Below, we find the proofs of Theorem 1, Theorem 2 and Theorem 3 given on the paper : H_∞ performance analysis of 2D continuous time varying delay systems.

Appendix 1: Proof of Theorem 1

To prove the asymptotic stability, we use similar arguments as [45] from which we borrow equality $(A.2)$ and using condition (7) we get

$$\int_{t_1+t_2=t}\left(\frac{\partial V^h(x^h_{t_1}(\cdot,t_2))}{\partial t_1}+\frac{\partial V^v(x^v_{t_2}(t_1,\cdot))}{\partial t_2}\right)\mathrm{d}s \quad = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_1+t_2=t}\Big(V^h(x^h_{t_1}(\cdot,t_2))+V^v(x^v_{t_2}(t_1,\cdot))\Big)\mathrm{d}s\right] - \sqrt{2}\Big(V^h(x^h_0(\cdot,t))+V^v(x^v_0(t,\cdot))\Big)<0$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_1+t_2=t}\Big(V^h(x^h_{t_1}(\cdot,t_2))+V^v(x^v_{t_2}(t_1,\cdot))\Big)\mathrm{d}s\right] < \sqrt{2}\Big(V^h(x^h_0(\cdot,t))+V^v(x^v_0(t,\cdot))\Big)$$

and integrating both sides we get

$$\begin{aligned}&\int_{t_1+t_2=t}\Big(V^h(x^h_{t_1}(\cdot,t_2))+V^v(x^v_{t_2}(t_1,\cdot))\Big)\mathrm{d}s\\ &<\sqrt{2}\int_0^t\Big(V^h(x^h_0(\cdot,\tau))+V^v(x^v_0(\tau,\cdot))\Big)\mathrm{d}\tau\\ &<\sqrt{2}c_2\int_0^T\Big(\|x^h_0(\cdot,\tau)\|^2_{cl}+\|x^v_0(\tau,\cdot)\|^2_{cl}\Big)\mathrm{d}\tau\\ &<\infty\end{aligned} \tag{17}$$

with $T=\max\{T_1,T_2\}$ and $\forall t\geq T$. T_1 and T_2 were introduced in the initial boundary conditions (2)-(3). With conditions (6a)-(6b) in mind, inequality (17) above implies that

$$\int_{t_1+t_2=t}\Big(\left\|x^h(t_1,t_2)\right\|^2+\|x^v(t_1,t_2)\|^2\Big)\,\mathrm{d}s<\infty$$

which implies necessarily that

$$\lim_{(t_1+t_2)=t\to\infty}\Big(\left\|x^h(t_1,t_2)\right\|^2+\|x^v(t_1,t_2)\|^2\Big)=0.$$

Appendix 2: Proof of Theorem 2

Lemma 1 *For symmetric matrices X_0, X_1, X_2, X_3, X_4 and a vector ξ_t, let $f(\alpha_1,\alpha_2) = \xi_t^T X_0 \xi_t + \alpha_1 \xi_t^T X_1 \xi_t + \alpha_2 \xi_t^T X_2 \xi_t + \alpha_1^2 \xi_t^T X_3 \xi_t + \alpha_2^2 \xi_t^T X_4 \xi_t$ with $X_3 \geq 0$ and $X_4 \geq 0$.*

$$\begin{cases} f(\alpha_{1min},\alpha_{2min}) \prec 0 \\ f(\alpha_{1max},\alpha_{2max}) \prec 0 \\ f(\alpha_{1max},\alpha_{2min}) \prec 0 \\ f(\alpha_{1min},\alpha_{2max}) \prec 0 \end{cases} \Rightarrow f(\alpha_1,\alpha_2) \prec 0 \quad \forall \alpha_1 \in [\alpha_{1min},\alpha_{1max}] \text{ and } \alpha_2 \in [\alpha_{2min},\alpha_{2max}] \tag{18}$$

Proof

$$\begin{cases} f(\alpha_1,\alpha_{2min}) \prec 0 \\ f(\alpha_1,\alpha_{2min}) \prec 0 \end{cases} \Rightarrow f(\alpha_1,\alpha_{2min}) \prec 0 \quad \forall \alpha_1 \in [\alpha_{1min},\alpha_{1max}] \tag{19}$$

For a fixed α_{2min}, the function $f(\alpha_1,\alpha_{2min})$ is a convex quadratic function on the variable α_1 since $\frac{d^2}{d\alpha^2} f(\alpha_1,\alpha_{2min}) = 2\xi_t^T X_3 \xi_t \geq 0$. Similar for α_{2max}, we have the following expressions

$$\begin{cases} f(\alpha_1,\alpha_{2max}) \prec 0 \\ f(\alpha_1,\alpha_{2max}) \prec 0 \end{cases} \Rightarrow f(\alpha_1,\alpha_{2max}) \prec 0 \quad \forall \alpha_1 \in [\alpha_{1min},\alpha_{1max}] \tag{20}$$

Thus, for a fixed $\alpha_1 \in [\alpha_{1min},\alpha_{1max}]$, we have

$$\begin{cases} f(\alpha_1,\alpha_{2min}) \prec 0 \\ f(\alpha_1,\alpha_{2max}) \prec 0 \end{cases} \Rightarrow f(\alpha_1,\alpha_2) \prec 0 \quad \forall \alpha_1 \in [\alpha_{1min},\alpha_{1max}] \quad \text{and} \quad \alpha_2 \in [\alpha_{2min},\alpha_{2max}] \tag{21}$$

This completes the proof.

Lemma 2 ([2]) *Let $W > 0$ and $g(s)$ be appropriate dimensional symmetric matrix and vector, respectively. Then, we have the following for all scalar valued function $\phi(s) \geq 0$, $\forall s \in [t_1, t_2]$*

(i)

$$-\int_{t_1}^{t_2} \phi(s) g^T(s) W g(s) ds \leq \gamma_1 \xi_t^T F_1^T W^{-1} F_1 \xi_t + 2\xi_t^T F_1^T \phi(s) g(s) ds,$$

(ii)

$$-\int_{t_1}^{t_2} \phi^2(s) g^T(s) W g(s) ds \leq \gamma_2 \xi_t^T F_2^T W^{-1} F_2 \xi_t + 2\xi_t^T F_2^T \phi(s) g(s) ds,$$

where matrices F_1, F_2 and a vector ξ_t (independent on the integral variable) are appropriate dimensional arbitary ones, and $\gamma_1 = \int_{t_1}^{t_2} \phi(s)ds$, $\gamma_2 = t_2 - t_1$.

Let us introduce some useful notations for the proof of Theorem 2.

$$\xi_t^h = \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1-\tau_1(t_1),t_2) \\ x^h(t_1-h_1,t_2) \\ \int_{t_1-h_1}^{t_1-\tau_1(t_1)} x^h(s,t_2)ds \\ \int_{t_1-\tau_1(t_1)}^{t_1} x^h(s,t_2)ds \end{bmatrix}, \xi_t^v = \begin{bmatrix} x^v(t_1,t_2) \\ x^v(t_1,t_2-\tau_2(t_2)) \\ x^v(t_1,t_2-h_2) \\ \int_{t_2-h_2}^{t_2-\tau_2(t_2)} x^v(t_1,k)dk \\ \int_{t_2-\tau_2(t_2)}^{t_2} x^v(t_1,k)dk \end{bmatrix}$$

Then, for $\omega(t_1,t_2)=0$, the system (8) can be rewritten as follows

$$\begin{aligned} \dot{x}^h(t_1,t_2) &= A_h\xi_t^h + A_{hv}\xi_t^v \\ \dot{x}^v(t_1,t_2) &= A_v\xi_t^v + A_{vh}\xi_t^h \end{aligned}$$

