)}{\theta_j^0}\right)^2}, \tag{3}$$

where $\hat{\theta}_j(i)$ is the j-th element of the estimated parameter vector obtained from the i-th realization, while the superscript '0' denotes the true parameter value.
For the CT approaches, the estimated model is a transfer function defined by

$$\hat{G}(s) = \frac{\hat{b}_0}{\hat{a}_0 + \hat{a}_1 s + s^2} \tag{4}$$

Table 1 – CT Monte Carlo results

		b_0	a_0	a_1	FIT (%)
	True	1	1	0.8	
SVF	Mean	0.840	0.848	0.623	93.76
	Std.	0.037	0.034	0.034	1.30
	NRMSE	0.164	0.156	0.226	
CTRPM	Mean	0.929	0.933	0.712	97.03
	Std.	0.034	0.031	0.034	0.92
	NRMSE	0.079	0.073	0.118	

Table 2 – DT Monte Carlo results

		b_0	b_1	a_0	a_1	FIT (%)
	True	0.018	0.019	0.852	- 1.815	
ARX	Mean	0.303	0.0006	-0.341	-0.355	80.23
	Std.	0.062	0.059	0.023	0.023	0.52
	NRMSE	16.283	3.285	1.401	0.805	
ARX	Mean	0.029	0.006	0.863	-1.828	95.87
with data	Std.	0.009	0.008	0.006	0.007	0.93
prefiltering	NRMSE	0.788	0.805	0.014	0.008	
DTRPM	Mean	0.047	-0.011	0.852	-1.816	97.99
	Std.	0.010	0.009	0.007	0.008	0.68
	NRMSE	1.720	1.675	0.008	0.004	

The identification is performed with the Matlab routine lssvf of the CONTSID toolbox [13] for the SVF method and with the routine lsctrpm downloaded from http://laii.univ-poitiers.fr/ouvrard/CTRPM/ for the CT RPM method. After empirical tests, the optimal design parameters are chosen ; the cut-off frequency of the routine lssvf is 0.9 rad/s and the reinitialization parameter of the routine lsctrpm is 20. The statistical results are presented in Table 1 with parameter mean values, standard deviations, FIT and NRMSE indexes.

In the DT case, the discretization with zero-order hold method of the true transfer function system (1) gives

$$G(z) = \frac{0.019z^{-1} + 0.018z^{-2}}{1 - 1.815z^{-1} + 0.852z^{-2}} \tag{5}$$

The estimated DT model has the same structure as (5). The DT system identification is performed with the routine arx of the System Identification Toolbox of Matlab for the ARX model estimate from prefiltered data or not, and with the routine lsdtrpm downloaded from http://laii.univ-poitiers.fr/ouvrard/DTRPM/ for the DT RPM method. The design parameter of the routine lsdtrpm is 20, the routine arx without prefiltering does not need design parameters and the routine arx with data prefiltering considers a prefilter given by the true denominator (unknown in practice), *i.e.* the prefilter $1/(1 + 0.8s + s^2)$, as it is advisable by [19]. The statistical results are presented in Table 2.

In both CT and DT cases, this illustrative example shows the interesting performance of the RPM approach. The bias of the CT RPM model is lower than the SVF-based model, which confirms the results about the RPM approach obtained in [14, 13, 23]. The DT RPM model gives an accurate estimate of poles and gain. The error on numerator parameters is due to the noise level and corresponds to an equivalent fast CT zero. The good fitting with the DT RPM model is explained by the embedded FIR filter. Without explicit data prefiltering, the ARX model estimates present a wrong fitting. The

use of a data prefiltering is well known for the ARX model estimation. In that case the order and the cut-off frequency of the prefilter are the design parameters of this approach. The differences between the ARX and DT RPM models have been described in [25].
Notice that for the above approaches an iterative instrumental variable approach [38] can be implemented to reduce the bias.

4 Continuous-time RPM model

Linear CT systems can be described by a differential equation, but the input-output time derivatives are unmeasurable. Three CT system identification method classes exist, based on filtering, integration or modulating functions [14], and allow an approximation of the time derivatives.
The CT RPM model approach appears to be based on an integration and thus belongs to the integral method class. However, it will be shown below that this approach can be written as a convolution with a filter. Therefore, it belongs also to the linear filter class of CT system identification methods.

4.1 Preliminaries - A simple case

For an easier understanding, consider a first order CT system defined by the following differential equation

$$\frac{dy_0(t)}{dt} = -a_0 y_0(t) + b_0 u(t) \tag{6}$$

where $y_0(t)$ is the true system output.
Define the n-th order CT partial moment of a signal $v(t)$ by

$$\mathcal{M}_n^v(T) = \int_0^T \frac{t^n}{n!} v(t)dt \tag{7}$$

Notice that the CT partial moment is the standard CT moment truncated to a finite interval $[0, T]$.
First, compute the first order CT partial moment of the differential equation (6)

$$\int_0^T t\frac{dy_0(t)}{dt}dt = -a_0 \int_0^T t y_0(t)dt + b_0 \int_0^T t u(t)dt \tag{8}$$

After an integration by parts $\int_{x_1}^{x_2} f(x)g'(x)dx = [f(x)g(x)]_{x_1}^{x_2} - \int_{x_1}^{x_2} f'(x)g(x)dx$ of the right-hand side term, the following output formulation is obtained

$$y_0(T) = -a_0\frac{\mathcal{M}_1^{y_0}(T)}{T} + b_0\frac{\mathcal{M}_1^{u}(T)}{T} + \frac{\mathcal{M}_0^{y_0}(T)}{T} \tag{9}$$

Second, because the true output $y_0(t)$ is inaccessible, it is substituted by the measured output $y(t)$ which is disturbed by a noise assumed with zero mean. Then, the corresponding output formulation has a variance depending on the interval defined by T. It can be shown that a value T_{opt} of T, linked to the system settling time, permits a minimum variance to be obtained (see Section A.3 in [31]) [3].
Thus, to keep this minimum variance at each time t and also to avoid an increasing computation time when T increases, the CT reinitialized partial moments have been introduced in [36]. The principle consists of using a sliding window of width $\widehat{T}$ for all t, where $\widehat{T}$ is an estimation of T_{opt}.
Let us define the n-th order CT RPM (*continuous-time reinitialized partial moment*) of a signal $v(t)$ by

$$M_n^v(t) = \int_0^{\widehat{T}} \tau^n v(t - \widehat{T} + \tau)d\tau \tag{10}$$

3. For the first order system defined by (6) and with the assumption of a zero-mean white noise, $T_{opt} = \frac{\sqrt{3}}{a_0}$.

where $\widehat{T}$ is the design parameter called the reinitialization parameter.
Third, substitute $y_0(t)$ by the measured output $y(t)$ in (9) and consider the CT RPM rather than the CT partial moment, the estimation $\widehat{y}(t)$ for all t built with $y(t)$ becomes

$$\widehat{y}(t) = -\widehat{a}_0 \frac{M_1^y(t)}{\widehat{T}} + \widehat{b}_0 \frac{M_1^u(t)}{\widehat{T}} + \frac{M_0^y(t)}{\widehat{T}} \tag{11}$$

Fourth, using the variable change $\mu = \widehat{T} - \tau$, the three CT RPM in the right-hand side term of the previous equation become

$$\begin{array}{l} M_1^v(t) = \widehat{T} \int_0^{\widehat{T}} \frac{\widehat{T}-\mu}{\widehat{T}} v(t-\mu)\, d\mu \quad \text{with } v = y \text{ or } u \\ M_0^y(t) = \widehat{T} \int_0^{\widehat{T}} \frac{1}{\widehat{T}} y(t-\mu)\, d\mu \end{array} \tag{12}$$

which are the following convolution products

$$\begin{array}{l} M_1^v(t) = \widehat{T}\, m(t) * v(t) \quad \text{with } v = y \text{ or } u \\ M_0^y(t) = \widehat{T} \left(\delta(t) - \frac{dm(t)}{dt}\right) * y(t) \end{array} \tag{13}$$

where

$$\begin{array}{l} m(t) = \begin{cases} \frac{\widehat{T}-t}{\widehat{T}} & \text{if } t \in \left[0, \widehat{T}\right] \\ 0 & \text{elsewhere} \end{cases} \\ \delta(t) \text{ is the Dirac function} \\ \text{and } * \text{ is the convolution product} \end{array} \tag{14}$$

Notice that $\delta(t)$ is introduced in the convolution product $M_0^y(t)$ to remove the Dirac function due to the derivative of the discontinuity of $m(t)$ at $t = 0$.
Then, the first order CT RPM model is described by convolution products

$$\widehat{y}(t) = -\widehat{a}_0(m(t) * y(t)) + \widehat{b}_0(m(t) * u(t)) + \left(\delta(t) - \frac{dm(t)}{dt}\right) * y(t) \tag{15}$$

where $m(t)$ is an implicit FIR filter and $\widehat{T}$ is the design parameter chosen in the neighbourhood of T_{opt}.
These are the four main steps which illustrate the origin of the CT RPM model. For the first order CT system, the computations are simple. For higher order systems, a fundamental equation must be introduced.

