Matlab routines: lsctrpm, ivctrpm

Régis Ouvrard[a], regis.ouvrard@univ-poitiers.fr

[a]Université de Poitiers, 2 rue Pierre Brousse, 86022 Poitiers Cedex, France

17 novembre 2010

Version 2

The Matlab routine lsctrpm computes the LS-estimates of continuous-time (CT) multi input multi output (MIMO) reinitialized partial moment (RPM) models. The Matlab routine ivctrpm computes the LS-estimates with instrumental variable (IV) of CT MIMO RPM models. This report defines the CT RPM model and describes the implementation.

1 Continuous-Time RPM model

Consider an n_a-th order system defined by the transfer function

$$G(s) = \frac{b_0 + b_1 s + \ldots + b_{n_b} s^{n_b}}{a_0 + a_1 s + \ldots + a_{n_a-1} s^{n_a-1} + s^{n_a}}, \quad n_a > n_b \tag{1}$$

The true response of this system can be modeled, from the input-output measurements $\{u(t), y(t)\}$ with $t = kt_s$, $k = 0, \ldots, N$ and t_s, the sampling period, by the CT RPM model defined by

$$\widehat{y}(t) = \sum_{j=0}^{n_b} \widehat{b}_j \beta_j^u(t) + \sum_{i=0}^{n_a-1} \widehat{a}_i \alpha_i^y(t) + \gamma^y(t) \tag{2}$$

where

$$\begin{aligned}
\beta_0^u(t) &= m(t) * u(t) \\
\alpha_0^y(t) &= -m(t) * y(t) \\
\beta_j^u(t) &= \tfrac{d^j m(t)}{dt^j} * u(t) \quad \text{for } 1 \leq j \leq n_b \\
\alpha_i^y(t) &= -\tfrac{d^i m(t)}{dt^i} * y(t) \quad \text{for } 1 \leq i < n_a \\
\gamma^y(t) &= \left(\delta(t) - \tfrac{d^{n_a} m(t)}{dt^{n_a}}\right) * y(t) \\
m(t) &= \tfrac{(\widehat{T}-t)^{n_a} t^{n_a-1}}{(n_a-1)!\widehat{T}^{n_a}} \quad \text{with} \quad t \in \left[0, \widehat{T}\right]
\end{aligned} \tag{3}$$

$m(t)$ is a FIR filter called CT RPM filter and $\widehat{T}$ is the design parameter called reinitialization interval.

The extension to the MIMO case with n_y outputs and n_u inputs is straightforward by considering n_y MISO (multi input single output) models.

2 Parameter estimation

The CT RPM model (2) can be rewritten in a linear regression form

$$\widehat{y}(t) = \varphi^T(t)\widehat{\theta}^{RPM} + \gamma^y(t) \tag{4}$$

where

$$\begin{aligned} \varphi(t) &= \left[\alpha_0^y(t), \ldots, \alpha_{n_a-1}^y(t), \beta_0^u(t), \ldots, \beta_{n_b}^u(t)\right]^T \\ \widehat{\theta}^{RPM} &= \left[\widehat{a}_0, \ldots, \widehat{a}_{n_a-1}, \widehat{b}_0, \ldots, \widehat{b}_{n_b}\right]^T \end{aligned} \tag{5}$$

The LS-estimate of $\widehat{\theta}^{RPM}$ is given by

$$\widehat{\theta}^{RPM} = \left[\sum_{k=\widehat{K}}^{N} \varphi(kt_s)\varphi^T(kt_s)\right]^{-1} \sum_{k=\widehat{K}}^{N} \varphi(kt_s)\left(y(kt_s) - \gamma^y(kt_s)\right) \tag{6}$$

where $\widehat{K}$ corresponds to $\widehat{T} = \widehat{K}t_s$.

3 Implementation

The Matlab routine lsctrpm implements (6) in MIMO case. This implementation is described in this subsection.

By referring to the CT RPM output model (2), $\alpha_i^y(t)$, $\beta_i^u(t)$ and $\gamma^y(t)$ are computed by performing the convolution products between $m(t)$ or its derivatives and the input-output signals. In practice, the following expressions are implemented

$$\begin{aligned} \alpha_i^y(t) &= -\int_0^{\widehat{T}} f_i(\mu)y(t - \widehat{T} + \mu)d\mu \\ \beta_i^u(t) &= \int_0^{\widehat{T}} f_i(\mu)u(t - \widehat{T} + \mu)d\mu \\ \gamma^y(t) &= -\int_0^{\widehat{T}} f_{n_a}(\mu)y(t - \widehat{T} + \mu)d\mu \end{aligned} \tag{7}$$

where

$$\begin{aligned} f_0(\mu) &= \frac{\mu^{n_a}(\widehat{T}-\mu)^{n_a-1}}{(n_a-1)!\widehat{T}^{n_a}} \\ f_i(\mu) &= -\frac{df_{i-1}(\mu)}{d\mu} \end{aligned} \tag{8}$$