Consider the vector function given by (4) with $V^h(x^h)$ and $V^v(x^v)$ are given by the following equations :

$$\begin{aligned} V^h(x_{t_1}^h(t_1,t_2)) = {} & V_1^h(x_{t_1}^h(t_1,t_2)) + V_2^h(x_{t_1}^h(t_1,t_2)) + V_3^h(x_{t_1}^h(t_1,t_2)) \\ & + V_4^h(x_{t_1}^h(t_1,t_2)) + V_5^h(x_{t_1}^h(t_1,t_2)) + V_6^h(x_{t_1}^h(t_1,t_2)) \end{aligned} \tag{22}$$

$$\begin{aligned} V^v(x_{t_2}^v(t_1,t_2)) = {} & V_1^v(x_{t_2}^v(t_1,t_2)) + V_2^v(x_{t_2}^v(t_1,t_2)) + V_3^v(x_{t_2}^v(t_1,t_2)) \\ & + V_4^v(x_{t_2}^v(t_1,t_2)) + V_5^v(x_{t_2}^v(t_1,t_2)) + V_6^v(x_{t_2}^v(t_1,t_2)) \end{aligned} \tag{23}$$

where

$$\begin{aligned} V_1^h(x_{t_1}^h(t_1,t_2)) &= \left[x^{hT}(t_1,t_2)\ \int_{t_1-h_1}^{t_1} x^{hT}(s,t_2)ds\right] P^h \begin{bmatrix} x^h(t_1,t_2) \\ \int_{t_1-h_1}^{t_1} x^h(s,t_2)ds \end{bmatrix} \\ V_2^h(x_{t_1}^h(t_1,t_2)) &= \int_{t_1-\tau_1(t_1)}^{t_1} \left[x^{hT}(t_1,t_2)\ x^{hT}(s,t_2)\right] Q_1^h \begin{bmatrix} x^h(t_1,t_2) \\ x^h(s,t_2) \end{bmatrix} ds \\ V_3^h(x_{t_1}^h(t_1,t_2)) &= \int_{t_1-h_1}^{t_1} \left[x^{hT}(t_1,t_2)\ x^{hT}(s,t_2)\right] Q_2^h \begin{bmatrix} x^h(t_1,t_2) \\ x^h(s,t_2) \end{bmatrix} ds \\ V_4^h(x_{t_1}^h(t_1,t_2)) &= \int_{t_1-h_1}^{t_1} (h_1-t_1+s) \left[x^{hT}(s,t_2)\ \dot{x}^{hT}(s,t_2)\right] Q_3^h \begin{bmatrix} x^h(s,t_2) \\ \dot{x}^h(s,t_2) \end{bmatrix} ds \\ V_5^h(x_{t_1}^h(t_1,t_2)) &= \int_{t_1-h_1}^{t_1} (h_1-t_1+s)^2 \dot{x}^{hT}(s,t_2) R_1^h \dot{x}^h(s,t_2) ds \\ V_6^h(x_{t_1}^h(t_1,t_2)) &= \int_{t_1-h_1}^{t_1} (h_1-t_1+s)^3 \dot{x}^{hT}(s,t_2) R_2^h \dot{x}^h(s,t_2) ds \end{aligned}$$

We have similar expressions of the Lyapunov functional $V^v(x_{t_2}^v(t_1,t_2))$ for the vertical direction.

The divergence of $V(x^h_{t_1}(\cdot,t_2), x^v_{t_2}(t_1,\cdot))$ *defined by* (4) *along the trajectories of the system is given by (5) with*

$$\begin{aligned}
\frac{\partial V^h(x^h_{t_1}(t_1,t_2))}{\partial t_1} &= \frac{\partial V^h_1(x^h_{t_1}(t_1,t_2))}{\partial t_1} + \frac{\partial V^h_2(x^h_{t_1}(t_1,t_2))}{\partial t_1} + \frac{\partial V3^h(x^h_{t_1}(t_1,t_2))}{\partial t_1} \\
&+ \frac{\partial V^h_4(x^h_{t_1}(t_1,t_2))}{\partial t_1} + \frac{\partial V^h_5(x^h_{t_1}(t_1,t_2))}{\partial t_1} + \frac{\partial V6^h(x^h_{t_1}(t_1,t_2))}{\partial t_1} \\
\frac{\partial V^v(x^v_{t_2}(t_1,t_2))}{\partial t_2} &= \frac{\partial V^v_1(x^v_{t_2}(t_1,t_2))}{\partial t_2} + \frac{\partial V^v_2(x^v_{t_2}(t_1,t_2))}{\partial t_2} + \frac{\partial V3^v(x^v_{t_2}(t_1,t_2))}{\partial t_2} \\
&+ \frac{\partial V^v_4(x^v_{t_2}(t_1,t_2))}{\partial t_2} + \frac{\partial V^v_5(x^v_{t_2}(t_1,t_2))}{\partial t_2} + \frac{\partial V6^v(x^v_{t_2}(t_1,t_2))}{\partial t_2}
\end{aligned}$$

Because of the similarities between the two dimensions, we will develop the calculations only in the horizontal dimension. Computing the derivative of $V^h_i(x^h_{t_1}(t_1,t_2))$, $i = 1,...6$, *we have :*