4.2 Generalization

4.2.1 Fundamental equation

The objective of the fundamental equation is to describe the behaviour of a linear system on the interval $[0, T]$ from input-output data as shown in Figure 1.
Consider a SISO linear system defined by the following minimal state-space representation

$$\begin{array}{l} \frac{dx(t)}{dt} = Ax(t) + Bu(t) \\ y_0(t) = Cx(t) + Du(t) \end{array} \tag{16}$$

where $x(t) \in \mathbb{R}^{n_a}$, $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. The system can also be represented by its transfer function and a free output due to the initial conditions $x(0)$

$$Y_0(s) = G(s)U(s) + L(s)x(0) \tag{17}$$

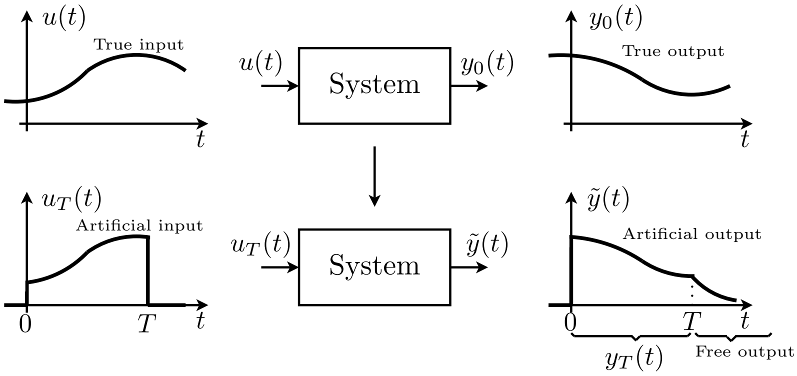

FIGURE 1 – Principle of the partial moment formulation in the CT domain

with

$$\begin{aligned} G(s) &= C\,(sI-A)^{-1}\,B + D \\ &= \frac{b_0+b_1 s+\ldots+b_{n_b}s^{n_b}}{a_0+a_1 s+\ldots+a_{n_a-1}s^{n_a-1}+s^{n_a}}, \quad n_a \geq n_b \\ L(s) &= C\,(sI-A)^{-1} = \frac{\left[\begin{array}{ccc} L_1(s) & \cdots & L_{n_a}(s) \end{array}\right]}{a_0+a_1 s+\ldots+a_{n_a-1}s^{n_a-1}+s^{n_a}} \end{aligned} \tag{18}$$

where $L_n(s) = l_{0,n} + l_{1,n}s + \ldots + l_{n_a-1,n}s^{n_a-1}$.
Suppose that the system excitation is an artificial input $u_T(t)$ defined as the true input on the interval $[0, T]$ and zero for $t > T$ as shown in Figure 1. Thus, the corresponding artificial output $\tilde{y}(t)$ is composed of two components : $y_T(t) = y_0(t)$ on $[0, T]$ and a free output due to $x(T)$ on $[T, \infty]$. The corresponding Laplace transform of $\tilde{y}(t)$ is given by

$$\tilde{Y}(s) = Y_T(s) + e^{-Ts}L(s)x(T) \tag{19}$$

According to (17), the above equation can also be written as follows

$$\tilde{Y}(s) = G(s)U_T(s) + L(s)x(0) \tag{20}$$

With both previous equations, the fundamental equation is deduced

$$G(s)U_T(s) + L(s)x(0) = Y_T(s) + e^{-Ts}L(s)x(T) \tag{21}$$

Both signals $Y_T(s)$ and $U_T(s)$ can be replaced by their corresponding Taylor series expansions in a neighbourhood of $s = 0$ defined by

$$\begin{aligned} Y_T(s) &= \sum_{n=0}^{\infty} (-1)^n s^n \mathcal{M}_n^{y_0}(T) \\ U_T(s) &= \sum_{n=0}^{\infty} (-1)^n s^n \mathcal{M}_n^{u}(T) \end{aligned} \tag{22}$$

with $\mathcal{M}_n^v(T)$, the partial moment defined by (7). Similarly, e^{-Ts} can be substituted by its Taylor series expansion at $s = 0$ given by

$$e^{-Ts} = \sum_{n=0}^{\infty} (-1)^n \frac{s^n}{n!} T^n \tag{23}$$

The fundamental equation can be arranged in a matrix form, where each row corresponds to a power of s, as follows

$$\mathbf{M}^u(T)\mathbf{b} + \mathbb{I}\mathbf{L}x(0) = \mathbf{M}^{y_0}(T)\mathbf{a} + \mathbf{T}\mathbf{L}x(T) \tag{24}$$

where $\mathbf{M}^u(T) \in \mathbb{R}^{\infty\times(n_b+1)}$, $\mathbf{M}^{y_0}(T) \in \mathbb{R}^{\infty\times(n_a+1)}$, $\mathbf{b} \in \mathbb{R}^{(n_b+1)\times 1}$, $\mathbf{a} \in \mathbb{R}^{(n_a+1)\times 1}$, $\mathbb{I} \in \mathbb{R}^{\infty\times n_a}$, $\mathbf{T} \in \mathbb{R}^{\infty\times n_a}$ and $\mathbf{L} \in \mathbb{R}^{n_a\times n_a}$. The above matrices and vectors are defined as follows

$$\mathbf{M}^v(T) = \begin{bmatrix} \mathcal{M}_0^v(T) & 0 & \dots & \dots & 0 \\ -\mathcal{M}_1^v(T) & \mathcal{M}_0^v(T) & 0 & & \vdots \\ \mathcal{M}_2^v(T) & -\mathcal{M}_1^v(T) & \mathcal{M}_0^v(T) & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & \end{bmatrix} \tag{25}$$

with $v = u$ or y_0,

$$\mathbf{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_{n_b} \end{bmatrix}, \ \mathbf{a} = \begin{bmatrix} a_0 \\ \vdots \\ a_{n_a-1} \\ 1 \end{bmatrix}, \ \mathbb{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \\ 0 & & \vdots & 0 \\ \vdots & & \vdots & \vdots \end{bmatrix} \tag{26}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -T & 1 & \ddots & \vdots \\ T^2/2! & -T & \ddots & 0 \\ -T^3/3! & T^2/2! & \ddots & 1 \\ \vdots & \vdots & & -T \\ \vdots & \vdots & & \vdots \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} l_{0,1} & \dots & l_{0,n_a} \\ l_{1,1} & \dots & l_{1,n_a} \\ \vdots & & \vdots \\ l_{n_a-1,1} & \dots & l_{n_a-1,n_a} \end{bmatrix} \tag{27}$$

4.2.2 Partial moment formulation

The fundamental equation (24) allows the definition of the state vector $x(t)$ at instants $t = 0$ and $t = T$ for any state-space representation and for any orders n_a and n_b. The objective is to obtain the expression of $y_0(T)$. To simplify the coefficients of $\mathbf{L}$, notice that a particular state-space representation with $y_0(T) = [\ 0 \ \ \dots \ \ 0 \ \ 1\ ]x(T)$ can be considered.
For an n_a-th order system, by using only the rows $n_a + 1$ to $2n_a$ of (24), a set of n_a equations is obtained without the components of $x(0)$. This set allows the computation of $y_0(T)$ as follows

$$\begin{aligned} y_0(T) \ &= \ b_0\boldsymbol{\beta}_0^u(T) + \dots + b_{n_b}\boldsymbol{\beta}_{n_b}^u(T) + a_0\boldsymbol{\alpha}_0^{y_0}(T) + \dots \\ &\quad +a_{n_a-1}\boldsymbol{\alpha}_{n_a-1}^{y_0}(T) + \boldsymbol{\gamma}^{y_0}(T) \end{aligned} \tag{28}$$

where $\boldsymbol{\beta}_n^u(T)$ is a function of the partial moments $\mathcal{M}_i^u(T)$ with $i = 0, \dots, 2n_a - 1$, and $\boldsymbol{\alpha}_n^{y_0}(T)$ and $\boldsymbol{\gamma}^{y_0}(T)$ are functions of the partial moments $\mathcal{M}_i^{y_0}(T)$ with $i = 0, \dots, 2n_a - 1$. These functions are dependent on the system structure, *i.e.* orders n_a and n_b, and are given by the fundamental equation (24). The example 1 shows the method to obtain $y_0(T)$ for a second order system.

Example 1 *Consider a CT linear system defined by the following transfer function*

$$G(s) = \frac{b_0 + b_1 s + b_2 s^2}{a_0 + a_1 s + s^2} \tag{29}$$

This transfer function can be written in a state-space representation (16) with the so-called observable canonical form

$$A = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}, \; B = \begin{bmatrix} b_0 - b_2 a_0 \\ b_1 - b_2 a_1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; D = b_2 \tag{30}$$

By selecting $n_a = 2$ and $n_b = 2$ in the fundamental equation (24), the first four rows of this equation become

$$\begin{aligned} &\begin{bmatrix} \mathcal{M}_0^u(T) & 0 & 0 \\ -\mathcal{M}_1^u(T) & \mathcal{M}_0^u(T) & 0 \\ \mathcal{M}_2^u(T) & -\mathcal{M}_1^u(T) & \mathcal{M}_0^u(T) \\ -\mathcal{M}_3^u(T) & \mathcal{M}_2^u(T) & -\mathcal{M}_1^u(T) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{M}_0^{y_0}(T) & 0 & 0 \\ -\mathcal{M}_1^{y_0}(T) & \mathcal{M}_0^{y_0}(T) & 0 \\ \mathcal{M}_2^{y_0}(T) & -\mathcal{M}_1^{y_0}(T) & \mathcal{M}_0^{y_0}(T) \\ -\mathcal{M}_3^{y_0}(T) & \mathcal{M}_2^{y_0}(T) & -\mathcal{M}_1^{y_0}(T) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ 1 \end{bmatrix} + \\ &\begin{bmatrix} 1 & 0 \\ -T & 1 \\ \frac{T^2}{2} & -T \\ \frac{-T^3}{6} & \frac{T^2}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} \end{aligned} \tag{31}$$