The following recursive form allows the computation of $f_i(\mu)$ for $i = 0, \ldots, n_a - 1$

$$f_i(\mu) = \frac{(-1)^i}{(n_a-1)!\widehat{T}^{n_a}} \sum_{j=0}^{i} (-1)^j \frac{i!}{j!(i-j)!} \frac{(n_a-1)!n_a!}{(n_a-j-1)!(n_a-i+j)!} \mu^{n_a-i+j}(\widehat{T} - \mu)^{n_a-j-1} \tag{9}$$

The integrations in (7) are computed using the Simpson's rule, *e.g.* for $\alpha_i^y(t)$

$$\begin{aligned} \alpha_i^y(t) \quad = \quad & -\frac{t_s}{3} \sum_{k=2}^{\widehat{K}} \left[f_i((k-2)t_s)\,y(t - (\widehat{K} - l + 2)t_s) + \right. \\ & \left. 4f_i((k-1)t_s)y(t - (\widehat{K} - l + 1)t_s) + f_i(kt_s)y(t - (\widehat{K} - l)t_s)\right] \end{aligned} \tag{10}$$

where k and $\widehat{K}$ are even.

The function $\beta_i^u(t)$ can be computed in a similar way to the expression given in (10). However, if $u(t)$ is a piecewise constant input, *e.g.* the input is generated by a digital to analog converter, the rectangle method can be implemented to compute the integration. Consequently, the following expressions is obtained

$$\beta_i^u(t) = \sum_{k=0}^{\widehat{K}-1} F_i^{rect}(kt_s) u(t - (\widehat{K} - k)t_s) \tag{11}$$

where the function $F_i^{rect}(kt_s)$ is given by

$$\begin{aligned} F_i^{rect}(kt_s) &= \tfrac{(-1)^i}{(n_a-1)!(\widehat{K}t_s)^{n_a}} \sum_{j=0}^{i} (-1)^j \tfrac{i!}{j!(i-j)!} \tfrac{(n_a-1)!n_a!}{(n_a-j-1)!(n_a-i+j)!} \\ & \sum_{r=0}^{n_a-j-1} (-1)^r \tfrac{(n_a-j-1)!}{r!(n_a-j-1-r)!} (\widehat{K}t_s)^{n_a-j-1-r} \left\{ \tfrac{((k+1)t_s)^{n_a-i+j+r+1} - (kt_s)^{n_a-i+j+r+1}}{n_a-i+j+r+1} \right\} \end{aligned} \tag{12}$$

4 Instrumental variable implementation

The instrumental variable iterative scheme can be used to remove the bias.

Consider the instrument built from an auxiliary model as follows

$$\Xi(s) = \frac{\widehat{b}_0 + b_1 s + \ldots + \widehat{b}_{n_b} s^{n_b}}{\widehat{a}_0 + \widehat{a}_1 s + \ldots + \widehat{a}_{n_a-1} s^{n_a-1} + s^{n_a}} U(s) \tag{13}$$

where $\Xi(s)$ and $U(s)$ are the Laplace transforms of time signals $\xi(t)$ and $u(t)$, respectively.

Hence, the IV regressor is built

$$\zeta(t) = \left[\alpha_0^\xi(t), \ldots, \alpha_{n_a-1}^\xi(t), \beta_0^u(t), \ldots, \beta_{n_b}^u(t) \right]^T \tag{14}$$

The IV-estimate is given by

$$\widehat{\theta}^{IV} = \left[\sum_{k=\widehat{K}}^{N} \zeta(kt_s) \varphi^T(kt_s) \right]^{-1} \sum_{k=\widehat{K}}^{N} \zeta(kt_s) \left(y(kt_s) - \gamma^y(kt_s) \right) \tag{15}$$

A few iterations of the IV-estimate must be performed to remove the bias.

5 Choice of the design parameter

The CT RPM model requires the selection of a design parameter, the reinitialization interval, $\widehat{T} = \widehat{K}t_s$.

Experiments show that the quality of the RPM model is not very sensitive to this choice. The selection of $\widehat{K}$ is not more difficult than the other design parameters of CT system identification methods.

The design parameter $\widehat{K}$ allows the adaptation of the RPM model to the nature of the noise :

- If the perturbation is a white output-error noise, an optimal reinitialization interval exists, namely $\widehat{K}_{wn}$, for which the variance of the error is minimal and the bias is highly reduced. Many experiments led to the following conclusion : the parameter $\widehat{K}$ should be selected such that the interval $[0, \widehat{K}t_s]$ is equivalent to the double of the main time constant for an aperiodic system or the double of the (zero to 90%) rising time for an oscillating system.

– If the perturbation is a coloured noise, the optimal reinitialization interval is on the interval $]n_a, \widehat{K}_{wn}[$.

In practice, the value of $\widehat{K}$ is selected as follows : $\widehat{K}$ is increased and a standard test, such as the quadratic criterion or the autocorrelation of the residuals, is evaluated to find the best $\widehat{K}$.