$$\begin{aligned}
\frac{\partial V^h_1(x^h_{t_1}(t_1,t_2))}{\partial t_1} &= 2\frac{d}{dt_1}\left[x^{hT}(t_1,t_2)\ \int_{t_1-h_1}^{t_1} x^{hT}(s,t_2)ds\right] P^h \begin{bmatrix} x^h(t_1,t_2) \\ \int_{t_1-h_1}^{t_1} x^h(s,t_2)ds \end{bmatrix} \\
&= 2\left[\dot{x}^{hT}(t_1,t_2)\ \frac{d}{dt_1}\int_{t_1-h_1}^{t_1} x^{hT}(s,t_2)ds\right] P^h \begin{bmatrix} x^h(t_1,t_2) \\ \int_{t_1-h_1}^{t_1} x^h(s,t_2)ds \end{bmatrix} \\
&= 2\left[\dot{x}^{hT}(t_1,t_2)\ x^{hT}(t_1,t_2) - x^{hT}(t_1-h_1,t_2)\right] P^h \begin{bmatrix} x^h(t_1,t_2) \\ \int_{t_1-h_1}^{t_1} x^h(s,t_2)ds \end{bmatrix} \\
\frac{\partial V^h_1(x^h_{t_1}(t_1,t_2))}{\partial t_1} &= 2\xi^{hT}_t \left[A^T_h\ e_{1h} - e_{3h}\right] P^h \left[e_{1h}\ e_{4h} + e_{5h}\right]^T \xi^h_t \\
&+ 2\xi^{hT}_t \left[e_{1h}\ e_{4h} + e_{5h}\right] P^h \left[A^T_{hv}\ e_{0v}\right]^T \xi^v_t
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V^h_2(x^h_{t_1}(t_1,t_2))}{\partial t_1} &= \frac{d}{dt_1}\int_{t_1-\tau_1(t_1)}^{t_1} \left[x^{hT}(t_1,t_2)\ x^{hT}(s,t_2)\right] Q^h_1 \begin{bmatrix} x^h(t_1,t_2) \\ x^h(s,t_2) \end{bmatrix} ds \\
&= \left[x^{hT}(t_1,t_2)\ x^{hT}(t_1,t_2)\right] Q^h_1 \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1,t_2) \end{bmatrix} \\
&- (1-\dot{\tau}_1(t_1))\left[x^{hT}(t_1,t_2)\ x^{hT}(t_1-\tau_1(t_1),t_2)\right] Q^h_1 \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1-\tau_1(t_1),t_2) \end{bmatrix} \\
&+ 2\int_{t_1-\tau_1(t_1)}^{t_1} \left[\dot{x}^{hT}(t_1,t_2)\ \frac{\partial}{\partial t_1}x^{hT}(s,t_2)\right] Q^h_1 \begin{bmatrix} x^h(t_1,t_2) \\ x^h(s,t_2) \end{bmatrix} ds \\
\frac{\partial V^h_2(x^h_{t_1}(t_1,t_2))}{\partial t_1} &= \xi^{hT}_t \left[e_{1h}\ e_{1h}\right] Q^h_1 \left[e_{1h}\ e_{1h}\right]^T \xi^h_t \\
&- \xi^{hT}_t(1-\dot{\tau}_1(t_1)) \left[e_{1h}\ e_{2h}\right] Q^h_1 \left[e_{1h}\ e_{2h}\right]^T \xi^h_t \\
&+ 2\xi^{hT}_t \left[A^T_h\ e_{0h}\right] Q^h_1 \left[\tau_1(t_1)e_{1h}\ e_{5h}\right]^T \xi^h_t \\
&+ 2\xi^{hT}_t \left[\tau_1(t_1)e_{1h}\ e_{5h}\right] Q^h_1 \left[A^T_{hv}\ e_{0v}\right]^T \xi^v_t
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial V_3^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= \frac{d}{dt_1}\int_{t_1-h_1}^{t_1} \left[x^{hT}(t_1,t_2)\ x^{hT}(s,t_2)\right] Q_2^h \begin{bmatrix} x^h(t_1,t_2) \\ x^h(s,t_2) \end{bmatrix} ds \\
&= \left[x^{hT}(t_1,t_2)\ x^{hT}(t_1,t_2)\right] Q_2^h \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1,t_2) \end{bmatrix} \\
&- \left[x^{hT}(t_1,t_2)\ x^{hT}(t_1-h_1,t_2)\right] Q_2^h \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1-h_1,t_2) \end{bmatrix} \\
&+ 2\int_{t_1-h_1}^{t_1} \left[\dot{x}^{hT}(t_1,t_2)\ \frac{\partial}{\partial t_1}x^{hT}(s,t_2)\right] Q_2^h \begin{bmatrix} x^h(t_1,t_2) \\ x^h(s,t_2) \end{bmatrix} ds \\
\frac{\partial V_3^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= \xi_t^{hT}\left[e_{1h}\ e_{1h}\right] Q_2^h \left[e_{1h}\ e_{1h}\right]^T \xi_t^h \\
&- \xi_t^{hT}\left[e_{1h}\ e_{3h}\right] Q_2^h \left[e_{1h}\ e_{3h}\right]^T \xi_t^h \\
&+ 2\xi_t^{hT}\left[A_h^T\ e_{0h}^T\right] Q_2^h \left[h_1 e_{1h}\ e_{4h}+e_{5h}\right]^T \xi_t^h \\
&+ 2\xi_t^{hT}\left[h_1 e_{1h}\ e_{4h}+e_{5h}\right] Q_2^h \left[A_{hv}^T\ e_{0v}\right]^T \xi_t^v
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_4^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= \frac{d}{dt_1}\int_{t_1-h_1}^{t_1} (h_1-t_1+s)\left[x^{hT}(s,t_2)\ \dot{x}^{hT}(s,t_2)\right] Q_3^h \begin{bmatrix} x^h(t_1,t_2) \\ \dot{x}^h(s,t_2) \end{bmatrix} ds \\
&= h_1\left[x^{hT}(t_1,t_2)\ \dot{x}^{hT}(t_1,t_2)\right] Q_3^h \begin{bmatrix} x^h(t_1,t_2) \\ \dot{x}^h(t_1,t_2) \end{bmatrix} \\
&+ \int_{t_1-h_1}^{t_1} \frac{\partial}{\partial t_1}\left\{(h_1-t_1+s)\left[x^{hT}(s,t_2)\ \dot{x}^{hT}(s,t_2)\right] Q_3^h \begin{bmatrix} x^h(s,t_2) \\ \dot{x}^h(s,t_2) \end{bmatrix}\right\} ds \\
\frac{\partial V_4^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= h_1\xi_t^{hT}\left[e_{1h}\ A_h^T\right] Q_3^h \left[e_{1h}\ A_h^T\right]^T \xi_t^h \\
&+ h_1\xi_t^{vT}\left[e_{0v}\ A_{hv}^T\right] Q_3^h \left[e_{0v}\ A_{hv}^T\right]^T \xi_t^v \\
&+ 2h_1\xi_t^{hT}\left[e_{1h}\ A_h^T\right] Q_3^h \left[e_{0v}\ A_{hv}^T\right]^T \xi_t^v \\
&- \int_{t_1-h_1}^{t_1} \zeta^{hT}\left[J_1\ J_2\right] Q_3^h \left[J_1\ J_2\right]^T \zeta^h ds
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_5^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= \frac{d}{dt_1}\int_{t_1-h_1}^{t_1}(h_1-t_1+s)^2\dot{x}^{hT}(s,t_2)R_1^h\dot{x}^h(s,t_2)ds \\
&= h_1^2\dot{x}^{hT}(t_1,t_2)R_1^h\dot{x}^h(t_1,t_2) \\
&- 2\int_{t_1-h_1}^{t_1}(h_1-t_1+s)\dot{x}^{hT}(s,t_2)R_1^h\dot{x}^h(s,t_2)ds \\
\frac{\partial V_5^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= h_1^2\xi_t^{hT}A_h^TR_1^hA_h\xi_t^h + h_1^2\xi_t^{vT}A_{hv}^TR_1^hA_{hv}\xi_t^v \\
&+ 2h_1^2\xi_t^{hT}A_h^TR_1^hA_{hv}\xi_t^v \\
&- 2\int_{t_1-h_1}^{t_1}(h_1-t_1+s)\dot{x}^{hT}(s,t_2)R_1^h\dot{x}^h(s,t_2)ds
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial V_6^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= \frac{d}{dt_1}\int_{t_1-h_1}^{t_1}(h_1-t_1+s)^3\dot{x}^{hT}(s,t_2)R_2^h\dot{x}^h(s,t_2)ds \\
&= h_1^3\dot{x}^{hT}(t_1,t_2)R_2^h\dot{x}^h(t_1,t_2) \\
&- 3\int_{t_1-h_1}^{t_1}(h_1-t_1+s)^2\dot{x}^{hT}(s,t_2)R_2^h\dot{x}^h(s,t_2)ds \\
\frac{\partial V_6^h(x_{t_1}^h(t_1,t_2))}{\partial t_1} &= h_1^3\xi_t^{hT}A_h^TR_2^hA_h\xi_t^h + h_1^3\xi_t^{vT}A_{hv}^TR_2^hA_{hv}\xi_t^v \\
&+ 2h_1^3\xi_t^{hT}A_h^TR_2^hA_{hv}\xi_t^v \\
&- 3\int_{t_1-h_1}^{t_1}(h_1-t_1+s)^2\dot{x}^{hT}(s,t_2)R_2^h\dot{x}^h(s,t_2)ds
\end{aligned}
$$

$$
\begin{aligned}
div(V(x^h_{t_1}(t_1,t_2),x^v_{t_2}(t_1,t_2))) = {} & 2\xi_t^{hT}\left[A_h^T\ e_{1h}-e_{3h}\right]P^h\left[e_{1h}\ e_{4h}+e_{5h}\right]^T\xi_t^h\\
&+2\xi_t^{hT}\left[e_{1h}\ e_{4h}+e_{5h}\right]P^h\left[A_{hv}^T\ e_{0v}\right]^T\xi_t^v\\
&+\xi_t^{hT}\left[e_{1h}\ e_{1h}\right]Q_1^h\left[e_{1h}\ e_{1h}\right]^T\xi_t^h\\
&-\xi_t^{hT}(1-\dot\tau_1(t_1))\left[e_{1h}\ e_{2h}\right]Q_1^h\left[e_{1h}\ e_{2h}\right]^T\xi_t^h\\
&+2\xi_t^{hT}\left[A_h^T\ e_{0h}\right]Q_1^h\left[\tau_1(t_1)e_{1h}\ e_{5h}\right]^T\xi_t^h\\
&+2\xi_t^{hT}\left[\tau_1(t_1)e_{1h}\ e_{5h}\right]Q_1^h\left[A_{hv}^T\ e_{0v}\right]^T\xi_t^v\\
&+\xi_t^{hT}\left[e_{1h}\ e_{1h}\right]Q_2^h\left[e_{1h}\ e_{1h}\right]^T\xi_t^h\\
&-\xi_t^{hT}\left[e_{1h}\ e_{3h}\right]Q_2^h\left[e_{1h}\ e_{3h}\right]^T\xi_t^h\\
&+2\xi_t^{hT}\left[h_1A_h^T\ e_{0h}\right]Q_2^h\left[e_{1h}\ e_{4h}+e_{5h}\right]^T\xi_t^h\\
&+2\xi_t^{hT}\left[h_1e_{1h}\ e_{4h}+e_{5h}\right]Q_2^h\left[A_{hv}^T\ e_{0v}\right]^T\xi_t^v\\
&+h_1\xi_t^{hT}\left[e_{1h}\ A_h^T\right]Q_3^h\left[e_{1h}\ A_h^T\right]^T\xi_t^h\\
&+h_1\xi_t^{vT}\left[e_{0v}\ A_{hv}^T\right]Q_3^h\left[e_{0v}\ A_{hv}^T\right]^T\xi_t^v\\
&+2h_1\xi_t^{hT}\left[e_{1h}\ A_h^T\right]Q_3^h\left[e_{0v}\ A_{hv}^T\right]^T\xi_t^v\\
&+h_1^2\xi_t^{hT}A_h^TR_1^hA_h\xi_t^h+h_1^2\xi_t^{vT}A_{hv}^TR_1^hA_{hv}\xi_t^v\\
&+2h_1^2\xi_t^{hT}A_h^TR_1^hA_{hv}\xi_t^v\\
&+h_1^3\xi_t^{hT}A_h^TR_2^hA_h\xi_t^h+h_1^3\xi_t^{vT}A_{hv}^TR_2^hA_{hv}\xi_t^v\\
&+2h_1^3\xi_t^{hT}A_h^TR_2^hA_{hv}\xi_t^v\\
&+2\xi_t^{vT}\left[A_v^T\ e_{1v}-e_{3v}\right]P^v\left[e_{1v}\ e_{4v}+e_{5v}\right]^T\xi_t^v\\
&+2\xi_t^{hT}\left[A_{vh}^T\ e_{0h}\right]P^v\left[e_{1v}\ e_{4v}+e_{5v}\right]^T\xi_t^v\\
&+\xi_t^{vT}\left[e_{1v}\ e_{1v}\right]Q_1^v\left[e_{1v}\ e_{1v}\right]^T\xi_t^v\\
&-\xi_t^{vT}(1-\dot\tau_2(t_2))\left[e_{1v}\ e_{2v}\right]Q_1^v\left[e_{1v}\ e_{2v}\right]^T\xi_t^v\\
&+2\xi_t^{vT}\left[A_v^T\ e_{0v}\right]Q_1^v\left[\tau_2(t_2)e_{1v}\ e_{5v}\right]^T\xi_t^v\\
&+2\xi_t^{hT}\left[A_{vh}^T\ e_{0h}\right]Q_1^v\left[\tau_2(t_2)e_{1v}\ e_{5v}\right]^T\xi_t^v\\
&+\xi_t^{vT}\left[e_{1v}\ e_{1v}\right]Q_2^v\left[e_{1v}\ e_{1v}\right]^T\xi_t^v\\
&-\xi_t^{vT}\left[e_{1v}^T\ e_{3v}^T\right]Q_2^v\left[e_{1v}\ e_{3v}\right]\xi_t^v\\
&+2\xi_t^{vT}\left[h_2A_v^T\ e_{0v}\right]Q_2^v\left[e_{1v}\ e_{4v}+e_{5v}\right]^T\xi_t^v\\
&+2\xi_t^{hT}\left[h_2A_{vh}^T\ e_{0h}\right]Q_2^v\left[e_{1v}\ e_{4v}+e_{5v}\right]^T\xi_t^v\\
&+h_2\xi_t^{vT}\left[e_{1v}\ A_v^T\right]Q_3^v\left[e_{1v}\ A_v^T\right]^T\xi_t^v\\
&+h_2\xi_t^{hT}\left[e_{0h}\ A_{vh}^T\right]Q_3^v\left[e_{0h}\ A_{vh}^T\right]^T\xi_t^h\\
&+2h_2\xi_t^{hT}\left[e_{0h}\ A_{vh}^T\right]Q_3^v\left[e_{1v}\ A_v^T\right]^T\xi_t^v+h_2^2\xi_t^{vT}A_v^TR_1^vA_v\xi_t^v\\
&+h_2^2\xi_t^{hT}A_{vh}^TR_1^vA_{vh}\xi_t^h\\
&+2h_2^2\xi_t^{hT}A_{vh}^TR_1^vA_v\xi_t^v\\
&+h_2^3\xi_t^{vT}A_v^TR_2^vA_v\xi_t^v+h_2^3\xi_t^{hT}A_{vh}^TR_2^hA_{vh}\xi_t^h\\
&+2h_2^3\xi_t^{hT}A_{vh}^TR_2^vA_v\xi_t^v+V_a(x^h_{t_1}(t_1,t_2),x^v_{t_2}(t_1,t_2))
\end{aligned}
\tag{24}
$$