By using the third and fourth rows, the following equation is obtained

$$y_0(T) = b_0 \boldsymbol{\beta}_0^u(T) + b_1 \boldsymbol{\beta}_1^u(T) + b_2 \boldsymbol{\beta}_2^u(T) + a_0 \boldsymbol{\alpha}_0^{y_0}(T) + a_1 \boldsymbol{\alpha}_1^{y_0}(T) + \boldsymbol{\gamma}^{y_0}(T) \tag{32}$$

where

$$\begin{aligned} \boldsymbol{\beta}_0^u(T) &= \frac{T\mathcal{M}_2^u(T) - 3\mathcal{M}_3^u(T)}{T^2/2} \\ \boldsymbol{\beta}_1^u(T) &= \frac{-T\mathcal{M}_1^u(T) + 3\mathcal{M}_2^u(T)}{T^2/2} \\ \boldsymbol{\beta}_2^u(T) &= \frac{T\mathcal{M}_0^u(T) - 3\mathcal{M}_1^u(T)}{T^2/2} + u(T) \\ \boldsymbol{\alpha}_0^{y_0}(T) &= -\left(\frac{T\mathcal{M}_2^{y_0}(T) - 3\mathcal{M}_3^{y_0}(T)}{T^2/2}\right) \\ \boldsymbol{\alpha}_1^{y_0}(T) &= -\left(\frac{-T\mathcal{M}_1^{y_0}(T) + 3\mathcal{M}_2^{y_0}(T)}{T^2/2}\right) \\ \boldsymbol{\gamma}^{y_0}(T) &= -\left(\frac{T\mathcal{M}_0^{y_0}(T) - 3\mathcal{M}_1^{y_0}(T)}{T^2/2}\right) \end{aligned} \tag{33}$$

■

4.2.3 Properties of the partial moment formulation

In the CT domain, the noise appears with the measurements of the signal. The signals $u(t)$ and $y(t)$ must be considered at sampling instants, *i.e.* $u(kt_s)$ and $y(kt_s)$ where t_s is the sampling period. The partial moments $\mathcal{M}_n^u(T)$ and $\mathcal{M}_n^y(T)$ must be evaluated from these DT data using a numerical integration method. Therefore, two types of errors must be considered. These are due to the noise and to the numerical integration method. The latter depends on the sampling period and the numerical integration method. In practice, the sampling period is small enough in relation to the system dynamics and Simpson's rule is used for the integrations. In this study, this numerical error is assumed to be negligible. This hypothesis has been *a posteriori* verified in [36].
Consider the disturbed output $y(t) = y_0(t) + w(t)$ at instants $t = kt_s$. The estimated signal $\widehat{y}(T)$ of $y_0(T)$, that is obtained from disturbed output $y(t)$, is given by

$$\widehat{y}(T) = b_0 \boldsymbol{\beta}_0^u(T) + \ldots + b_{n_b} \boldsymbol{\beta}_{n_b}^u(T) + a_0 \boldsymbol{\alpha}_0^y(T) + \ldots + a_{n_a-1} \boldsymbol{\alpha}_{n_a-1}^y(T) + \boldsymbol{\gamma}^y(T) \tag{34}$$

By noting that $\boldsymbol{\alpha}_i^y(T) = \boldsymbol{\alpha}_i^{y_0}(T) + \boldsymbol{\alpha}_i^w(T)$ and $\boldsymbol{\gamma}^y(T) = \boldsymbol{\gamma}^{y_0}(T) + \boldsymbol{\gamma}^w(T)$, the corresponding estimation error defined by $\varepsilon(T) = \widehat{y}(T) - y_0(T)$ is given by

$$\varepsilon(T) = a_0\boldsymbol{\alpha}_0^w(T) + \ldots + a_{n_a-1}\boldsymbol{\alpha}_{n_a-1}^w(T) + \boldsymbol{\gamma}^w(T) \tag{35}$$

This estimation error depends on the coefficients a_n of the denominator. The statistical study depends on the numerical integration method which computes the functions $\boldsymbol{\alpha}_n^w(T)$ and $\boldsymbol{\gamma}^w(T)$. The general case is very complex. But, the study of a small order system allows the generalization of the properties. Moreover, the computation of the statistical properties of $\varepsilon(T)$ is simplified by implementing the rectangle method to perform the numerical integration. A second order system case is studied in Example 2.

Example 2 *Consider the oscillating CT system defined by (1) with sampling period $t_s = 0.2s$. By using the rectangle method, the following approximation can be made in the equation (33) for the noise part $\boldsymbol{\alpha}_n^w(T)$ and $\boldsymbol{\gamma}^w(T)$*

$$\begin{aligned}
\mathcal{M}_0^w(T) &\equiv \mathcal{M}_0^w(Kt_s) = t_s \sum_{i=0}^{K-1} w(it_s) \\
\mathcal{M}_1^w(T) &\equiv \mathcal{M}_1^w(Kt_s) = t_s^2 \sum_{i=0}^{K-1} i w(it_s) \\
\mathcal{M}_2^w(T) &\equiv \mathcal{M}_2^w(Kt_s) = t_s^3 \sum_{i=0}^{K-1} \frac{i^2}{2} w(it_s) \\
\mathcal{M}_3^w(T) &\equiv \mathcal{M}_3^w(Kt_s) = t_s^4 \sum_{i=0}^{K-1} \frac{i^3}{6} w(it_s)
\end{aligned} \tag{36}$$

With the hypothesis of a zero-mean white noise $w(kt_s)$, it can be proved that the statistical mean $E\{\varepsilon(Kt_s)\}$ of the estimation error is zero. An expression of the variance $var\{\varepsilon(Kt_s)\}$ is obtained and is plotted in Figure 2. It is clear that the variance is minimal at $T_{opt} = K_{opt}t_s = 20t_s$.

■

Actually, the estimation error variance of the first order system (6) is minimal at $T = T_{opt} = \sqrt{3}/a_0$. For an oscillating second order system, it is at $T = T_{opt} = 2t_m$ where t_m is the (zero to 90 percent) rising time. For higher order systems or using of another integration method, the analytical expression of the variance $var\{\varepsilon(Kt_s)\}$ in terms of the system parameters can be complex. It is not straightforward to give the value which leads to the minimum variance. A discussion about the use of different integration methods with an example is given in Section A.3 in [31]. It is shown that the same value of T_{opt} is obtained by using the rectangular and Simpson methods.
As a conclusion, the output model (28) based on the partial moment is characterized by the property of minimum variance for the interval $[0, T_{opt}]$. Hence, the reinitialized partial moments are introduced to keep this minimum variance property at each time t. The design parameter $\widehat{T}$ introduced in RPM must be chosen in the neighbourhood of T_{opt}.

4.2.4 CT RPM formulation

The output CT RPM model is defined from the output model (28) by substituting the partial moments by the RPM given by (10) and the true output y_0 by the disturbed output y. This is further clarified by studying a second order system.

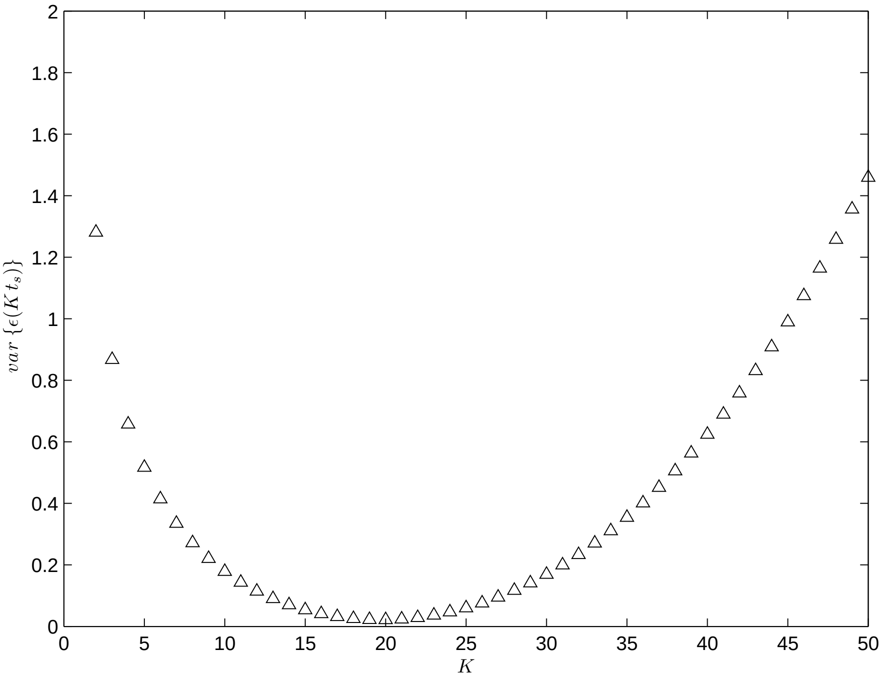

FIGURE 2 – Second order system. Dependence of the estimation error variance on K

Example 3 *Consider again the second order CT system introduced in Example 1.*
By using the RPM and the disturbed output, the output model given by (32) becomes

$$\widehat{y}(t) = \widehat{b}_0\beta_0^u(t) + \widehat{b}_1\beta_1^u(t) + \widehat{b}_2\beta_2^u(t) + \widehat{a}_0\alpha_0^y(t) + \widehat{a}_1\alpha_1^y(t) + \gamma^y(t) \tag{37}$$

where the functions in the right-hand side term are functions of RPM, e.g. $\alpha_0^y(T)$ *can be expressed as*

$$\begin{aligned} \alpha_0^y(t) &= \frac{M_3^y(t)}{\widehat{T}^2} - \frac{M_2^y(t)}{\widehat{T}} \\ &= \int_0^{\widehat{T}} \frac{\tau^3 y(t-\widehat{T}+\tau)}{\widehat{T}^2} d\tau - \int_0^{\widehat{T}} \frac{\tau^2 y(t-\widehat{T}+\tau)}{\widehat{T}} d\tau \\ &= \int_0^{\widehat{T}} f_0(\tau) y(t-\widehat{T}+\tau) d\tau \end{aligned} \tag{38}$$