where $V_a(x^h_{t_1}(t_1,t_2), x^v_{t_2}(t_1,t_2))$ *represent the sum of integral terms expressed by the following expression:*

$$\begin{aligned}
V_a(x^h_{t_1}(t_1,t_2), x^v_{t_2}(t_1,t_2)) = &-\int_{t_1-h_1}^{t_1} \zeta^{hT} \begin{bmatrix} J_1^T & J_2^T \end{bmatrix} Q_3^h \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \zeta^h ds \\
&- 2\int_{t_1-h_1}^{t_1} (h_1 - t_1 + s)\dot{x}^{hT}(s,t_2) R_1^h \dot{x}^h(s,t_2) ds \\
&- 3\int_{t_1-h_1}^{t_1} (h_1 - t_1 + s)^2\dot{x}^{hT}(s,t_2) R_2^h \dot{x}^h(s,t_2) ds \\
&-\int_{t_2-h_2}^{t_2} \zeta^{vT} \begin{bmatrix} J_1^T & J_2^T \end{bmatrix} Q_3^v \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \zeta^v dk \\
&- 2\int_{t_2-h_2}^{t_2} (h_2 - t_2 + k)\dot{x}^{vT}(t_1,k) R_1^v \dot{x}^v(t_1,k) dk \\
&- 3\int_{t_2-h_2}^{t_2} (h_2 - t_2 + k)^2\dot{x}^{vT}(t_1,k) R_2^v \dot{x}^v(t_1,k) dk
\end{aligned}$$

where $\zeta^T(s,t_2) = \begin{bmatrix} x^T(s,t_2) & \dot{x}(s,t_2) \end{bmatrix}$ *and* $\zeta^T(t_1,k) = \begin{bmatrix} x^T(t_1,k) & \dot{x}(t_1,k) \end{bmatrix}$.

$$\begin{aligned}
V_a(x^h_{t_1}(t_1,t_2), x^v_{t_2}(t_1,t_2)) \leq &-\int_{t_1-h_1}^{t_1-\tau_1(t_1)} \Big\{ \zeta^{hT} \begin{bmatrix} J_1^T & J_2^T \end{bmatrix} Q_3^h \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \zeta^h ds \\
&+2(h_1 - t_1 + s)\dot{x}^{hT}(s,t_2) R_1^h \dot{x}^h(s,t_2) ds \\
&+3(h_1 - t_1 + s)^2\dot{x}^{hT}(s,t_2) R_2^h \dot{x}^h(s,t_2) ds \Big\} \\
&-\int_{t_1-\tau_1(t_1)}^{t_1} \Big\{ \zeta^{hT} \begin{bmatrix} J_1^T & J_2^T \end{bmatrix} Q_3^h \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \zeta^h ds \\
&+2(\tau_1(t_1) - t_1 + s)\dot{x}^{hT}(s,t_2) R_1^h \dot{x}^h(s,t_2) ds \\
&+3(\tau_1(t_1) - t_1 + s)^2\dot{x}^{hT}(s,t_2) R_2^h \dot{x}^h(s,t_2) ds \Big\} \\
&-\int_{t_2-h_2}^{t_2-\tau_2(t_2)} \Big\{ \zeta^{vT} \begin{bmatrix} J_1^T & J_2^T \end{bmatrix} Q_3^v \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \zeta^v dk \\
&+2(h_2 - t_2 + k)\dot{x}^{vT}(t_1,k) R_1^v \dot{x}^v(t_1,k) dk \\
&+3(h_2 - t_2 + k)^2\dot{x}^{vT}(t_1,k) R_2^v \dot{x}^v(t_1,k) dk \Big\} \\
&-\int_{t_2-\tau_2(t_2)}^{t_2} \Big\{ \zeta^{vT} \begin{bmatrix} J_1^T & J_2^T \end{bmatrix} Q_3^v \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \zeta^v dk \\
&+2(\tau_2(t_2) - t_2 + k)\dot{x}^{vT}(t_1,k) R_1^v \dot{x}^v(t_1,k) dk \\
&+3(\tau_2(t_2) - t_2 + k)^2\dot{x}^{vT}(t_1,k) R_2^v \dot{x}^v(t_1,k) dk \Big\} \\
&= \hat{V}_a(x^h_{t_1}(t_1,t_2), x^v_{t_2}(t_1,t_2))
\end{aligned}$$

Using $_-(h_1 - t_1 + s) \leq -(\tau_1(t_1) - t_1 + s) \forall s \in [t_1 - \tau_1(t_1), t_1]$ *and* $_-(h_2 - t_2 + k) \leq -(\tau_2(t_2) - t_2 + k) \forall k \in [t_2 - \tau_2(t_2), t_2]$ *and using the property of* *a quadratic convex function and upper bounds of the integrals of quadratic multiplied by scalar functions for both horizontal and vertical directions. Thus, we apply lemma 2 and we have*

$$
\begin{aligned}
&\hat{V}_a(x^h_{t_1}(t_1,t_2), x^v_{t_2}(t_1,t_2)) \leq (h_1-\tau_1(t_1))\xi_t^{hT}F_1^{hT}(Q_3^h)^{-1}F_1^h\xi_t^h \\
&+ 2\xi_t^{hT}F_1^{hT}\left[e_{4h}\ e_{2h}-e_{3h}\right]\xi_t^h \\
&+ (h_1-\tau_1(t_1))^2\xi_t^{hT}F_2^{hT}(R_1^h)^{-1}F_2^h\xi_t^h + 4\xi_t^{hT}F_2^{hT}\left[(h_1-\tau_1(t_1))e_{2h}-e_{4h}\right]\xi_t^h \\
&+ 3(h_1-\tau_1(t_1))\xi_t^{hT}F_3^{hT}(R_2^h)^{-1}F_3^h\xi_t^h + 6\xi_t^{hT}F_3^{hT}\left[(h_1-\tau_1(t_1))e_{2h}-e_{4h}\right]\xi_t^h \\
&+ \tau_1(t_1)\xi_t^{hT}F_4^{hT}(Q_3^h)^{-1}F_4^h\xi_t^h + 2\xi_t^{hT}F_4^{hT}\left[e_{5h}\ e_{1h}-e_{2h}\right]\xi_t^h \\
&+ \tau_1(t_1)^2\xi_t^{hT}F_5^{hT}(R_1^h)^{-1}F_5^h\xi_t^h + 4\xi_t^{hT}F_5^{hT}\left[\tau_1(t_1)e_{1h}-e_{5h}\right]\xi_t^h \\
&+ 3\tau_1(t_1)\xi_t^{hT}F_6^{hT}(R_2^h)^{-1}F_6^h\xi_t^h + 6\xi_t^{hT}F_6^{hT}\left[\tau_1(t_1)e_{1h}-e_{5h}\right]\xi_t^h \\
&+ (h_2-\tau_2(t_2))\xi_t^{vT}F_1^{vT}(Q_3^v)^{-1}F_1^v\xi_t^v + 2\xi_t^{vT}F_1^{vT}\left[e_{4v}\ e_{2v}-e_{3v}\right]\xi_t^v \\
&+ (h_2-\tau_2(t_2))^2\xi_t^{vT}F_2^{vT}(R_1^v)^{-1}F_2^v\xi_t^v + 4\xi_t^{vT}F_2^{vT}\left[(h_2-\tau_2(t_2))e_{2v}-e_{4v}\right]\xi_t^v \\
&+ 3(h_2-\tau_2(t_2))\xi_t^{vT}F_3^{vT}(R_2^v)^{-1}F_3^v\xi_t^v + 6\xi_t^{vT}F_3^{vT}\left[(h_2-\tau_2(t_2))e_{2v}-e_{4v}\right]\xi_t^v \\
&+ \tau_2(t_2)\xi_t^{vT}F_4^{vT}(Q_3^v)^{-1}F_4^v\xi_t^v + 2\xi_t^{vT}F_4^{vT}\left[e_{5v}\ e_{1v}-e_{2v}\right]\xi_t^v \\
&+ \tau_2(t_2)^2\xi_t^{vT}F_5^{vT}(R_1^v)^{-1}F_5^v\xi_t^v + 4\xi_t^{vT}F_5^{vT}\left[\tau_2(t_2)e_{1v}-e_{5v}\right]\xi_t^v \\
&+ 3\tau_2(t_2)\xi_t^{vT}F_6^{vT}(R_2^v)^{-1}F_6^v\xi_t^v + 6\xi_t^{vT}F_6^{vT}\left[\tau_2(t_2)e_{1v}-e_{5v}\right]\xi_t^v
\end{aligned}
$$

Combining the previous inequality with (24) yield the following condition for both directions

$$
\begin{aligned}
div(V(x^h_{t_1}(t_1,t_2), x^v_{t_2}(t_1,t_2))) \leq \left[\xi_t^{hT}\ \xi_t^{vT}\right]\{&\Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 \\
&+\tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\}\begin{bmatrix}\xi_t^h \\ \xi_t^v\end{bmatrix}
\end{aligned}
$$

where Φ_0, Φ_1 and Φ_2 are given as follows

$$
\begin{aligned}
\Phi_0 &= \Psi_0 + \gamma_0 = \begin{bmatrix}\Psi_0^h + \gamma_0^h & \Psi_0^{hv} \\ * & \Psi_0^v + \gamma_0^v\end{bmatrix} \\
\Phi_1 &= \Psi_1 + \gamma_1^h = \begin{bmatrix}\Psi_1^h + \gamma_1^h & \Psi_1^{hv} \\ * & 0\end{bmatrix} \\
\Phi_2 &= \Psi_2 + \gamma_2^v = \begin{bmatrix}0 & \Psi_2^{hv} \\ * & \Psi_2^v + \gamma_2^v\end{bmatrix}
\end{aligned}
$$

with Ψ_0, Ψ_1, Ψ_2 are given in Theorem 2 and γ_0^h, γ_0^v, γ_1^h, γ_2^v, γ_3^h and γ_3^v are given by the following equations

$$\begin{aligned}
\gamma_0^h &= h_1 F_1^{hT}(Q_3^h)^{-1}F_1^h + h_1^2 F_2^{hT}(R_1^h)^{-1}F_2^h + 3h_1 F_3^{hT}(R_2^h)^{-1}F_3^h \\
\gamma_0^v &= h_2 F_1^{vT}(Q_3^v)^{-1}F_1^v + h_2^2 F_2^{vT}(R_1^v)^{-1}F_2^v + 3h_2 F_3^{vT}(R_2^v)^{-1}F_3^v \\
\gamma_1^h &= -F_1^{hT}(Q_3^h)^{-1}F_1^h - 2h_1 F_2^{hT}(R_1^h)^{-1}F_2^h - 3h_1 F_3^{hT}(R_2^h)^{-1}F_3^h \\
&\quad + F_4^{hT}(Q_3^h)^{-1}F_4^h + 3F_6^{hT}(R_2^h)^{-1}F_6^h \\
\gamma_2^v &= -F_1^{vT}(Q_3^v)^{-1}F_1^v - 2h_2 F_2^{vT}(R_1^v)^{-1}F_2^v - 3h_2 F_3^{vT}(R_2^v)^{-1}F_3^v \\
&\quad + F_4^{vT}(Q_3^v)^{-1}F_4^v + 3F_6^{vT}(R_2^v)^{-1}F_6^v \\
\gamma_3^h &= F_2^{hT}(R_1^h)^{-1}F_2^h + F_5^{hT}(R_1^h)^{-1}F_5^h \\
\gamma_3^v &= F_2^{vT}(R_1^v)^{-1}F_2^v + F_5^{vT}(R_1^v)^{-1}F_5^v
\end{aligned}$$

Consider the scalar valued function given by the following equation

$$\begin{bmatrix}\xi_t^{hT} & \xi_t^{vT}\end{bmatrix}\left\{\Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\right\}\begin{bmatrix}\xi_t^h \\ \xi_t^v\end{bmatrix}$$

which can be written as follows

$$\begin{aligned}
&a_0 + \tau_1(t_1)a_{11} + \tau_2(t_2)a_{12} + \tau_1(t_1)^2 a_{21} + \tau_2(t_2)^2 a_{22} \prec 0 \\
&a_0 + \begin{bmatrix}\tau_1(t_1) & \tau_2(t_2)\end{bmatrix}\begin{bmatrix}a_{11} \\ a_{12}\end{bmatrix} + \begin{bmatrix}\tau_1(t_1) & \tau_2(t_2)\end{bmatrix}\begin{bmatrix}a_{21} & 0 \\ 0 & a_{22}\end{bmatrix}\begin{bmatrix}\tau_1(t_1) \\ \tau_2(t_2)\end{bmatrix} \prec 0
\end{aligned}$$

with

$$\begin{aligned}
a_0 &= \xi_t^{hT}(\Psi_0^h + \gamma_0^h)\xi_t^h + 2\xi_t^{hT}\Psi_0^{hv}\xi_t^v + \xi_t^{vT}(\Psi_0^v + \gamma_0^v)\xi_t^v \\
a_{11} &= \xi_t^{hT}(\Psi_1^h + \gamma_1^h)\xi_t^h + 2\xi_t^{hT}\Psi_1^{hv}\xi_t^v \\
a_{12} &= \xi_t^{vT}(\Psi_2^v + \gamma_2^v)\xi_t^v + 2\xi_t^{hT}\Psi_2^{hv}\xi_t^v \\
a_{21} &= \xi_t^{hT}\gamma_3^h\xi_t^h \\
a_{22} &= \xi_t^{vT}\gamma_3^v\xi_t^v
\end{aligned}$$

Note that the scalar valued function $a_0 + \tau_1(t_1)a_{11} + \tau_2(t_2)a_{12} + \tau_1^2(t_1)a_{21} + \tau_2^2(t_2)a_{22}$ *is a quadratic function on the scalars* $\tau_1(t_1)$ *and* $\tau_2(t_2)$*. The coefficients of second order are* $F_2^{hT}(R_1^h)^{-1}F_2^h + F_5^{hT}(R_1^h)^{-1}F_5^h \geq 0$ *and* $F_2^{vT}(R_1^v)^{-1}F_2^v + F_5^{vT}(R_1^v)^{-1}F_5^v \geq 0$ *since* $R_1^h \succ 0$ *and* $R_1^v \succ 0$*. Thus, the function* $a_0 + \tau_1(t_1)a_{11} + \tau_2(t_2)a_{12} + \tau_1(t_1)^2 a_{21} + \tau_2(t_2)^2 a_{22}$ *is a convex quadratic function of* $\tau_1(t_1)$ *and* $\tau_2(t_2)$*.*

Applying lemma 1, we get

$$\begin{cases}
\left\{\Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\right\}_{\tau_1(t_1)=0 \quad \text{and} \quad \tau_2(t_2)=0} \prec 0 \\
\left\{\Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\right\}_{\tau_1(t_1)=h_1 \quad \text{and} \quad \tau_2(t_2)=h_2} \prec 0 \\
\left\{\Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\right\}_{\tau_1(t_1)=h_1 \quad \text{and} \quad \tau_2(t_2)=0} \prec 0 \\
\left\{\Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\right\}_{\tau_1(t_1)=0 \quad \text{and} \quad \tau_2(t_2)=h_2} \prec 0
\end{cases} \tag{25}$$

which implies that

$$\mathcal{W}(t_1,t_2) = \Phi_0 + \tau_1(t_1)\Phi_1 + \tau_2(t_2)\Phi_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v < 0,$$

$\forall\tau_1(t_1) \in [0,h_1]$ *and* $\forall\tau_2(t_2) \in [0,h_2]$.