Notice that the term $1/n!$ *in the definition of partial moment (7) disappears in the definition of RPM (10). It explains the different functions between (33) and the above equation.*
In the same way, all functions can be defined as follows

$$\begin{aligned} \beta_0^u(t) &= -\int_0^{\widehat{T}} f_0(\tau)u(t-\widehat{T}+\tau)d\tau \\ \beta_1^u(t) &= -\int_0^{\widehat{T}} f_1(\tau)u(t-\widehat{T}+\tau)d\tau \\ \beta_2^u(t) &= -\int_0^{\widehat{T}} f_2(\tau)u(t-\widehat{T}+\tau)d\tau + u(t) \\ \alpha_0^y(t) &= \int_0^{\widehat{T}} f_0(\tau)y(t-\widehat{T}+\tau)d\tau \\ \alpha_1^y(t) &= \int_0^{\widehat{T}} f_1(\tau)y(t-\widehat{T}+\tau)d\tau \\ \gamma^y(t) &= \int_0^{\widehat{T}} f_2(\tau)y(t-\widehat{T}+\tau)d\tau \end{aligned} \tag{39}$$

where

$$\begin{aligned} f_0(\tau) &= \frac{\tau^2(\tau-\widehat{T})}{\widehat{T}^2} \\ f_1(\tau) &= \frac{\tau(2\widehat{T}-3\tau)}{\widehat{T}^2} \\ f_2(\tau) &= \frac{2(3\tau-\widehat{T})}{\widehat{T}^2} \end{aligned} \tag{40}$$

■

4.2.5 Representation with FIR filter

The implementation for any n_a-th order system can be simplified by rewriting $\beta_n^u(t)$, $\alpha_n^y(t)$ and $\gamma^y(t)$ as responses of a FIR filter. In this context, a second order system is studied in the following example.

Example 4 *Consider again the same second order CT system. The variable change $\mu = \widehat{T} - \tau$ in (39) yields*

$$\begin{aligned} \beta_0^u(t) &= m(t) * u(t) \\ \beta_1^u(t) &= \frac{dm(t)}{dt} * u(t) \\ \beta_2^u(t) &= \frac{d^2m(t)}{dt^2} * u(t) + u(t) \\ \alpha_0^y(t) &= -m(t) * y(t) \\ \alpha_1^y(t) &= -\frac{dm(t)}{dt} * y(t) \\ \gamma^y(t) &= \left(\delta(t) - \frac{d^2m(t)}{dt^2}\right) * y(t) \end{aligned} \tag{41}$$

where

$$m(t) = \frac{(\widehat{T}-t)^2 t}{\widehat{T}^2} \qquad \textit{with} \qquad t \in \left[0, \widehat{T}\right] \tag{42}$$

■

This example can be generalized to an n_a-th order system defined by the transfer function $G(s)$ in (18). The true response $y_0(t)$ to the input $u(t)$ of this system can be modeled by the CT RPM model defined by

$$\widehat{y}(t) = \sum_{j=0}^{n_b} \widehat{b}_j \beta_j^u(t) + \sum_{i=0}^{n_a-1} \widehat{a}_i \alpha_i^y(t) + \gamma^y(t) \tag{43}$$

where

$$\begin{aligned} \beta_0^u(t) &= m(t) * u(t) \\ \alpha_0^y(t) &= -m(t) * y(t) \\ \beta_j^u(t) &= \frac{d^j m(t)}{dt^j} * u(t) \quad \text{for } 1 \le j \le n_b \\ \alpha_i^y(t) &= -\frac{d^i m(t)}{dt^i} * y(t) \quad \text{for } 1 \le i < n_a \\ \gamma^y(t) &= \left(\delta(t) - \frac{d^{n_a} m(t)}{dt^{n_a}}\right) * y(t) \\ m(t) &= \frac{(\widehat{T}-t)^{n_a} t^{n_a-1}}{(n_a-1)!\widehat{T}^{n_a}} \quad \text{with} \quad t \in \left[0, \widehat{T}\right] \end{aligned} \tag{44}$$

$m(t)$ is a FIR filter called the CT RPM filter.

4.2.6 Parameter estimation

The CT RPM model (43) can be rewritten in a linear regression form

$$\widehat{y}(t) = \varphi^T(t)\widehat{\theta}^{RPM} + \gamma^y(t) \tag{45}$$

where

$$\begin{aligned}\varphi(t) &= \left[\alpha_0^y(t), \ldots, \alpha_{n-1}^y(t), \beta_0^u(t), \ldots, \beta_m^u(t)\right]^T \\ \widehat{\theta}^{RPM} &= \left[\widehat{a}_0, \ldots, \widehat{a}_{n-1}, \widehat{b}_0, \ldots, \widehat{b}_m\right]^T\end{aligned} \tag{46}$$

Assuming that N discrete values of the input-output signals are measured, the least-squares estimate of $\widehat{\theta}^{RPM}$ is given by

$$\widehat{\theta}^{RPM} = \left[\sum_{i=\widehat{K}}^{N} \varphi(it_s)\varphi^T(it_s)\right]^{-1} \sum_{i=\widehat{K}}^{N} \varphi(it_s)\left(y(it_s) - \gamma^y(it_s)\right) \tag{47}$$

where $\widehat{K}$ corresponds to $\widehat{T} = \widehat{K}t_s$ that is an estimation of T_{opt}.

Remarque 1 *Different equation error methods can be applied, such as the iterative instrumental variable approach with an auxiliary model [38, 39], to eliminate the bias.*

Remarque 2 *Notice that the implicit FIR filter and the above least-squares estimate allow the removal of the transient effect of an infinite impulse response filter. Effectively, in the least-squares estimate (47), the $\widehat{K}$ first measurements are not considered.*

Remarque 3 *The MISO transfer function model with a common denominator can be considered. In that case, the regressor and the parameter vector in (45) become*

$$\begin{aligned}\varphi(t) &= \left[\alpha_0^y(t), \ldots, \alpha_{n-1}^y(t), \beta_0^{u_1}(t), \ldots, \beta_{m_1}^{u_1}(t), \ldots, \beta_0^{u_{n_u}}(t), \ldots, \beta_{m_{n_u}}^{u_{n_u}}(t)\right]^T \\ \widehat{\theta}^{RPM} &= \left[\widehat{a}_0, \ldots, \widehat{a}_{n-1}, \widehat{b}_0^1, \ldots, \widehat{b}_{m_1}^1, \ldots, \widehat{b}_0^{n_u}, \ldots, \widehat{b}_{m_{n_u}}^{n_u}\right]^T\end{aligned} \tag{48}$$

where n_u is the considered input number.
The MIMO case with n_y outputs can be considered as n_y MISO models. Notice that, in the general MIMO case, the variance of the estimated model may increase.

4.2.7 Implementation

The Matlab routines lsctrpm and ivctrpm, that implement (47) and the iterative instrumental variable approach for MIMO systems, respectively, can be downloaded from http://laii.univ-poitiers.fr/ouvrard/CTRPM. The implementation is described in this subsection.
By referring to the CT RPM output model (43), $\alpha_i^y(t)$, $\beta_i^u(t)$ and $\gamma^y(t)$ are computed by performing the convolution products between $m(t)$ or its derivatives and the input-output signals. In practice, the following expressions are implemented

$$\begin{aligned}\alpha_i^y(t) &= -\int_0^{\widehat{T}} f_i(\mu)y(t-\widehat{T}+\mu)d\mu \\ \beta_i^u(t) &= \int_0^{\widehat{T}} f_i(\mu)u(t-\widehat{T}+\mu)d\mu \\ \gamma^y(t) &= -\int_0^{\widehat{T}} f_{n_a}(\mu)y(t-\widehat{T}+\mu)d\mu\end{aligned} \tag{49}$$

where

$$\begin{aligned}f_0(\mu) &= \frac{\mu^{n_a}(\widehat{T}-\mu)^{n_a-1}}{(n_a-1)!\widehat{T}^{n_a}} \\ f_i(\mu) &= -\frac{df_{i-1}(\mu)}{d\mu}\end{aligned} \tag{50}$$

The following recursive form allows the computation of $f_i(\mu)$ for $i = 0, \ldots, n_a - 1$

$$f_i(\mu) = \frac{(-1)^i}{(n_a-1)!\widehat{T}^{n_a}} \sum_{j=0}^{i} (-1)^j \frac{i!}{j!(i-j)!} \frac{(n_a-1)!n_a!}{(n_a-j-1)!(n_a-i+j)!} \mu^{n_a-i+j}(\widehat{T}-\mu)^{n_a-j-1} \tag{51}$$

The integrations in (49) are computed using Simpson's rule, *e.g.* for $\alpha_i^y(t)$

$$\begin{array}{rcl} \alpha_i^y(t) & = & -\frac{t_s}{3}\sum\limits_{k=2}^{\widehat{K}}\left[f_i((k-2)t_s)\,y(t-(\widehat{K}-l+2)t_s)+\right. \\ & & \left. 4f_i((k-1)t_s)y(t-(\widehat{K}-l+1)t_s)+f_i(kt_s)y(t-(\widehat{K}-l)t_s)\right] \end{array} \tag{52}$$

where k and $\widehat{K}$ are even.

The function $\beta_i^u(t)$ can be computed in a similar way to the expression given in (52). However, if $u(t)$ is a piecewise constant input, *e.g.* the input is generated by a digital to analog converter, the rectangle method has to be implemented to compute the integration. Consequently, the following expression is obtained

$$\beta_i^u(t) = \sum_{k=0}^{\widehat{K}-1} F_i^{sq}(kt_s)u(t-(\widehat{K}-k)t_s) \tag{53}$$

where the function $F_i^{sq}(kt_s)$ is given by

$$\begin{array}{l} F_i^{sq}(kt_s) = \frac{(-1)^i}{(n_a-1)!(\widehat{K}t_s)^{n_a}}\sum\limits_{j=0}^{i}(-1)^j\frac{i!}{j!(i-j)!}\frac{(n_a-1)!n_a!}{(n_a-j-1)!(n_a-i+j)!} \\ \sum\limits_{r=0}^{n_a-j-1}(-1)^r\frac{(n_a-j-1)!}{r!(n_a-j-1-r)!}(\widehat{K}t_s)^{n_a-j-1-r}\left\{\frac{((k+1)t_s)^{n_a-i+j+r+1}-(kt_s)^{n_a-i+j+r+1}}{n_a-i+j+r+1}\right\} \end{array} \tag{54}$$

4.3 Applications

The RPM model properties have been used for two decades in different application fields such as electrical engineering [4, 5, 7] or electronics [8]. Generally, the RPM model parameters are used to initialize an optimization algorithm in physical parameter estimations.

The CT RPM model has been included in the CONTSID Matlab toolbox [13] and the routine is called lsrpm. This Matlab toolbox can be downloaded from http://www.cran.uhp-nancy.fr/contsid/. A comparison with other methods implemented in the CONTSID Matlab toolbox is described in [14, 13, 23], where the interesting performance of the RPM model has been shown.

A subspace-based method for CT MIMO systems identification has been proposed in [24]. This approach more precisely consists of introducing the RPM FIR filter to build a particular sampled input-output algebraic relationship to which a MOESP-like algorithm can be applied.

The CT RPM model has been compared with five other CT system identification methods (Section 2.3.1 in [31]) in terms of initialization of optimization algorithms. This study highlights the suitable initialization performed by the CT RPM model.

New algorithms have been introduced in Chapter 3 in [31]. These so-called pseudo-output error algorithms are optimization algorithms based on pseudo-sensitivity functions with a filter, such as the RPM FIR filter, which gives a global asymptotic convergence if a positive realness condition is satisfied [32, 34].

5 Discrete-time RPM model

If, in the CT framework, an integration of the differential equation is obvious to approximate the unmeasurable derivatives, in the DT framework, the application of an integration (a summation) is less natural. However, it will be shown in the following that this approach can also be rewritten as an implicit data filtering, and for the DT identification methods, an explicit data filtering is often used. This is an implicit suitable property of the DT RPM model.

5.1 Preliminaries

In the DT case, the approach to introducing the DT RPM model follows the same steps as in the CT case.
Define the n-th order DT partial moment of a sequence $v(k)$ by

$$\mathcal{C}_n^v(K) = \sum_{k=n}^{K} \frac{k!}{(k-n)!} v(k) \tag{55}$$

Notice that the DT partial moment is the standard factorial DT moment truncated to a finite interval $[n, K]$.
The computation of the DT partial moment of the system difference equation gives an output formulation with the minimum variance property for a particular time window $K = K_{opt}$. To keep this minimum variance at each instant k, the n-th order DT RPM (*discrete-time reinitialized partial moment*) of a signal $v(k)$ has been introduced and is defined by

$$C_n^v(k) = \sum_{j=n}^{\widehat{K}-1} A_j^n v(k - \widehat{K} + j) \tag{56}$$

where $A_j^n = \frac{j!}{(j-n)!}$ and $\widehat{K}$ is an estimation of K_{opt}. $\widehat{K}$ is the design parameter called the reinitialization parameter [4].
The computation of the DT RPM of the system difference equation gives the DT RPM model. See the following section for the general case.

5.2 Generalization

5.2.1 Fundamental equation

As in the CT case, the fundamental equation allows consideration of the general case. The objective is to describe the behaviour of a DT linear system on the interval $[0, K-1]$ based on the input-output data as shown in Figure 1 but with discrete-time signals.
Consider a SISO linear system defined by the following minimal state-space representation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y_0(k) &= Cx(k) + Du(k) \end{aligned} \tag{57}$$

where $x(k) \in \mathbb{R}^{n_a}$, $y_0(k) \in \mathbb{R}$ and $u(k) \in \mathbb{R}$. The system can also be represented by its z transfer function and a free output due to the initial conditions $x(0)$

$$Y_0(z) = G(z)U(z) + L(z)x(0) \tag{58}$$

with

$$\begin{aligned} G(z) &= C\,(zI - A)^{-1}\,B + D = \frac{b_0 + b_1 z^{-1} + \ldots + b_{n_b} z^{-n_b}}{1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a}} \\ L(z) &= C\,(zI - A)^{-1} = \frac{\left[\begin{array}{ccc} L_1(z) & \cdots & L_{n_a}(z) \end{array}\right]}{1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a}} \end{aligned} \tag{59}$$

where $L_n(z) = l_{0,n} + l_{1,n} z^{-1} + \ldots + l_{n_a-1,n} z^{-(n_a-1)}$.
Suppose that the system excitation is an artificial input $u_K(k)$. It is defined as the true input on the interval $[0, K-1]$ and is equal to zero for $k \geq K$. Also, assume that $x(0)$ is the initial condition. The

4. Notice that both CT and DT design parameters, $\widehat{T}$ in (10) and $\widehat{K}$ in (56), are linked by $\widehat{T} = \widehat{K}\, t_s$.

corresponding artificial output $\tilde{y}(k)$ consists of $y_K(k) = y_0(k)$ on the interval $[0, K-1]$ and a free output due to $x(K)$ on the interval $[K, \infty[$. The corresponding z transform of $\tilde{y}(k)$ yields

$$\tilde{Y}(z) = Y_K(z) + z^{-K} L(z) x(K) \tag{60}$$

According to (58), $\tilde{y}(k)$ in the z domain can also be written as follows

$$\tilde{Y}(z) = G(z) U_K(z) + L(z) x(0) \tag{61}$$

With both previous equations, the fundamental equation is deduced

$$G(z) U_K(z) + L(z) x(0) = Y_K(z) + z^{-K} L(z) x(K) \tag{62}$$

Both signals $Y_K(z)$ and $U_K(z)$ can be substituted by their corresponding Taylor series expansions in a neighbourhood of $z^{-1} = 1$ defined by

$$\begin{aligned} Y_K(z) &= \sum_{n=0}^{\infty} \frac{(z^{-1}-1)^n}{n!} \mathcal{C}_n^{y_0}(K-1) \\ U_K(z) &= \sum_{n=0}^{\infty} \frac{(z^{-1}-1)^n}{n!} \mathcal{C}_n^{u}(K-1) \end{aligned} \tag{63}$$

with $\mathcal{C}_n^v(K-1)$, the partial moment defined by (55). The fundamental equation can be arranged in a matrix form, where each row corresponds to a power of z, as follows

$$\mathbf{C}^u(K-1)\mathbf{N}_b\mathbf{b} + \mathbb{I}\mathbf{N}_L\mathbf{L}x(0) = \mathbf{C}^{y_0}(K-1)\mathbf{N}_a\mathbf{a} + \mathbf{K}\mathbf{N}_L\mathbf{L}x(K) \tag{64}$$

where $\mathbf{C}^u(K-1) \in \mathbb{R}^{\infty\times(n_b+1)}$, $\mathbf{C}^{y_0}(K-1) \in \mathbb{R}^{\infty\times(n_a+1)}$, $\mathbf{b} \in \mathbb{R}^{(n_b+1)\times 1}$, $\mathbf{a} \in \mathbb{R}^{(n_a+1)\times 1}$, $\mathbb{I} \in \mathbb{R}^{\infty\times n_a}$, $\mathbf{N}_b \in \mathbb{R}^{(n_b+1)\times(n_b+1)}$, $\mathbf{N}_a \in \mathbb{R}^{(n_a+1)\times(n_a+1)}$, $\mathbf{N}_L \in \mathbb{R}^{n_a\times n_a}$, $\mathbf{K} \in \mathbb{R}^{\infty\times n_a}$ and $\mathbf{L} \in \mathbb{R}^{n_a\times n_a}$. The above matrices and vectors are defined as follows

$$\mathbf{C}^v(K-1) = \begin{bmatrix} \mathcal{C}_0^v(K-1) & 0 & \ldots & \ldots & 0 \\ \mathcal{C}_1^v(K-1) & \mathcal{C}_0^v(K-1) & 0 & & \vdots \\ \mathcal{C}_2^v(K-1) & 2\mathcal{C}_1^v(K-1) & \mathcal{C}_0^v(K-1) & 0 & \vdots \\ \mathcal{C}_3^v(K-1) & 3\mathcal{C}_2^v(K-1) & 3\mathcal{C}_1^v(K-1) & \mathcal{C}_0^v(K-1) & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{65}$$

with $v = u$ or y_0,

$$\mathbf{b} = \begin{bmatrix} b_0 \\ \vdots \\ b_{n_b} \end{bmatrix}, \ \mathbf{a} = \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_{n_a} \end{bmatrix}, \ \mathbb{I} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \\ 0 & & \vdots & 0 \\ \vdots & & \vdots & \vdots \end{bmatrix} \tag{66}$$

$$\mathbf{N}_m = \begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & \ldots \\ 0 & 1 & 2 & 3 & \ldots & \ldots \\ 0 & 0 & 2 & 6 & \ldots & N_{i,j} \\ 0 & 0 & 0 & 6 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & & \end{bmatrix} \tag{67}$$

with $m = b$, a or L, and $N_{i,j} = \frac{(j-1)!}{(j-i)!}$ for the i-th row and the j-th column,

$$\mathbf{K} = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots \\ K & 1 & 0 & \dots & \dots \\ K(K-1) & 2K & 1 & 0 & \dots \\ K(K-1)(K-2) & 3K(K-1) & 3K & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{68}$$

$$\mathbf{L} = \begin{bmatrix} l_{0,1} & \dots & l_{0,n_a} \\ l_{1,1} & \dots & l_{1,n_a} \\ \vdots & & \vdots \\ l_{n_a-1,1} & \dots & l_{n_a-1,n_a} \end{bmatrix} \tag{69}$$