Consequently, we can write

$$div(V(x^h_{t_1}(\cdot,t_2), x^v_{t_2}(t_1,\cdot))) \leq \begin{bmatrix}\xi_t^{hT} & \xi_t^{vT}\end{bmatrix} \mathcal{W}(t_1,t_2) \begin{bmatrix}\xi_t^h \\ \xi_t^v\end{bmatrix} < 0 \qquad (26)$$

which shows that condition (5) *of Theorem 1 is satisfied.*

Conditions (25) can be written as:

$$\Phi_0 \prec 0 \qquad (27)$$

$$\Phi_0 + h_1\Phi_1 + h_2\Phi_2 + h_1^2\gamma_3^h + h_2^2\gamma_3^v \prec 0 \qquad (28)$$

$$\Phi_0 + h_1\Phi_1 + h_1^2\gamma_3^h \prec 0 \qquad (29)$$

$$\Phi_0 + h_2\Phi_2 + h_2^2\gamma_3^v \prec 0 \qquad (30)$$

Applying Schur complement to (27), (28), (29) and (30), we get $\Gamma_1 \prec 0$ *in (9),* $\Gamma_2 \prec 0$ *in (10),* $\Gamma_3 \prec 0$ *in (11) and* $\Gamma_4 \prec 0$ *in (12). In order to complete the proof that the* $2D$ *continuous time varying delay system is asymptotically stable according to Theorem 1, we need to show that, in addition to* (26)*, the two components of the vector function (4) satisfies conditions (6a) and (6b). First, for the horizontal direction,* $V^h(x^h_{t_1}(t_1,t_2))$*, one can show easily that*

$$V^h(x^h_{t_1}(\cdot,t_2)) \geq \lambda_{min}(P^h)\left\|x^h(t_1,t_2)\right\|^2.$$

Similar argument can be applied for $V^v(x^h_{t_2}(t_1,\cdot))$ *and* c_1 *can be deduced easily. To show that the functional* $V^h(x^h_{t_1}(\cdot,t_2))$ *given by equation (22) has an upper bound, let us consider for instance* $V_1^h(x^h_{t_1}(\cdot,t_2))$ *which can be bounded as follows:*

$$\begin{aligned}
V_1^h(x^h_{t_1}(\cdot,t_2)) &= \begin{bmatrix}x^{hT}(t_1,t_2) & \int_{t_1-h_1}^{t_1} x^{hT}(s,t_2)ds\end{bmatrix} P^h \begin{bmatrix}x^h(t_1,t_2) \\ \int_{t_1-h_1}^{t_1} x^h(s,t_2)ds\end{bmatrix} \\
&\leq \lambda_{max}(P^h)\left\{x^{hT}(t_1,t_2)x^h(t_1,t_2) + \int_{t_1-h_1}^{t_1} x^{hT}(s,t_2)ds \int_{t_1-h_1}^{t_1} x^h(s,t_2)ds\right\} \\
&\leq \lambda_{max}(P^h)\left\{\left\|x^h(t_1,t_2)\right\|^2 + h_1\int_{t_1-h_1}^{t_1} x^{hT}(s,t_2)x^h(s,t_2)ds\right\} \\
&\qquad \text{thanks to lemma 1 in [32]} \\
&\leq \lambda_{max}(P^h)\left\{\left\|x^h(t_1,t_2)\right\|^2 + h_1^2\left\|x^h_{t_1}(\cdot,t_2)\right\|_c^2\right\} \\
&\leq c_{21}\left\|x^h_{t_1}(\cdot,t_2)\right\|_{cl}^2 \quad \textit{with} \quad c_{21} = (1+h_1^2)\lambda_{max}(P^h)
\end{aligned}$$

Consider now a component of $V^h(x^h_{t_1}(\cdot,t_2))$ containing a derivative such as $V^h_5(x^h_{t_1}(\cdot,t_2))$ which can be bounded as follows:

$$\begin{aligned}
V^h_5(x^h_{t_1}(\cdot,t_2)) &= \int_{t_1-h_1}^{t_1} (h_1-t_1+s)^2 \dot{x}^{hT}(s,t_2) R^h_1 \dot{x}^h(s,t_2) ds \\
&\leq h_1^3 \lambda_{max}(R^h_1) \int_{t_1-h_1}^{t_1} \left\| \dot{x}^h(s,t_2) \right\|^2 ds \\
&\leq h_1^3 \lambda_{max}(R^h_1) \left\| \left(\frac{\partial}{\partial t_1} x^h \right)_{t_1} (\cdot,t_2) \right\|_c^2 \\
&\leq c_{25} \left(\left\| x^h_{t_1}(\cdot,t_2) \right\|_{cl}^2 \right) \quad \textit{with} \quad c_{25} = h_1^3 \lambda_{max}(R^h_1)
\end{aligned}$$

Thus, performing similar operations on the other terms of $V^h(x^h_{t_1}(\cdot,t_2))$ and those of $V^v(x^v_{t_2}(t_1,\cdot))$, one can deduce easily that there exists a $c_2 \in \mathbb{R}_+$ such as the upper bound in conditions (6a) and (6b) is satisfied. Therefore, since $V^h(x^v_{t_2}(\cdot,t_2))$ and $V^v(x^v_{t_2}(t_1,\cdot))$ satisfy the conditions of Theorem 1, the 2D continuous time varying delay system is asymptotically stable.

This completes the proof.

Appendix 3: Proof of Theorem 3

The system (1) can be written as

$$\begin{aligned}
\dot{x}^h(t_1,t_2) &= A_h \xi^h_t + A_{hv} \xi^v_t + B_{\omega_1} \omega(t_1,t_2) \\
\dot{x}^v(t_1,t_2) &= A_v \xi^v_t + A_{vh} \xi^h_t + B_{\omega_2} \omega(t_1,t_2) \\
z(t_1,t_2) &= H_1 x^h(t_1,t_2) + H_2 x^v(t_1,t_2) + L\omega(t_1,t_2)
\end{aligned} \tag{31}$$