In the matrices $\mathbf{C}^v(K-1)$ and $\mathbf{K}$, the numbers, which appear in the lower triangular part, are given by $\frac{(i-1)!}{(i-j)!(j-1)!}$ for the i-th row and the j-th column

5.2.2 Partial moment formulation and its properties

The fundamental equation (64) allows the computation of $y_0(K)$ for any state-space representation and for any orders n_a and n_b by considering the rows n_a+1 to $2n_a$. To simplify the coefficients of $\mathbf{L}$, consider a particular state-space representation which yields $y_0(K) = [\begin{array}{cccc} 0 & \dots & 0 & 1 \end{array}]x(K)$.
The partial moment formulation of $y_0(K)$ is given by

$$y_0(K) = b_0\boldsymbol{\beta}_0^u(K) + \dots + b_{n_b}\boldsymbol{\beta}_{n_b}^u(K) + a_1\boldsymbol{\alpha}_1^{y_0}(K) + \dots + a_{n_a}\boldsymbol{\alpha}_{n_a}^{y_0}(K) + \boldsymbol{\gamma}^{y_0}(K) \tag{70}$$

where $\boldsymbol{\beta}_n^u(K)$ is a function of the partial moments $\mathcal{C}_i^u(K-1)$ with $i = 0, \dots, 2n_a - 1$, and $\boldsymbol{\alpha}_n^{y_0}(K)$ and $\boldsymbol{\gamma}^{y_0}(K)$ are functions of the partial moments $\mathcal{C}_i^{y_0}(K-1)$ with $i = 0, \dots, 2n_a - 1$. These functions are given by the fundamental equation (64) for any orders n_a and n_b. The example 5 gives the way to obtain $y_0(K)$ for a second order system.

Example 5 *Consider a DT linear system defined by the following z transfer function*

$$G(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{71}$$

This transfer function can be written in a state-space representation (57) with

$$A = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix}, \; B = \begin{bmatrix} b_2 - b_0 a_2 \\ b_1 - b_0 a_1 \end{bmatrix}, \\ C = [\begin{array}{cc} 0 & 1 \end{array}], \; D = b_0, \; x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{72}$$

By selecting $n_a = 2$ and $n_b = 2$, the first four rows of the fundamental equation (64) become

$$
\begin{aligned}
&\begin{bmatrix} \mathcal{C}_0^u(K-1) & 0 & 0 \\ \mathcal{C}_1^u(K-1) & \mathcal{C}_0^u(K-1) & 0 \\ \mathcal{C}_2^u(K-1) & 2\mathcal{C}_1^u(K-1) & \mathcal{C}_0^u(K-1) \\ \mathcal{C}_3^u(K-1) & 3\mathcal{C}_2^u(K-1) & 3\mathcal{C}_1^u(K-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\
&+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{C}_0^{y_0}(K-1) & 0 & 0 \\ \mathcal{C}_1^{y_0}(K-1) & \mathcal{C}_0^{y_0}(K-1) & 0 \\ \mathcal{C}_2^{y_0}(K-1) & 2\mathcal{C}_1^{y_0}(K-1) & \mathcal{C}_0^{y_0}(K-1) \\ \mathcal{C}_3^{y_0}(K-1) & 3\mathcal{C}_2^{y_0}(K-1) & 3\mathcal{C}_1^{y_0}(K-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} \\
&+ \begin{bmatrix} 1 & 0 \\ K & 1 \\ K(K-1) & 2K \\ K(K-1)(K-2) & 3K(K-1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(K) \\ x_2(K) \end{bmatrix}
\end{aligned} \tag{73}
$$

The use of the third and fourth rows yields the following equation

$$
y_0(K) = b_0\boldsymbol{\beta}_0^u(K) + b_1\boldsymbol{\beta}_1^u(K) + b_2\boldsymbol{\beta}_2^u(K) + a_1\boldsymbol{\alpha}_1^{y_0}(K) + a_2\boldsymbol{\alpha}_2^{y_0}(K) + \boldsymbol{\gamma}^{y_0}(K) \tag{74}
$$

where

$$
\begin{aligned}
\boldsymbol{\beta}_0^u(K) &= \tfrac{-C_3^u(K-1)+(K-1)C_2^u(K-1)}{K(K-1)} + u(K) \\
\boldsymbol{\beta}_1^u(K) &= \tfrac{-C_3^u(K-1)+(K-4)C_2^u(K-1)+2(K-1)C_1^u(K-1)}{K(K-1)} \\
\boldsymbol{\beta}_2^u(K) &= \tfrac{-C_3^u(K-1)+(K-7)C_2^u(K-1)+2(2K-5)C_1^u(K-1)+2(K-1)C_0^u(K-1)}{K(K-1)} \\
\boldsymbol{\alpha}_1^{y_0}(K) &= \tfrac{C_3^{y_0}(K-1)-(K-4)C_2^{y_0}(K-1)-2(K-1)C_1^{y_0}(K-1)}{K(K-1)} \\
\boldsymbol{\alpha}_2^{y_0}(K) &= \tfrac{C_3^{y_0}(K-1)-(K-7)C_2^{y_0}(K-1)-2(2K-5)C_1^{y_0}(K-1)-2(K-1)C_0^{y_0}(K-1)}{K(K-1)} \\
\boldsymbol{\gamma}^{y_0}(K) &= \tfrac{C_3^{y_0}(K-1)-(K-1)C_2^{y_0}(K-1)}{K(K-1)}
\end{aligned} \tag{75}
$$

■

The estimated signal $\widehat{y}(K)$ of $y_0(K)$, that is obtained from the disturbed output measurements $y(k) = y_0(k) + w(k)$, is given by

$$
\widehat{y}(K) = \widehat{b}_0\boldsymbol{\beta}_0^u(K) + \ldots + \widehat{b}_{n_b}\boldsymbol{\beta}_{n_b}^u(K) + \widehat{a}_1\boldsymbol{\alpha}_1^y(K) + \ldots + \widehat{a}_{n_a}\boldsymbol{\alpha}_{n_a}^y(K) + \boldsymbol{\gamma}^y(K) \tag{76}
$$

As in the CT case, it was shown [36, 31] that any linear stable DT system can be represented by the output model (76) with a minimum variance by selecting $K = K_{opt}$. In practice, the choice of K_{opt} corresponds to T_{opt}/t_s discussed in Section 4.2.3 for a first or second order CT system. Hence, the reinitialized partial moments are introduced to estimate the output model with minimum variance at each instant k.

5.2.3 DT RPM formulation

The output DT RPM model can be defined from the output model (76) as follows

$$
\widehat{y}(k) = \widehat{b}_0\beta_0^u(k) + \ldots + \widehat{b}_{n_b}\beta_{n_b}^u(k) + \widehat{a}_1\alpha_1^y(k) + \ldots + \widehat{a}_{n_a}\alpha_{n_a}^y(k) + \gamma^y(k) \tag{77}
$$

where $\beta_n^u(k)$, $\alpha_n^y(k)$ and $\gamma^y(k)$ are functions of the RPM.
It has been shown in [36] that the RPM can be expressed in a recursive form

$$C_n^v(k) = C_n^v(k-1) + A_{\widehat{K}}^n v(k-1) - nC_{n-1}^v(k) \tag{78}$$

with

$$C_0^v(k) = C_0^v(k-1) + v(k-1) - v(k-1-\widehat{K}) \tag{79}$$

Similarly, the functions $\beta_n^u(k)$, $\alpha_n^y(k)$ and $\gamma^y(k)$ can be formulated in a recursive form. In this context, a second order system is studied in the following example.

Example 6 *Consider again the second order DT system introduced in Example 5.*
The output model as given by (74) can be formulated using the RPM in their recursive expressions

$$\begin{aligned} C_3^v(k) &= C_3^v(k-1) + A_{\widehat{K}}^3 v(k-1) - 3C_2^v(k) \\ C_2^v(k) &= C_2^v(k-1) + A_{\widehat{K}}^2 v(k-1) - 2C_1^v(k) \\ C_1^v(k) &= C_1^v(k-1) + A_{\widehat{K}}^1 v(k-1) - C_0^v(k) \end{aligned} \tag{80}$$

Then the RPM model is obtained

$$\widehat{y}(k) = \widehat{b}_0\beta_0^u(k) + \widehat{b}_1\beta_1^u(k) + \widehat{b}_2\beta_2^u(k) + \widehat{a}_1\alpha_1^y(k) + \widehat{a}_2\alpha_2^y(k) + \gamma^y(k) \tag{81}$$

with the following recursive expressions

$$\begin{aligned} \beta_0^u(k) &= \frac{-C_3^u(k)+(\widehat{K}-1)C_2^u(k)}{\widehat{K}(\widehat{K}-1)} + u(k) \\ \beta_1^u(k) &= \beta_0^u(k-1) \\ \beta_2^u(k) &= \beta_1^u(k-1) \\ \gamma^y(k) &= \frac{C_3^y(k)-(\widehat{K}-1)C_2^y(k)}{\widehat{K}(\widehat{K}-1)} \\ \alpha_1^y(k) &= \gamma^y(k-1) - y(k-1) \\ \alpha_2^y(k) &= \alpha_1^y(k-1) \end{aligned} \tag{82}$$

■

This approach can be extended to an n_a-th order system. Similar recursive expressions can be derived. Consequently, the implementation is simplified and the computational time is reduced.

5.2.4 Representation with FIR filter

The computational time can be further reduced by expressing the functions $\beta_n^u(k)$, $\alpha_n^y(k)$ and $\gamma^y(k)$ as responses of a FIR filter. In this context, consider the following example.

Example 7 *Consider again the same second order DT system (71) and use the recursive equations as given in (82).*
$\beta_0^u(k)$ and $\gamma^y(k)$ can be rewritten as follows

$$\begin{aligned} \beta_0^u(k) &= \frac{-\sum\limits_{i=3}^{\widehat{K}-1} i(i-1)(i-2)u(k-\widehat{K}+i)+(\widehat{K}-1)\sum\limits_{i=2}^{\widehat{K}-1} i(i-1)u(k-\widehat{K}+i)}{\widehat{K}(\widehat{K}-1)} + u(k) \\ \gamma^y(k) &= \frac{\sum\limits_{i=3}^{\widehat{K}-1} i(i-1)(i-2)y(k-\widehat{K}+i)-(\widehat{K}-1)\sum\limits_{i=2}^{\widehat{K}-1} i(i-1)y(k-\widehat{K}+i)}{\widehat{K}(\widehat{K}-1)} \end{aligned} \tag{83}$$

and can be reformulated

$$\begin{aligned}\beta_0^u(k) &= \sum_{i=0}^{\widehat{K}-2} m_i u(k-i)\\ \gamma^y(k) &= -\sum_{i=0}^{\widehat{K}-2} m_i y(k-i) + m_0 y(k)\end{aligned} \tag{84}$$

where

$$m_i = \frac{(i+1)(\widehat{K}-i)(\widehat{K}-i-1)}{\widehat{K}(\widehat{K}-1)} \tag{85}$$

The other functions can be deduced from $\beta_0^u(k)$ and $\gamma^y(k)$ using the recursive expressions.

■

This example can be generalized to an n_a-th order system defined by the z transfer function $G(z)$ in (59). The true response $y_0(k)$ to the input $u(k)$ of this system can be modeled by the DT RPM model defined by

$$\widehat{y}(k) = \sum_{m=0}^{n_b} \widehat{b}_m \beta_m^u(k) + \sum_{n=1}^{n_a} \widehat{a}_n \alpha_n^y(k) + \gamma^y(k) \tag{86}$$

where

$$\begin{aligned}\beta_0^u(k) &= \sum_{i=0}^{\widehat{K}-n_a} m_i u(k-i)\\ \beta_m^u(k) &= \beta_{m-1}^u(k-1),\ \ m=1,\dots,n_b\\ \gamma^y(k) &= -\sum_{i=1}^{\widehat{K}-n_a} m_i y(k-i)\\ \alpha_1^y(k) &= \gamma^y(k-1) - y(k-1)\\ \alpha_n^y(k) &= \alpha_{n-1}^y(k-1),\ \ n=1,\dots,n_a\\ m_i &= \tfrac{(i+1)(i+2)\dots(i+n_a-1)A_{\widehat{K}-i}^{n_a}}{(n_a-1)!A_{\widehat{K}}^{n_a}}\end{aligned} \tag{87}$$

5.2.5 Parameter estimation

The DT RPM model (86) can be rewritten in a linear regression form

$$\widehat{y}(k) = \phi^T(k)\widehat{\theta}^{RPM} + \gamma^y(k) \tag{88}$$

where

$$\begin{aligned}\widehat{\theta}^{RPM} &= \left[\widehat{a}_1, \cdots, \widehat{a}_{n_a}, \widehat{b}_0, \cdots, \widehat{b}_{n_b}\right]^T\\ \phi(k) &= \left[\alpha_1^y(k), \cdots, \alpha_{n_a}^y(k), \beta_0^u(k), \cdots, \beta_{n_b}^u(k)\right]^T\end{aligned} \tag{89}$$

Assuming that N values of the input-output signals are measured, the least-squares estimate of $\widehat{\theta}^{RPM}$ is given by

$$\widehat{\theta}^{RPM} = \left[\sum_{k=\widehat{K}-n_a}^{N} \phi(k)\phi^T(k)\right]^{-1} \sum_{k=\widehat{K}-n_a}^{N} \phi(k)(y(k)-\gamma^y(k)) \tag{90}$$

where $\widehat{K}$ is an estimation of K_{opt}.

Remarque 4 *As in the CT case, different equation error methods can be applied. Moreover, the implicit FIR filter and the above implementation allow the removal of the transient effect of an infinite impulse response filter because in the least-squares estimate (90), the $\widehat{K}-n_a$ first measurements are not considered.*

Remarque 5 *The MISO transfer function model with a common denominator can be considered. In that case, the parameter vector and the regressor in (89) become*

$$\begin{aligned} \widehat{\theta}^{RPM} &= \left[\widehat{a}_1, \ldots, \widehat{a}_{n_a}, \widehat{b}_0^1, \ldots, \widehat{b}_{n_b^1}^1, \ldots, \widehat{b}_0^{n_u}, \ldots, \widehat{b}_{n_b^{n_u}}^{n_u}\right]^T \\ \phi(k) &= \left[\alpha_1^y(k), \ldots, \alpha_{n_a}^y(k), \beta_0^{u_1}(k), \ldots, \beta_{n_b^1}^{u_1}(k), \ldots, \beta_0^{u_{n_u}}(k), \ldots, \beta_{n_b^{n_u}}^{u_{n_u}}(k)\right]^T \end{aligned} \tag{91}$$

where n_u is the considered input number.
The MIMO case with n_y outputs can be considered as n_y MISO models.

Contrary to the CT case, the implementation of the DT RPM model estimation is straightforward. The Matlab routines lsdtrpm and ivdtrpm, that implement (90) and the iterative instrumental variable approach for MIMO systems, respectively, can be downloaded from http://laii.univ-poitiers.fr/ouvrard/DTRPM.

5.3 Applications

Until now, the RPM models have been used mainly in the CT area. But, recent papers [28, 29] have highlighted the problem of convergence of discrete-time optimization algorithms, particularly, the OE, PEM and N4SID algorithms [5]. In [19], it has been shown that these problems are due to the bias introduced by the ARX model which is used as the initial value for the optimization algorithms. This is a typical problem of bad initialization, *i.e.*, a convergence to a secondary optimum. The solution proposed in [19] to reduce the bias of the ARX model is to use a low-pass data filter. The DT RPM model does not have this problem because it presents an implicit embedded filter which plays the same role as the explicit filter for the ARX model. A bias analysis and a comparison study of the DT RPM and ARX models is given in Section 2.3.2 in [31] and in [33, 25].
Notice also, as for the CT case, that the new pseudo-output error algorithms introduced in Chapter 3 in [31] and in [34, 32] can be applied in the DT framework. The DT RPM FIR filter can be used and its properties used to good account.

6 Choice of the design parameter

The CT RPM and the DT RPM models require the selection of a design parameter, the reinitialization parameters, $\widehat{T}$ and $\widehat{K}$, respectively. Indeed, these two design parameters are linked because $\widehat{T} = \widehat{K}t_s$. Taking into account this link, only the term $\widehat{K}$ and the DT case are mentioned in this section.
A wide experience in RPM handling has shown that the quality of the RPM model is not very sensitive to this choice (See [31], for instance). The selection of $\widehat{K}$ is not more difficult than the selection of the cutoff frequency and the order of the recommended data filter of an ARX model [19], or any design parameters of CT system identification methods [14].
The design parameter $\widehat{K}$ allows the adaptation of the RPM model to the nature of the noise :

- If the perturbation is a white output-error noise, *i.e.* the structure of the system belongs to the OE model set, then an optimal reinitialization parameter exists, namely $\widehat{K}_{wn}$, for which the variance of the error is minimal and the bias is highly reduced. For a small system order, $\widehat{K}_{wn}$ can be computed and many experiments [31] led to the following conclusion : the parameter $\widehat{K}$ should be selected such that the value $\widehat{K}t_s$ (t_s, sampling time) is equivalent to the double of the main time constant for an aperiodic system or the double of the (zero to 90%) rising time for an oscillating system. But for an higher system order, the optimal value $\widehat{K}_{wn}$ can be only

5. Here the OE, PEM and N4SID algorithms refer to the Matlab procedures with the same names in the System Identification Toolbox.

found empirically by increasing $\widehat{K}$ progressively. If $\widehat{K} = \widehat{K}_{wn}$, the RPM model is close to an OE model, *i.e.* the implicit RPM filter is close to the ideal filter of Steiglitz-McBride [30]. For more details see Section 2.3.2 in [31] and [33, 25].