Similar to the proof given for the Theorem 2, one gets

$$
\begin{aligned}
div(V(x^h_{t_1}(t_1,t_2),x^v_{t_2}(t_1,t_2))) \leq\ & 2\xi_t^{hT}\left[A_h^T\ e_{1h}-e_{3h}\right]P^h\left[e_{1h}\ e_{4h}+e_{5h}\right]^T\xi_t^h \\
&+\xi_t^{hT}\left[e_{1h}\ e_{1h}\right]Q_1^h\left[e_{1h}\ e_{1h}\right]^T\xi_t^h \\
&-\xi_t^{hT}(1-d_1)\left[e_{1h}\ e_{2h}\right]Q_1^h\left[e_{1h}\ e_{2h}\right]^T\xi_t^h \\
&+2\xi_t^{hT}\left[A_h^T\ e_{0h}\right]Q_1^h\left[\tau_1(t_1)e_{1h}\ e_{5h}\right]^T\xi_t^h \\
&+\xi_t^{hT}\left[e_{1h}\ e_{1h}\right]Q_2^h\left[e_{1h}\ e_{1h}\right]^T\xi_t^h \\
&-\xi_t^{hT}\left[e_{1h}\ e_{3h}\right]Q_2^h\left[e_{1h}\ e_{3h}\right]^T\xi_t^h \\
&+h_2\xi_t^{hT}\left[e_{0h}\ A_{vh}^T\right]Q_3^v\left[e_{0h}\ A_{vh}^T\right]^T\xi_t^h \\
&+h_2^2\xi_t^{hT}A_{vh}^TR_1^vA_{vh}\xi_t^h \\
&+2\xi_t^{hT}\left[h_1A_h^T\ e_{0h}\right]Q_2^h\left[e_{1h}\ e_{4h}+e_{5h}\right]^T\xi_t^h \\
&+h_1\xi_t^{hT}\left[e_{1h}\ A_h^T\right]Q_3^h\left[e_{1h}\ A_h^T\right]^T\xi_t^h \\
&+h_2^3\xi_t^{hT}A_{vh}^TR_2^hA_{vh}\xi_t^h+h_1^2\xi_t^{hT}A_h^TR_1^hA_h\xi_t^h \\
&+h_1^3\xi_t^{hT}A_h^TR_2^hA_h\xi_t^h+2\xi_t^{hT}F_1^{hT}\left[e_{4h}\ e_{2h}-e_{3h}\right]\xi_t^h \\
&+h_1^2\xi_t^{vT}A_{hv}^TR_1^hA_{hv}\xi_t^v+h_1^3\xi_t^{vT}A_{hv}^TR_2^hA_{hv}\xi_t^v \\
&+2\xi_t^{vT}\left[A_v^T\ e_{1v}-e_{3v}\right]P^v\left[e_{1v}\ e_{4v}+e_{5v}\right]^T\xi_t^v \\
&+\xi_t^{vT}\left[e_{1v}\ e_{1v}\right]Q_1^v\left[e_{1v}\ e_{1v}\right]^T\xi_t^v \\
&+h_1\xi_t^{vT}\left[e_{0v}\ A_{hv}^T\right]Q_3^h\left[e_{0v}\ A_{hv}^T\right]^T\xi_t^v \\
&-\xi_t^{vT}(1-d_2)\left[e_{1v}\ e_{2v}\right]Q_1^v\left[e_{1v}\ e_{2v}\right]^T\xi_t^v \\
&+2\xi_t^{vT}\left[A_v^T\ e_{0v}\right]Q_1^v\left[\tau_2(t_2)e_{1v}\ e_{5v}\right]^T\xi_t^v \\
&+\xi_t^{vT}\left[e_{1v}\ e_{1v}\right]Q_2^v\left[e_{1v}\ e_{1v}\right]^T\xi_t^v \\
&-\xi_t^{vT}\left[e_{1v}^T\ e_{3v}^T\right]Q_2^v\left[e_{1v}\ e_{3v}\right]\xi_t^v \\
&+2\xi_t^{vT}\left[h_2A_v^T\ e_{0v}\right]Q_2^v\left[e_{1v}\ e_{4v}+e_{5v}\right]^T\xi_t^v \\
&+h_2\xi_t^{vT}\left[e_{1v}\ A_v^T\right]Q_3^v\left[e_{1v}\ A_v^T\right]^T\xi_t^v \\
&+h_2^2\xi_t^{vT}A_v^TR_1^vA_v\xi_t^v+h_2^3\xi_t^{vT}A_v^TR_2^vA_v\xi_t^v \\
&+2\xi_t^{vT}F_1^{vT}\left[e_{4v}\ e_{2v}-e_{3v}\right]\xi_t^v
\end{aligned}
$$

$$
\begin{aligned}
&+ h_1\omega^T \left[e_{0\omega}\ B_{\omega_1}^T\right] Q_3^h \left[e_{0\omega}\ B_{\omega_1}^T\right]^T \omega\\
&+ h_1^2\omega^T B_{\omega_1}^T R_1^h B_{\omega_1}\omega + h_1^3\omega^T B_{\omega_1}^T R_2^h B_{\omega_1}\omega\\
&+ h_2\omega^T \left[e_{0\omega}\ B_{\omega_2}^T\right] Q_3^v \left[e_{0\omega}\ B_{\omega_2}^T\right]^T \omega\\
&+ h_2^2\omega^T B_{\omega_2}^T R_1^v B_{\omega_2}\omega + h_2^3\omega^T B_{\omega_2}^T R_2^v B_{\omega_2}\omega\\
&+ 2\xi_t^{hT} \left[e_{1h}\ e_{4h} + e_{5h}\right] P^h \left[A_{hv}^T\ e_{0v}\right]^T \xi_t^v\\
&+ 2\xi_t^{hT} \left[\tau_1(t_1)e_{1h}\ e_{5h}\right] Q_1^h \left[A_{hv}^T\ e_{0v}\right]^T \xi_t^v\\
&+ 2h_1\xi_t^{hT} \left[e_{1h}\ A_h^T\right] Q_3^h \left[e_{0v}\ A_{hv}^T\right]^T \xi_t^v + 2h_1^2\xi_t^{hT} A_h^T R_1^h A_{hv}\xi_t^v\\
&+ 2h_1^3\xi_t^{hT} A_h^T R_2^h A_{hv}\xi_t^v\\
&+ 2\xi_t^{hT} \left[A_{vh}^T\ e_{0h}\right] P^v \left[e_{1v}\ e_{4v} + e_{5v}\right]^T \xi_t^v\\
&+ 2\xi_t^{hT} \left[h_2 A_{vh}^T\ e_{0h}\right] Q_2^v \left[e_{1v}\ e_{4v} + e_{5v}\right]^T \xi_t^v\\
&+ 2h_2\xi_t^{hT} \left[e_{0h}\ A_{vh}^T\right] Q_3^v \left[e_{1v}\ A_v^T\right]^T \xi_t^v\\
&+ 2h_2^2\xi_t^{hT} A_{vh}^T R_1^v A_v\xi_t^v + 2h_2^3\xi_t^{hT} A_{vh}^T R_2^v A_v\xi_t^v\\
&+ 2\xi_t^{hT} \left[h_1 e_{1h}\ e_{4h} + e_{5h}\right] Q_2^h \left[A_{hv}^T\ e_{0v}\right]^T \xi_t^v\\
&+ 2\xi_t^{hT} \left[A_{vh}^T\ e_{0h}\right] Q_1^v \left[\tau_2(t_2)e_{1v}\ e_{5v}\right]^T \xi_t^v\\
&+ 2\xi_t^{hT} \left[e_{1h}\ e_{4h} + e_{5h}\right] P^h \left[B_{\omega_1}^T\ e_{0\omega}\right]^T \omega\\
&+ 2\xi_t^{hT} \left[\tau_1(t_1)e_{1h}\ e_{5h}\right] Q_1^h \left[B_{\omega_1}^T\ e_{0\omega}\right]^T \omega\\
&+ 2\xi_t^{hT} \left[h_1 e_{1h}\ e_{4h} + e_{5h}\right] Q_2^h \left[B_{\omega_1}^T\ e_{0\omega}\right]^T \omega\\
&+ 2h_1\xi_t^{hT} \left[e_{1h}\ A_h^T\right] Q_3^h \left[e_{0\omega}\ B_{\omega_1}^T\right]^T \omega\\
&+ 2h_1^2\xi_t^{hT} A_h^T R_1^h B_{\omega_1}\omega + 2h_1^3\xi_t^{hT} A_h^T R_2^h B_{\omega_1}\omega\\
&+ 2h_2\xi_t^{hT} \left[e_{1h}\ A_{vh}^T\right] Q_3^v \left[e_{0\omega}\ B_{\omega_2}^T\right]^T \omega\\
&+ 2h_2^3\xi_t^{hT} A_{vh}^T R_2^v B_{\omega_2}\omega + 2h_2^3\xi_t^{hT} A_{vh}^T R_2^v B_{\omega_2}\omega
\end{aligned}
$$

$$
\begin{aligned}
&+ 2h_1\xi_t^{vT}\left[e_{0v}\ A_{hv}^T\right]Q_3^h\left[e_{0\omega}\ B_{\omega_1}^T\right]^T\omega + 2h_1^2\xi_t^{vT}A_{hv}^TR_1^hB_{\omega_1}\omega + 2h_1^3\xi_t^{vT}A_{hv}^TR_2^hB_{\omega_1}\omega\\
&+ 2\xi_t^{vT}\left[e_{1v}\ e_{4v}+e_{5v}\right]P^v\left[B_{\omega_2}^T\ e_{0\omega}\right]^T\omega + 2\xi_t^{vT}\left[\tau_2(t_2)e_{1v}\ e_{5v}\right]Q_1^v\left[B_{\omega_2}^T\ e_{0\omega}\right]^T\omega\\
&+ 2\xi_t^{vT}\left[h_2e_{1v}\ e_{4v}+e_{5v}\right]Q_2^v\left[B_{\omega_2}^T\ e_{0\omega}\right]^T\omega + 2h_2\xi_t^{vT}\left[e_{0v}\ A_v^T\right]Q_3^v\left[e_{0\omega}\ B_{\omega_2}^T\right]^T\omega\\
&+ 2h_2^2\xi_t^{vT}A_v^TR_1^vB_{\omega_2}\omega + 2h_2^3\xi_t^{vT}A_v^TR_2^vB_{\omega_2}\omega\\
&+ (h_1-\tau_1(t_1))\xi_t^{hT}F_1^{hT}(Q_3^h)^{-1}F_1^h\xi_t^h\\
&+ (h_1-\tau_1(t_1))^2\xi_t^{hT}F_2^{hT}(R_1^h)^{-1}F_2^h\xi_t^h + 4\xi_t^{hT}F_2^{hT}\left[(h_1-\tau_1(t_1))e_{2h}-e_{4h}\right]\xi_t^h\\
&+ 3(h_1-\tau_1(t_1))\xi_t^{hT}F_3^{hT}(R_2^h)^{-1}F_3^h\xi_t^h + 6\xi_t^{hT}F_3^{hT}\left[(h_1-\tau_1(t_1))e_{2h}-e_{4h}\right]\xi_t^h\\
&+ \tau_1(t_1)\xi_t^{hT}F_4^{hT}(Q_3^h)^{-1}F_4^h\xi_t^h + 2\xi_t^{hT}F_4^{hT}\left[e_{5h}\ e_{1h}-e_{2h}\right]\xi_t^h\\
&+ \tau_1(t_1)^2\xi_t^{hT}F_5^{hT}(R_1^h)^{-1}F_5^h\xi_t^h + 4\xi_t^{hT}F_5^{hT}\left[\tau_1(t_1)e_{1h}-e_{5h}\right]\xi_t^h\\
&+ 3\tau_1(t_1)\xi_t^{hT}F_6^{hT}(R_2^h)^{-1}F_6^h\xi_t^h + 6\xi_t^{hT}F_6^{hT}\left[\tau_1(t_1)e_{1h}-e_{5h}\right]\xi_t^h\\
&+ (h_2-\tau_2(t_2))\xi_t^{vT}F_1^{vT}(Q_3^v)^{-1}F_1^v\xi_t^v\\
&+ (h_2-\tau_2(t_2))^2\xi_t^{vT}F_2^{vT}(R_1^v)^{-1}F_2^v\xi_t^v + 4\xi_t^{vT}F_2^{vT}\left[(h_2-\tau_2(t_2))e_{2v}-e_{4v}\right]\xi_t^v\\
&+ 3(h_2-\tau_2(t_2))\xi_t^{vT}F_3^{vT}(R_2^v)^{-1}F_3^v\xi_t^v + 6\xi_t^{vT}F_3^{vT}\left[(h_2-\tau_2(t_2))e_{2v}-e_{4v}\right]\xi_t^v\\
&+ \tau_2(t_2)\xi_t^{vT}F_4^{vT}(Q_3^v)^{-1}F_4^v\xi_t^v + 2\xi_t^{vT}F_4^{vT}\left[e_{5v}\ e_{1v}-e_{2v}\right]\xi_t^v\\
&+ \tau_2(t_2)^2\xi_t^{vT}F_5^{vT}(R_1^v)^{-1}F_5^v\xi_t^v + 4\xi_t^{vT}F_5^{vT}\left[\tau_2(t_2)e_{1v}-e_{5v}\right]\xi_t^v\\
&+ 3\tau_2(t_2)\xi_t^{vT}F_6^{vT}(R_2^v)^{-1}F_6^v\xi_t^v + 6\xi_t^{vT}F_6^{vT}\left[\tau_2(t_2)e_{1v}-e_{5v}\right]\xi_t^v
\end{aligned}
$$

Computing $z^T(t_1,t_2)z(t_1,t_2)-\gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2)$, we have

$$
\begin{aligned}
z^T(t_1,t_2)z(t_1,t_2)-\gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2) &= \xi_t^{hT}\tilde{H}_1\xi_t^h + \xi_t^{vT}\tilde{H}_2\xi_t^v\\
&\quad + \omega^TL^TL\omega - \gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2)
\end{aligned}
$$

with $\tilde{H}_1 = diag\left\{H_1^TH_1\quad 0\quad 0\quad 0\quad 0\right\}$ *and* $\tilde{H}_2 = diag\left\{H_2^TH_2\quad 0\quad 0\quad 0\quad 0\right\}$

Thus, we have

$$
div(V(x_{t_1}^h(t_1,t_2),x_{t_2}^v(t_1,t_2))) + z^T(t_1,t_2)z(t_1,t_2) - \gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2) < 0 \tag{32}
$$

which is equivalent to the following inequality

$$
div(V(x_{t_1}^h(t_1,t_2),x_{t_2}^v(t_1,t_2))) + \left[\xi_t^{hT}\ \xi_t^{vT}\ \omega^T\right]\begin{bmatrix}\tilde{H}_1 & 0 & 0\\ * & \tilde{H}_2 & 0\\ * & * & L^TL-\gamma^2I\end{bmatrix}\begin{bmatrix}\xi_t^h\\ \xi_t^v\\ \omega\end{bmatrix} < 0 \tag{33}
$$

Thus, we have

$$
\left[\xi_t^{hT}\ \xi_t^{vT}\ \omega^T\right]\left\{\Gamma_0 + \tau_1(t_1)\Gamma_1 + \tau_2(t_2)\Gamma_2 + \tau_1(t_1)^2\gamma_3^h + \tau_2(t_2)^2\gamma_3^v\right\}\begin{bmatrix}\xi_t^h\\ \xi_t^v\\ \omega\end{bmatrix} < 0
$$

where Γ_0, Γ_1 and Γ_2 are given by the following equations :

$$\Gamma_0 = \Xi_0 + \gamma_0, \Gamma_1 = \Xi_1 + \gamma_1, \Gamma_2 = \Xi_2 + \gamma_2$$

with Ξ_0, Ξ_1 and Ξ_2 are given in Theorem 3 and γ_0, γ_1, γ_2, γ_3^h and γ_3^v are given previously in the proof of theorem 2.

As in the proof of Theorem 2, by using lemma 1 and Schur complement, one gets the LMIs given in Theorem 3.

For a prescribed scalar $\gamma \succ 0$, the performance index is defined as

$$J = \int_0^\infty \int_0^\infty (z^T(t_1,t_2)z(t_1,t_2) - \gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2))dt_1dt_2$$

When $\omega(t_1,t_2) = 0$, (32) means $div(V(x_{t_1}^h(t_1,t_2), x_{t_2}^v(t_1,t_2))) < 0$, therefore the system (31) is asymptotically stable in the case of $\omega(t_1,t_2) = 0$.

When $\omega(t_1,t_2) \neq 0$, integrating twice both sides of (32) from 0 to ∞ and from 0 to ∞ yields :

$$\int_0^\infty \int_0^\infty \left[z^T(t_1,t_2)z(t_1,t_2) - \gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2)\right] dt_1dt_2 \tag{34}$$

$$< -\int_0^\infty \int_0^\infty div(V(x_{t_1}^h(t_1,t_2), x_{t_2}^v(t_1,t_2)))dt_1dt_2 \tag{35}$$

From (5), it can be obtained that :

$$\int_0^\infty \int_0^\infty div(V(x_{t_1}^h(t_1,t_2), x_{t_2}^v(t_1,t_2)))dt_1dt_2 = \int_0^\infty \int_0^\infty \frac{\partial V^h(x_{t_1}^h(t_1,t_2))}{\partial t_1}dt_1dt_2 + \int_0^\infty \int_0^\infty \frac{\partial V^v(x_{t_2}^v(t_1,t_2))}{\partial t_2}dt_1dt_2$$

Then, the inequatilty (34) can be written as :

$$\int_0^\infty \int_0^\infty \frac{\partial V^h(x_{t_1}^h(t_1,t_2))}{\partial t_1}dt_1dt_2 + \int_0^\infty \int_0^\infty \frac{\partial V^v(x_{t_2}^v(t_1,t_2))}{\partial t_2}dt_2dt_1 + J < 0$$

$$\int_0^\infty \left[V^h(x_{t_1}^h(\infty,t_2)) - V^h(x_{t_1}^h(0,t_2))\right] dt_2 + \int_0^\infty \left[V^v(x_{t_2}^v(t_1,\infty)) - V^v(x_{t_2}^v(t_1,0))\right] dt_1 + J < 0$$

$$\int_0^\infty V^h(x_{t_1}^h(\infty,t_2))dt_2 - \int_0^\infty V^h(x_{t_1}^h(0,t_2))dt_2 + \int_0^\infty V^v(x_{t_2}^v(t_1,\infty))dt_1 - \int_0^\infty V^v(x_{t_2}^v(t_1,0))dt_1 + J < 0$$

Note that under zero initial conditions, we have $\int_0^\infty V^h(x_{t_1}^h(0,t_2))dt_2 = 0$ *and* $\int_0^\infty V^v(x_{t_2}^v(t_1,0))dt_1 = 0$ *while* $\int_0^\infty V^h(x_{t_1}^h(\infty,t_2))dt_2 > 0$ *and* $\int_0^\infty V^v(x_{t_2}^v(t_1,\infty))dt_1 > 0$

Therefore $J < 0$ *which can be written as*

$$\int_0^\infty \int_0^\infty z^T(t_1,t_2)z(t_1,t_2)dt_1dt_2 < \gamma^2\omega^T(t_1,t_2)\omega(t_1,t_2)dt_1dt_2$$

That is

$$\|z(t_1,t_2)\|_2 < \gamma\,\|\omega(t_1,t_2)\|_2$$

This completes the proof.