– If the perturbation is a white equation-error noise, *i.e.* the structure of the system belongs to the ARX model set, then the reinitialization parameter must be equal to n_a. Consequently, the RPM model is equivalent to an ARX model and the estimation is unbiased.
– If the perturbation is a coloured noise, *i.e.* the structure of the system does not belong to the ARX model set or the OE model set, then the optimal reinitialization parameter is in the interval $]n_a, \widehat{K}_{wn}[$.

In practice, the value of $\widehat{K}$ can be selected empirically as follows : $\widehat{K}$ is increased and a standard test, such as the quadratic criterion or the autocorrelation of the residuals, is evaluated to find the best $\widehat{K}$. Notice again that an iterative instrumental variable technique with an auxiliary model [38, 39] is recommended and can also be applied to eliminate the bias in all cases.

7 Conclusion

This paper presents a complete description of the reinitialized partial moments in both CT and DT cases. The RPM has originated from the partial moment formulation. This formulation had been introduced to avoid the restriction to impulse or step input and the necessity to calculate moments on an infinite time interval, two problems which limited the application of moments in system identification. The complete details of the partial moment formulation are described in this paper. The developments that lead to the RPM and the implementation are given. Some advice on use allows the tuning of the design parameter. The RPM model applications are listed. Generally, the RPM models give a suitable output error method initialization in both CT and DT domains. The implicit embedded FIR filter yields certain properties to this kind of equation error approach. In continuous-time, the filter allows approximating of the unmeasurable input-output derivatives. In discrete-time, the filter plays the same role as the explicit filter for the ARX model estimation.

Références

[1] B. Abdelmadjid, N. Maamri, and J. C. Trigeassou. The moments in control : a tool for analysis, reduction and design. *International Journal of Computers, Communications and Control*, 2 (1) :82–103, 2007.

[2] B. Abdelmadjid, N. Maamri, and J. C. Trigeassou. Design of a mimo pid robust controller using moments based approach. *International Journal of Computers, Communications and Control*, 3 (2) :125–134, 2008.

[3] C. Bruni, A. Isidori, and A. Ruberti. A method of realization based on the moments of the impulse-response matrix. *IEEE Trans. Automat. Contr.*, 14-2 :203–204, April 1969.

[4] P. Coirault, J. C. Trigeassou, J. P. Gaubert, and G. Champenois. Parameter estimation of an induction machine using reinitialized partial moments. In *IEEE Conference on Control Applications*, pages 979–984, 1995.

[5] P. Coirault, J. C. Trigeassou, D. Kerignard, and J. P. Gaubert. Recursive parameter identification of an induction machine using a non linear programming method. In *IEEE Conference on Control Applications*, pages 644–649, 1996.

[6] H. Cramér. *Mathematical methods of statistics.* NJ : Princeton University Press, Princeton, 1946.

[7] J. P. Da Cunha, G. Champenois, and J. D. Gabano. Comparison of parameter estimation algorithms for continuous time models applied to d.c. motor thermal supervision. *Electric Machines and Power Systems*, 26 (8) :839–853, 1996.

[8] M. Djamai, E. Tohme, R. Ouvrard, and S. Bachir. Continuous-time model identification using reinitialized partial moments - application to power amplifier modeling. In *14th IFAC Symposium on System Identification*, pages 1270–1275, Newcastle, Australia, 2006.

[9] W. C. Elmore. The transient response of damped linear networks with particular regard to wideband amplifiers. *J. Applied Physics*, 19 :55–63, January 1948.

[10] M. Fliess and H. Sira-Ramírez. An algebraic framework for linear identification. *ESAIM Control Optim Calc. Var.*, 9 :151–168, 2003.

[11] M. Fliess and H. Sira-Ramírez. Reconstructeur d'état. *C. R. Acad. Sci. Paris*, Ser. I 338, 2004.

[12] M. Fliess and H. Sira-Ramírez. *Identification of continuous-time models from sampled data*, chapter Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques, pages 383–391. Springer, 2008.

[13] H. Garnier, M. Gilson, and E. Huselstein. Developments for the matlab contsid toolbox. In *13th IFAC Symposium on System Identification*, pages 1007–1012, Rotterdam, The Netherlands, 2003.

[14] H. Garnier, M. Mensler, and A. Richard. Continuous-time model identification from sampled data : implementation issues and performance evaluation. *Int. J. Control*, 76(13) :1337–1357, 2003.

[15] H. Garnier and L. Wang. *Identification of continuous-time models from sampled data*. Springer-Verlag, London, 2008.

[16] W. Krajewski, A. Lepschy, and U. Viaro. Model reduction by matching markov parameters, time moments, and impulse-response energies. *IEEE Trans. Automat. Contr.*, 40 (5) :949–953, 1995.

[17] I. D. Landau. Unbiased recursive identification using model reference adaptive techniques. *IEEE Trans. Automat. Contr.*, AC-21(2) :194–202, April 1976.

[18] L. Ljung. *System identification - Theory for the User*. Prentice-Hall, Upper Saddle River, N.J., 2nd edition, 1999.

[19] L. Ljung. Initialisation aspects for subspace and output-error identification methods. In *European Control Conference*, Cambridge, U.K., 2003.

[20] L. Ljung. Experiments with identification of continuous time models. In *15th IFAC Symposium on System Identification*, pages 1175–1180, Saint-Malo, France, 2009.

[21] L. Ljung and T. Söderström. *Theory and practice of recursive identification*. MIT Press, London, England, 1983.

[22] C. Lyzell, M. Enqvist, and L. Ljung. Handling certain structure information in subspace identification. In *15th IFAC Symposium on System Identification*, pages 90–95, Saint-Malo, France, 2009.

[23] M. Mensler. *Analyse et étude comparative de méthodes d'identification des systèmes à représentation continue. Développement d'une boîte à outils logicielle*. PhD thesis, Université Henri Poincaré, Nancy 1, France, 1999.

[24] G. Mercère, R. Ouvrard, M. Gilson, and H. Garnier. Subspace based methods for continuous-time model identification of mimo systems from filtered sampled data. In *European Control Conference*, Kos, Greece, 2007.

[25] R. Ouvrard, E. Tohme, T. Poinot, and A. Abche. Model based on the reinitialised partial moments for initialising output-error identification methods. *IET Control Theory and Applications*, 4-9 :1725–1738, September 2010.

[26] R. Ouvrard and J. C. Trigeassou. On embedded fir filter models for identifying continuous-time and discrete-time transfer functions : the rpm approach. *Int. J. Control*, 84(3) :616–632, March 2011.

[27] S. Puri and H. Takeda. Minimal realization of a symmetric transfer-function matrix using moments. *IEEE Trans. Automat. Contr.*, 18-3 :305–306, June 1973.

[28] G. P. Rao and H. Garnier. Numerical illustrations of the relevance of direct continuous-time model identification. In *In Proceedings of 15th IFAC world congress*, Barcelona, Spain, 2002.

[29] G. P. Rao and H. Garnier. Identification of continuous-time systems : direct or indirect ? *Systems Science*, 30-3 :25–50, 2004.

[30] K. Steiglitz and L. McBride. A technique for the identification of linear systems. *IEEE Trans. Automat. Contr.*, 10-4 :461–464, 1965.

[31] E. Tohme. *Initialization of Output Error Identification Algorithms*. PhD thesis, Université de Poitiers, France, http://laii.univ-poitiers.fr/THESES/elie_tohme.pdf, 2008.

[32] E. Tohme, R. Ouvrard, A. Abche, J. C. Trigeassou, T. Poinot, and G. Mercere. A methodology to enhance the convergence of output error identification algorithms. In *European Control Conference*, Kos, Greece, 2007.

[33] E. Tohme, R. Ouvrard, T. Poinot, J. C. Trigeassou, and A. Abche. Initialization of output-error identification methods - comparison between arx and rpm models. In *15th IFAC Symposium on System Identification*, pages 302–307, Saint-Malo, France, 2009.

[34] E. Tohme, R. Ouvrard, J. C. Trigeassou, and A. Abche. A pseudo-output error algorithm to improve global convergence. In *The 32nd Annual Conference of the IEEE Industrial Electronics Society*, pages 659–664, Paris, France, 2006.

[35] J. C. Trigeassou. *Identification et commande de processus mono-entrée mono-sortie par la méthode des moments - Expérimentation sur calculatrice programmable*. PhD thesis, Université de Nantes, France, 1980.

[36] J. C. Trigeassou. *Contribution à l'extension de la méthode des moments en automatique. Application à l'identification des systèmes linéaires*. Thèse d'etat, Université de Poitiers, France, 1987.

[37] P. Young. In flight dynamic checkout : a discussion. *IEEE Transactions on Aerospace*, AS2 :1106–1111, 1964.

[38] P. Young. An instrumental variable for real-time identification of a noisy process. *Automatica*, 6 :271–287, 1970.

[39] P. Young. The refined instrumental variable method : unified estimation of discrete and continuous-time transfer function models. *APII-Journal Européen des Systèmes Automatisés*, 42 :149–179, 2008.

[40] Y. Zou and W. P. Heath. Conditions for attaining the global minimum in maximum likelihood system identification. In *15th IFAC Symposium on System Identification*, pages 1110–1115, Saint-Malo, France, 2009.